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ON THE STRUCTURE OF THE SOLUTION SET OF EVOLUTION INCLUSIONS WITH TIME–DEPENDENT SUBDIFFERENTIALS

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Abstract. In this paper we consider evolution inclusions driven by a time dependent subdifferential operator and a set-valued perturbation term. First we show that the problem with a convex-valued, h^*-u.s.c. orientor field (i.e. perturbation term) has a nonempty solution set which is an R_δ-set in $C(T,H)$, in particular then compact and acyclic. For the non convex problem (i.e. the orientor field is non convex-valued), without assuming that the functional $\varphi(t,x)$ of the subdifferential is of compact type, we show that for every initial datum $\xi \in \overline{\text{dom}}\varphi(0,\cdot)$ the solution set $S(\xi)$ is nonempty and we also produce a continuous selector for the multifunctions $\xi \to S(\xi)$. Some examples of distributed parameter systems are also included.

1. Introduction

In a recent paper (cf. [**21**]) we established the nonemptiness and path-connectedness of the solution set of an evolution inclusion driven by a time-dependent subdifferential $\partial\varphi(t,x)$ and a nonconvex-valued, h-Lipschitz in x multivalued perturbation term $F(t,x)$.

In this paper first we consider time-dependent subdifferential evolution inclusions with a convex valued perturbation term $F(t,x)$ satisfying a more general continuity hypothesis in the x-variable. For such evolution inclusions we show that the solution set is R_δ in $C(T,H)$, in particular then nonempty, compact and connected. Then we return to the nonconvex problem with an h-Lipschitz in x perturbation term $F(t,x)$. Without assuming $\varphi(t,\cdot)$ is of compact type (which in the time invariant case means that $\partial\varphi(\cdot)$ generates a compact semigroup of contractions), we establish the nonemptiness of the solution set $S(\xi)$ and in addition we generate a continuous selector for the multifunction $\xi \mapsto S(\xi)$. Finally we present some examples of parabolic distributed parameter systems illustrating the applicability of our results.

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The problem of connectedness of the solution set of differential inclusions in R^N was investigated by many authors. The first to establish that this set is R_δ in $C(T, R^N)$ were Himmelberg-Van Vleck (cf. [**12**]) for autonomous systems and De Blasi-Myjak (cf. [**7**]) for nonautonomous systems. The result of De Blasi-Myjak was extended to systems with state constraints by Hu-Papageorgiou (cf. [**14**]). We should also mention the remarkable recent work of De Blasi-Pianigiani (cf. [**8**], [**9**]) who developed the so-called "Baire category method" to study the structure of the extremal solutions of differential inclusions in $\mathbb{R}^N$ and in Banach spaces. However their compactness and continuity hypotheses preclude the applicability of their results to systems involving unbounded operators (i.e. distributed parameter systems with multivalued terms). There is also the very recent work of De Blasi-Pianigiani-Staicu (cf. [**10**]) which deals with semilinear evolution inclusion monitored by a time invariant unbounded linear operator generating a C_0-semigroup. Extension of the results of De Blasi-Pianigiani-Staicu can be found in Hu-Lakshmikantham-Papageorgiou (cf. [**13**]). We should also mention the work of Ballotti (cf. [**1**]) which deals with semilinear evolution equation driven by a time invariant linear operator which is the generator of a C_0-semigroup and by a single-valued perturbation term $f(t, x)$ which is jointly continuous. Our work here extends that of Ballotti. Finally on the question of existence of continuous selectors for the solution multifunction $\xi \mapsto S(\xi)$ there are the works of Cellina (cf. [**5**]) for differential inclusions in R^N and of Staicu (cf. [**23**]) for evolution inclusions driven by a time invariant maximal monotone operator on a Hilbert space.

2. Mathematical Preliminaries

Let X be a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : A \text{ nonempty, closed and (convex)}\},$$
$$P_{(w)k(c)}(X) = \{A \subseteq X : A \text{ nonempty, (weakly-)compact (convex)}\}.$$

If (Ω, Σ, μ) is a finite measure space, a multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$, the function $\omega \mapsto d(x, F(\omega)) = \inf\big\{\|x - z\| : z \in F(\omega)\big\}$ is measurable. If $F(\cdot)$ is measurable, then $\operatorname{Gr} F = \big\{(\omega, x) \in \Omega \times X : x \in F(\omega)\big\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability), while the converse is true if Σ is μ-complete. By S_F^p $(1 \le p \le \infty)$ we will denote the set of all measurable selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(\Omega, X)$; i.e. $S_F^p = \big\{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \;\; \mu\text{-a.e.}\big\}$. In general this set may be empty. It is easy to check using Aumann's selection Theorem (cf. [**24**, Theorem 5.10]), that for a graph measurable multifunction $F\colon \Omega \to 2^X \setminus \{\emptyset\}$, S_F^p is nonempty if and only if the function $\omega \mapsto \inf\{\|z\| : z \in F(\omega)\}$ belongs to $L^p(\Omega, \mathbb{R}_+)$. Recall that a subset K of $L^p(\Omega, X)$ is decomposable if for every triple $(f, g, A) \in K \times K \times \Sigma$, we have $f\chi_A + g\chi_{A^c} \in K$, where χ_A denotes the characteristic function of the set A. Clearly S_F^p is decomposable.

A subset A of X is said to be a absolute retract if, given any metric space Y and a closed $B \subseteq Y$ and a continuous function $f\colon B \to A$, there exists a continuous extensions $\tilde{f}\colon Y \to A$ of f. Then A is said to be a R_δ-set if $A = \cap_{n\geq 1} A_n$ for a decreasing sequence of compact absolute retracts A_n of X (cf. [**15**]). Every R_δ-set is acyclic.

Recall that on $P_f(X)$ we define a generalized metric, known in the literature as the "Hausdorff metric", by setting, for A, $B \in P_f(X)$,

$$h(A,B) = \max\bigl\{\sup\{d(a,B) : a \in A\},\quad \sup\{d(b,A) : b \in B\}\bigr\}$$

(where $d(a,B) = \inf\{\|a-b\| : b \in B\}$; similarly for $d(b,A)$). A multifunction $F\colon T \to P_f(X)$ is said to be Hausdorff continuous (h-continuous) if it is continuous from T into the metric space $(P_f(X), h)$. Moreover, F is said to be Hausdorff upper semicontinuous (h^*-u.s.c.) if, for every $t \in T$ and for all $\varepsilon > 0$ there exists $\delta > 0$, such that $|t - t'| < \delta \Rightarrow F(t') \subseteq F(t) + \varepsilon B_1$, where B_1 is the unit ball in X.

Let $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We will say that $\varphi(\cdot)$ is proper, if it not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and l.s.c. (usually this family of $\mathbb{R}$-valued functions is denoted by $\Gamma_0(X)$). By dom φ we will denote the effective domain of $\varphi(\cdot)$; i.e. dom $\varphi = \{x \in X : \varphi(x) < \infty\}$. The subdifferential of $\varphi(\cdot)$ at x, is the set

$$\partial\varphi(x) = \{x^* \in X^* : (x^*, y - x) \leq \varphi(y) - \varphi(x),\ \forall\, y \in \operatorname{dom} \varphi\}$$

(in this definition, by $(\cdot,\cdot)$ we denote the duality brackets for the pair (X, X^*)). It is well-known that if $\varphi(\cdot)$ is Gateaux differentiable at x, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $\varphi \in \Gamma_0(X)$ is of compact type, if for all $\lambda \in \mathbb{R}_+$, the level set $\{x \in X : \varphi(x) + \|x\|^2 \leq \lambda\}$ is compact.

Let $T = [0,b]$ and H a separable Hilbert space. We consider the following multivalued Cauchy problem:

$$(1) \qquad -\dot{x}(t) \in \partial\varphi(t, x(t)) + F(t, x(t)),\ \text{a.e. on } T,\quad x(0) = \xi,\quad \xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot).$$

By a solution of (1) we mean a function $x \in C(T, H)$ such that $x(\cdot)$ is absolutely continuous on any closed subinterval of $(0,b)$ and with the property:

1) $x(t) \in \operatorname{dom}\, \varphi(t,\cdot)$, a.e. on T;

2) $\exists\, f \in L^2(T,H)$ such that $f(t) \in F(t,x(t))$ and $-\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t)$, a.e. on T;

3) $x(0) = \xi$.

Recall that an absolutely continuous functions $x\colon (0,b) \to H$ is differentiable almost everywhere (see [**2**, Theorem 2.1, p. 16]) and so in problem (1) the derivative $\dot{x}(\cdot)$ is a strong derivative.

Following S. Yotsutani (cf. [**26**]) we make the following hypothesis on the function $\varphi(t,x)$, which will be in effect throughout this paper:

$H(\varphi)$: $\qquad \varphi\colon T \times X \to \mathbb{R} \cup \{+\infty\}$ is a function such that

(i) $\forall\, t \in T$, $x \mapsto \varphi(t,x)$ is proper, convex, l.s.c;

(ii) for every integer $r > 0$, there exist $K_r > 0$, an absolutely continuous function $g_r\colon T \to R$ with $\dot{g}_r \in L^{\beta}(T)$ and a function of bounded variation $h_r\colon T \to R$ such that if $t \in T$, $x \in \operatorname{dom}\varphi(t,\cdot)$ with $\|x\| \le r$ and $s \in [t,b]$, then there exists $\hat{x} \in \operatorname{dom}\varphi(s,\cdot)$ satisfying

$$\|x - \hat{x}\| \le |g_r(s) - g_r(t)|(\varphi(t,x) + K_r)^{\alpha}$$

and

$$\varphi(s,\hat{x}) \le \varphi(t,x) + |h_r(s) - h_r(t)|(\varphi(t,x) + K_r)$$

where $\alpha \in [0,1]$ and $\beta = 2$ if $\alpha \in \left[0,\frac{1}{2}\right]$ or $\beta = 1/(1-\alpha)$ if $\alpha \in \left[\frac{1}{2},1\right]$.

Remark. Hypothesis $H(\varphi)$(ii) allows the effective domain $\operatorname{dom}\varphi(t,\cdot)$ of $\varphi(t,\cdot)$ to vary in a regular way with respect to $t \in T$ without excluding the possibility that $\operatorname{dom}\varphi(t,\cdot) \cap \operatorname{dom}\varphi(s,\cdot) = \emptyset$ if $t \neq s$. This hypothesis, which is suitable for the analysis of obstacle problems, has its origin (in more restrictive form) in the works of Kemnochi (cf. [**16**]) and Yamada (cf. [**25**]).

3. Topological Structure of the Solution Set

Our first result establishes the nonemptiness and the topological structure of the solution set $S(\xi) \subseteq C(T,H)$ of (1). We will need the following hypothesis on the orienter field F:

$H(F)$: $F\colon T \times H \to P_{fc}(H)$ is a multifunction such that

(i) $\forall\, x \in H$, $t \mapsto F(t,x)$ is measurable;

(ii) $\forall\, t \in T$, $x \mapsto F(t,x)$ is h^*-u.s.c.;

(iii) $\exists\, a, c \in L^2(T,\mathbb{R}_+)$:

$$\|F(t,x)\| = \sup\{\|z\| : z \in F(t,x)\} \le a(t) + c(t)\|x\|, \quad \text{a.e. in } T, \forall\, x \in H.$$

In the proof of the structural theorem concerning $S(\xi)$ we will need some auxiliary results which we state next.

The first is an approximation lemma which can be proved as Proposition 4.1. of De Blasi (cf. [**6**]), with some appropriate modifications to accomodate for the presence of $t \in T$.

Lemma 1. *Let $F\colon T \times X \to P_{fc}(H)$ be a multifunction such that*

(i) *$\forall\, x \in H$, $t \mapsto F(t,x)$ is measurable;*

(ii) *$\forall\, t \in T$, $x \mapsto F(t,x)$ is h^*-u.s.c.;*

(iii) *$\exists\, \psi \in L^2(T,\mathbb{R}_+) : \|F(t,x)\| \le \psi(t)$, a.e. in T, $\forall\, x \in H$;*

then there exists a sequence of multifunctions $F_n: T \times H \to P_{fc}(H)$, $n \geq 1$, with the properties:

I) $\forall n \geq 1$ *and* $\forall x \in H$ *there exist* $k_n(x) > 0$ *and* $\varepsilon_n > 0$ *such that if* $x_1, x_2 \in \overline{B}_{\varepsilon_n}(x) = \{y \in H : \|x-y\| \leq \varepsilon_n\}$, *then* $h(F_n(t,x_1), F_n(t,x_2)) \leq k_n(x)\psi(t)\|x_1 - x_2\|$ *a.e. on* T *(i.e.* $F_n(t,\cdot)$ *is locally h-Lipschitz);*

II) $F(t,x) \subseteq \cdots \subseteq F_n(t,x) \subseteq F_{n+1}(t,x) \subseteq \ldots$, $\forall (t,x) \in T \times H$;

III) $\|F_n(t,x)\| \leq \psi(t)$ *a.e. in* T, $\forall x \in H$;

IV) $F_n(t,x) \xrightarrow{h} F(t,x)$ *as* $n \to \infty$ *for every* $(t,x) \in T \times H$

and finally there exists maps $u_n: T \times H \to H$, $n \geq 1$, *measurable in* t, *locally-Lipschitz in* x *(as* $F_n(t,\cdot)$*) and* $u_n(t,x) \in F_n(t,x)$ *for every* $(t,x) \in T \times H$.

Moreover if $F(t,\cdot)$ *is h-continuous, then for every* $n \geq 1$, $t \to F_n(t,x)$ *is measurable.*

Proof. For each (fixed) $n \geq 1$ $\mathcal{S}_n = \left\{b\left(x, \frac{1}{3^n}\right)\right\}_{x \in H}$ is an open cover of H. Let $\mathcal{P}_n = \{\gamma_x^n\}_{x \in H}$ be a locally-Lipschitz partition of unity subordinate to $\mathcal{S}_n$.

Let $x_0 \in H$. Then there exist $k_n(x_0)$ and $\varepsilon_n > 0$ such that

$$N(n,x_0) = \left\{x \in H : \operatorname{supp} \gamma_x^n \cap \overline{B}_{\varepsilon_n}(x_0) \neq \emptyset\right\}$$

is finite and for every $x_1, x_2 \in \overline{B}_{\varepsilon_n}(x_0)$ we have

$$\sum_{x_k^n \in N(n,x_0)} \left|\gamma_{x_k^n}^n(x_1) - \gamma_{x_k^n}^n(x_2)\right| \leq k_n(x_0)\|x_1 - x_2\|.$$

Define

$$F_n(t,x_0) = \overline{\sum_{x_k^n \in N(n,x_0)} \gamma_{x_k^n}^n(x_0) G_k^n(t)}$$

where $G_k^n(t) = \overline{\operatorname{co}} F\left(t, B\left(x_k^n, \frac{2}{3^n}\right)\right)$. Evidently since the function $x \mapsto \gamma_x^n(x_0)$ vanish outside the set $N(n,x_0)$ we have $F_n(t,x_0) = \overline{\sum_{x \in H} \gamma_x^n(x_0) G_x^n(t)}$ with $G_x^n(t) = \overline{\operatorname{co}} F\left(t, B\left(x, \frac{2}{3^n}\right)\right)$.

Now define

$$u_n(t,x_0) = \sum_{x_k^n \in N(n,x_0)} \gamma_{x_k^n}^n(x_0) f_k^n(t)$$

with $f_k^n(\cdot)$ being a measurable selector of $G_k^n(\cdot)$. It exists since by hypothesis $t \mapsto F(t,x)$ is measurable. Clearly for every $x \in \overline{B}_{\varepsilon_n}(x_0)$ we have

$$F_n(t,x) = \overline{\sum_{x_k^n \in N(n,x_0)} \gamma_{x_k^n}^n(x) G_k^n(t)} \quad \text{and} \quad u_n(t,x) = \sum_{x_k^n \in N(n,x_0)} \gamma_{x_k^n}^n(x) f_k^n(t).$$

So for every $x_1, x_2 \in \overline{B}_{\varepsilon_n}(x_0)$ and almost all $t \in T$ we have

$$h(F_n(t,x_1), F_n(t,x_2)) \leq k_n(x_0)\psi(t)\|x_1 - x_2\|$$

and

$$\|u_n(t,x_1) - u_n(t,x_2)\| \le k_n(x_0)\psi(t)\|x_1 - x_2\|$$

which establishes the local h-Lipshitzness of the approximating multifunctions $F_n(t,\cdot)$, $n \ge 1$, and the local Lipshitzness of the sector $u_n(t,\cdot)$, $n \ge 1$.

Next we will show that $\{F_n(t,x)\}_{n\ge1}$ is a increasing sequence. Fix $y \in H$ and let $M(n,y)$ (resp. $M(n+1,y)))$ be the nonempty finite set of all functions $\gamma_x^n \in \mathcal{P}_n$ (resp. $\gamma_x^{n+1} \in \mathcal{P}_{n+1}$) whose support contains y. Let $1 \le j \le |M(n,y)|$ and $1 \le k \le |M(n+1,y)|$. We will show that $G^{n+1}_{x_k^{n+1}}(t) \subseteq G^n_{x_j^n}(t)$, $t \in T$. To this end observe that since $y \in \operatorname{supp} \gamma^n_{x_j^n}$ and $y \in \operatorname{supp} \gamma^{n+1}_{x_k^{n+1}}$, we have $y \in B\left(x_k^{n+1}, \frac{1}{3^{n+1}}\right)$ and $y \in B\left(x_j^n, \frac{1}{3^n}\right)$.

So if $z \in B\left(x_k^{n+1}, \frac{2}{3^{n+1}}\right)$ we have

$$\|z - x_j^n\| \le \|z - x_k^{n+1}\| + \|x_k^{n+1} - y\| + \|y - x_j^n\| < \frac{2}{3^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{3^n} = \frac{2}{3^n}$$

and so $B\left(x_k^{n+1}, \frac{2}{3^{n+1}}\right) \subseteq B\left(x_j^n, \frac{2}{3^n}\right)$ which in turn implies that $G^{n+1}_{x_k^{n+1}}(t) \subseteq G^n_{x_j^n}(t)$, $t \in T$. From this fact and the definition of $F_n(t,y)$ we immediately deduce that $F_n(t,y) \subseteq F_{n+1}(t,y)$. In a similar fashion we can also get that $F(t,y) \subseteq F_n(t,y)$ for every $n \ge 1$ and every $(t,y) \in T \times H$.

Now we will show that $F_n(t,x) \xrightarrow{h} F(t,x)$ as $n \to \infty$ for every $(t,x) \in T \times H$. Since by hypothesis $F(t,\cdot)$ is h^*-u.s.c. given $\varepsilon > 0$ we can find $\delta(t,x,\varepsilon) > 0$ such that $F(t,y) \subseteq F(t,x) + \frac{\varepsilon}{3}B_1$ for every $y \in H$ with $\|x - y\| \le \delta$.

Let $N_0(t,x,\varepsilon) \ge 1$ be such that for $n \ge N_0$ we have $\frac{1}{3^n} \le \frac{\delta}{3}$. Let $v \in H$ be such that $\gamma_v^n \in M(n,x)$ and let $y \in B\left(v, \frac{2}{3^n}\right)$. For $n \ge N_0$ we have

$$\|x - y\| \le \|x - v\| + \|v - y\| < \frac{1}{3^n} + \frac{2}{3^n} = \frac{1}{3^{n-1}} \le \delta.$$

Thus for $n \ge N_0$, for $v \in H$ for which $\gamma_v^n \in M(n,x)$ and for $y \in B\left(v, \frac{2}{3^n}\right)$ we have $F(t,y) \subseteq F(t,x) + \frac{\varepsilon}{3}B_1$. So $G_v^n(t) \subseteq \overline{F(t,x) + \frac{\varepsilon}{3}B_1} \subseteq F(t,x) + \frac{2\varepsilon}{3}B_1$ for all $n \ge N_0$ and all $v \in H$ such that $\gamma_v^n \in M(n,x)$. Thus we get that $F_n(t,x) \subseteq \overline{F(t,x) + \frac{2\varepsilon}{3}B_1} \subseteq F(t,x) + \varepsilon B_1$ for $n \ge N_0$ and so we conclude that $F_n(t,x) \xrightarrow{h} F(t,x)$ as $n \to \infty$.

Finally if $F(t,\cdot)$ is h-continuous, then for $\{y_m\}_{m\ge1}$ a dense subset of $B\left(x_k^n, \frac{2}{3^n}\right)$ we have that $G_k^n(t) = \overline{\mathrm{co}}\left(\cup_{m\ge1} F(t,y_m)\right)$. Invoking Proposition 2.3 and Theorem 9.1 of [**11**], we deduce that $t \mapsto G_k^n(t)$ is measurable and so $t \mapsto F_n(t,x)$ is measurable. □

Now let $p\colon L^2(T,H) \to C(T,H)$ be the map which to each $g \in L^2(T,H)$ assigns the unique solution of

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + g(t), \quad \text{a.e. on } T, \quad x(0) = \xi \in \operatorname{dom} \varphi(0,\cdot).$$

The existence (and of course uniqueness) of the solution of the above Cauchy problem follows from the result of [**26**].

We have the following result concerning the solution map $p(\cdot)$.

Lemma 2. *If hypothesis $H(\varphi)$ holds, if for every $t \in T$ $\varphi(t, \cdot)$ is of compact-type and $\xi \in \operatorname{dom} \varphi(0, \cdot)$, then $p(\cdot)$ is completely continuous (hence compact).*

Proof. Let $g_n \to g$ weakly in $L^2(T, H)$. For economy in the notation set $x_n = p(g_n)$, $n \geq 1$, and $x = p(g)$. Exploiting the monotonicity of the subdifferential we get

$$\begin{aligned} & (-\dot{x}_n(t) + \dot{x}(t), x(t) - x_n(t)) \leq (g_n(t) - g(t), x(t) - x_n(t)) \text{ a.e. on } T, \\ \Rightarrow \quad & \frac{1}{2}\frac{d}{dt}\|x_n(t) - x(t)\|^2 \leq \|g_n(t) - g(t)\| \, \|x_n(t) - x(t)\| \text{ a.e. on } T, \\ \Rightarrow \quad & \frac{1}{2}\|x_n(t) - x(t)\|^2 \leq \int_0^t \|g_n(s) - g(s)\| \, \|x_n(s) - x(s)\| \, ds, \ \forall\, t \in T. \end{aligned}$$

Using Lemma A.5, p. 157, of [**4**], we get that

$$\|x_n(t) - x(t)\| \leq \int_0^t \|g_n(s) - g(s)\| \, ds, \quad \forall\, t \in T.$$

Since $g_n \to g$ weakly in $L^2(T, H)$ we can find $M_1 > 0$ such that $\|g_n\|_1$, $\|g\|_1 \leq M_1$, $n \geq 1$. So for all $n \geq 1$ and all $t \in T$ we have

$$\|x_n(t)\| \leq \|x\|_\infty + 2M_1 = M_2 < \infty\,.$$

Moreover from inequality (7.9), p. 645 of [**26**], we see that there exists $M_3 > 0$ such that for all $t \in T$ and all $n \geq 1$ we have

$$\varphi(t, x_n(t)) \leq M_3,$$

(in fact M_3 depends only on the total variation of $h_{M_2}(\cdot)$, on $\|g_{M_2}\|_\beta$, on ξ, on $\varphi(0, \xi)$ and on M_1; see [**26**]). So for every $t \in T$ we have

$$\begin{aligned} & \{x_n(t)\}_{n\geq 1} \subseteq \left\{y \in H : \|y\|^2 + \varphi(t, y) \leq M_2^2 + M_3 = M_4\right\} \\ \Rightarrow \quad & \overline{\{x_n(t)\}}_{n\geq 1} \in P_k(H) \quad \text{(recall that } \varphi(t, \cdot) \text{ is of compact-type).} \end{aligned}$$

Also if s, $t \in T$, $s \leq t$ we have

$$\|x_n(t) - x_n(s)\| = \left\|\int_s^t \dot{x}_n(\tau) \, d\tau\right\| \leq \int_s^t \|\dot{x}_n(\tau)\| \, d\tau \leq \left(\sqrt{t - s}\right) \sup \|\dot{x}_n\|_2\,.$$

But from inequality (7.5) p. 645 of [**26**], we know that $\sup \|\dot{x}_n\|_2 = M_5 < +\infty$ with $M_5 > 0$ depending only on M_1, $\|\xi\|$ and $\varphi(0, \xi)$. Thus we deduce that $\{x_n\}_{n\geq 1}$

is equicontinuous. Invoking the Arzela-Ascoli theorem we get that $\{x_n\}_{n\geq 1}$ is relatively compact in $C(T,H)$. By passing to a subsequence if necessary, we may assume that $x_n \to y$ in $C(T,H)$ and $\dot{x}_n \to \dot{y}$ weakly in $L^2(T,H)$.

Let $\Phi\colon L^2(T,H) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\Phi(x) = \begin{cases} \int_0^b \varphi(t,x(t))\,dt, & \text{if } \varphi(\cdot,x(\cdot)) \in L^1(T) \\ +\infty, & \text{otherwise} \end{cases}$$

(note that by Lemma 3.4, p. 629 of [**26**], for every $x \in L^2(T,H)$, $t \mapsto \varphi(t,x(t))$ is measurable, moreover Corollary 4.1, p. 633 tell us that $\operatorname{dom}\Phi \neq \emptyset$). It is well-known (see for example [**26**, Lemma 4.4, p. 634]) that

$$\partial\Phi(x) = \big\{v \in L^2(T,H) : v(t) \in \partial\varphi(t,x(t)) \ \text{ a.e. on } T\big\}.$$

For every $n \geq 1$ $[x_n, -\dot{x}_n - g_n] \in \operatorname{Gr}\partial\Phi$ since $\partial\Phi(\cdot)$ is a maximal monotone operator, $\operatorname{Gr}\partial\Phi$ is demiclosed. Therefore $[y, -\dot{y} - g] \in \operatorname{Gr}\partial\Phi$ and so $-\dot{y}(t) \in \partial\varphi(t,y(t)) + g(t)$ a.e. on T, $y(0) = \xi$, i.e. $y = p(g) = x$. So we conclude that $x_n \to x$ in $C(T,H)$, proving that indeed $p(\cdot)$ is completely continuous.

Now observe that if x_1, $x_2 \in C(T,H)$ are solutions of

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + g(t), \text{a.e. on } T, \quad x(0) = \xi_1 \in \overline{\operatorname{dom}}\,\varphi(0,\cdot),$$
$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + g(t), \text{a.e. on } T, \quad x(0) = \xi_2 \in \overline{\operatorname{dom}}\,\varphi(0,\cdot),$$

respectively, then $\|x_1(t) - x_2(t)\| \leq \|\xi_1 - \xi_2\|$, $\forall\, t \in T$. This is an immediate consequence of the monotonicity of the subdifferential operator.

If $\hat{p}\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \times L_2(T,H) \to C(T,H)$ is the map which to each $(\xi, g) \in \overline{\operatorname{dom}}\,\varphi(0,\cdot) \times L^2(T,H)$ assigns the unique solution of

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + g(t), \text{a.e. on } T, \quad x(0) = \xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot),$$

(cf. [**26**]), then we have an alternative version of Lemma 2.

Lemma 2'. *If hypothesis $H(\varphi)$ holds, if for every $t \in T$ $\varphi(t,\cdot)$ is of compact-type and if for every $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$ there exist M, $r > 0$ such that $\varphi(0,x) \leq M$, $\forall\, x \in \operatorname{dom}\,\varphi(0,\cdot)$ with $|x - \xi| < r$, then the solution map $\hat{p}$ is sequentially continuous by considering on $L^2(T,H)$ the weak topology.*

The proof of this lemma is the same as that of the previous, hence is omitted.

Now we are ready for the result on the nonempiness and topological structure of the solution set $S(\xi)$.

Theorem 3. *If hypotheses $H(\varphi)$, $H(F)$ hold, if for every $t \in T$ $\varphi(t, \cdot)$ is of compact type and $\xi \in \operatorname{dom} \varphi(0, \cdot)$, then $S(\xi)$ is a R_δ-set in $C(T, H)$.*

Proof. First let us derive an a priori bound for the elements in $S(\xi)$. So let $x \in S(\xi)$ and let $y \in C(T, H)$ be the unique solution of

$$-\dot{y}(t) \in \partial\varphi(t, y(t)), \text{ a.e. on } T, \quad y(0) = \xi.$$

Let $f \in S^2_{F(\cdot, x(\cdot))}$ be such that $x = p(f)$. As before from the monotonicity of the subdifferential operator we have

$$\begin{aligned} &(-\dot{x}(t) + \dot{y}(t), y(t) - x(t)) \leq (f(t), y(t) - x(t) \quad \text{a.e. on } T, \\ \Rightarrow \quad &\frac{1}{2}\|x(t) - y(t)\|^2 \leq \int_0^t \|f(s)\| \, \|y(s) - x(s)\| \, ds, \quad \forall\, t \in T. \end{aligned}$$

Once again Lemma A.5, p. 157, of [**4**], tell us that

$$\begin{aligned} &\|x(t) - y(t) \leq \int_0^t \|f(s)\| \, ds \leq \int_0^t (a(s) + c(s)\|x(s)\|) \, ds, \quad \forall\, t \in T \\ \Rightarrow \quad &\|x(t)\| \leq \|y\|_\infty + \int_0^t (a(s) + c(s)\|x(s)\|) \, ds, \quad \forall\, t \in T. \end{aligned}$$

Using Gronwall's lemma, we deduce that there exists $M > 0$ such that $\|x(t)\| \leq M$ for every $t \in T$ and every solutions $x \in S(\xi)$. Thus without any los of generality put $\psi(t) = a(t) + c(t)M$, $\psi \in L^2(T, \mathbb{R}_+)$, we may assume that $|F(t, x)| \leq \psi(t)$, a.e. in T, $\forall\, x \in C(T, H)$ (otherwise replace $F(t, x)$ by $F(t, r_M(x))$ with $r_M(\cdot)$ being the M-radial retraction on H; note that $t \mapsto F(t, r_M(x))$ is measurable, $x \mapsto F(t, r_M(x))$ is h^*-u.s.c. and in addition $\|F(t, r_M(x))| \leq \psi(t)$ a.e. on T, with $\psi \in L^2(T, R^+)$).

Now let $F_n \colon T \times H \to P_{fc}(H)$, $n \geq 1$, be a sequence of multifunction as postulated by Lemma 1. For fixed $n \geq 1$ we consider the following multivalued Cauchy problem

$$-\dot{x}(t) \in \partial\varphi(t, x(t)) + F_n(t, x(t)), \text{a.e. on } T, \quad x(0) = \xi, \quad \xi \in T. \tag{2}$$

We already known that problem (2) above has a nonempty set $S_n(\xi)$ of solutions (cf. [**21**]).

Note that $S_n(\xi) \subseteq p(V)$, with $V = \{h \in L^2(T, H) : \|h(t)\| \leq \psi(t) \text{ a.e. in } T\}$ and from Lemma 2 we known that $\overline{p(V)}$ is compact in $C(T, H)$. Also if $x_m \in S_n(\xi)$, $m \geq 1$, and $x_m \to x$ in $C(T, H)$ as $m \to \infty$ we have that $x_m = p(f_m)$ with $f_m \in S^2_{F_n(\cdot, x_m(\cdot))}$. We may assume that $f_m \to f$ weakly in $L^2(T, H)$ and $f \in S^2_{F_n(\cdot, x(\cdot))}$. Moreover Lemma 2 tell us that $x_m \to p(f) = x$ in $C(T, H)$. Thus $S_n(\xi)$ is closed hence compact in $C(T, H)$.

We also claim that, for every $n \geq 1$, $S_n(\xi)$ is contractible. Let $u_n(t,x)$ be the Caratheordory (in fact locally Lipschitz in x) sector of $F_n(t,x)$ (cf. Lemma 1). Given $r \in [0,b)$ and $x \in S_n(\xi)$, let $z(r,x)(\cdot) \in C(T,H)$ be the unique solution of

$$-\dot{z}(t) \in \partial\varphi(t,z(t)) + u_n(t,z(t)), \text{a.e. on } [r,b], \quad z(r) = x(r).$$

For $r = b$ $z(b,x)(b) = x(b)$. Define $h\colon T \times S_n(\xi) \to S_n(\xi)$ by

$$h(r,x)(t) = \begin{cases} x(t), & \text{for } 0 \leq t \leq r, \\ z(r,x)(t), & \text{for } r \leq t \leq b. \end{cases}$$

Evidently $h(0,x) = z_0$ and also $h(b,x) = x$ with $z_0 \in C(T,H)$ being the unique solution of

$$-\dot{z}(t) \in \partial\varphi(t,z(t)) + u_n(t,z(t)), \text{a.e. on } T, \quad z(0) = \xi.$$

If we can show that $h(\cdot,\cdot)$ is continuous we will have established the contractibility of $S_n(\xi)$ in $C(T,H)$. To this end let $\{(r_m,x_m)\}_m \subseteq T \times S_n(\xi)$, with $(r_m,x_m) \to (r,x)$ in $T \times S_n(\xi)$. We consider two distinct cases:

Case I: $r_m \geq r$ for every $m \geq 1$.

Let $v_m(t) = h(r_m,x_m)(t)$, $t \in T$. Evidently $v_m \in S_n(\xi)$, $m \geq 1$, and so by passing to a subsequence if necessary, we may assume that $v_m \to v$ in $C(T,H)$. From the definition of $h(\cdot,\cdot)$ we see that for $t \in [0,r]$ we have $v(t) = x(t)$. Let $y \in C(T,H)$ be the unique solution of

$$-\dot{y}(t) \in \partial\varphi(t,y(t)) + u_n(t,v(t)), \text{a.e. on } [r,b], \quad y(r) = v(r).$$

Let $N \geq 1$. Then for all $m \geq N$ large enough we have that $-\dot{v}_m(t) \in \partial\varphi(t,v_m(t)) + u_n(t,v_m(t))$ a.e. on $[r_N,b]$. As before via the monotonicity of the subdifferential operator we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y(t)-v_m(t)\|^2 \leq \|u_n(t,v(t)) - u_n(t,v_m(t))\|\,\|y(t)-v_m(t)\| \text{ a.e. on } [r_N,b] \\ \Rightarrow \quad &\frac{1}{2}\|y(t)-v_m(t)\|^2 \leq \frac{1}{2}\|y(r_N)-v_m(r_N)\|^2 \\ &\qquad + \int_{r_N}^t \|u_n(s,v(s)) - u_n(s,v_m(s))\|\,\|y(s)-v_m(s)\|\,ds\,. \end{aligned}$$

An application of Lemma A.5 of [**4**] gives us

$$\|y(t)-v_m(t)\| \leq \|y(r_N)-v_m(r_N)\| + \int_{r_N}^t \|u_n(s,v(s)) - u_n(s,v_m(s))\|\,ds.$$

Passing to the limit as $m \to \infty$, we get that

$$\|y(t) - v(t)\| \leq \|y(r_N) - v(r_N)\| \quad \text{for } t \in [r_N, b].$$

Note that as $N \to \infty$ we have $y(r_N) \to x(r)$ and $v(r_N) \to v(r) = x(r)$. Since $N \geq 1$ was arbitrary we conclude that $y(t) = v(t)$ for $t \in [r, b]$. Hence $v = h(r, x)$ and so $h(r_m, x_m) \to h(r, x)$ in $C(T, H)$ as $m \to \infty$.

Case II: $r_m \leq r$ for every $m \geq 1$.

Keeping the notation introduced in the analysis of case I, we see that $v(t) = x(t)$ for $t \in [0, r]$.

Moreover the same arguments as in case I, given us that

$$\|y(t) - v_m(t)\| \leq \|y(r) - v_m(r)\| + \int_r^t \|u_n(s, v(s)) - u_n(s, v_m(s))\| \, ds \quad \text{for } t \in [r, b],$$

and by passing to the limit as $m \to \infty$

$$\|y(t) - v(t)\| \leq \|y(r) - v(r)\| \quad \text{for } t \in [r, b].$$

But $y(r) = x(r) = v(r)$. So $y(t) = v(t)$ for $t \in [r, b]$. Hence $v = h(r, x)$ and so again we have $h(r_m, x_m) \to h(r, x)$ in $C(T, H)$ as $m \to \infty$.

In general we can always find a subsequence $\{r_m\}_{m \geq 1}$ satisfying case I or case II. Thus we have proved the continuity of the map $h(\cdot, \cdot)$. So, for every $n \geq 1$, $S_n(\xi)$ is compact and contractible in $C(T, H)$. We claim that $S(\xi) = \cap_{n \geq 1} S_n(\xi)$. Clearly $S(\xi) \subseteq \cap_{n \geq 1} S_n(\xi)$. Let $x \in \cap_{n \geq 1} S_n(\xi)$. Then by definition $x = p(f_n)$ with $f_n \in S^2_{F_n(\cdot, x(\cdot))}$, $n \geq 1$. Evidently $\{f_n\}_{n \geq 1}$ is bounded in $L^2(T, H)$. So by passing to a subsequence, if necessary, we may assume that $f_n \to f$ weakly in $L^2(T, H)$. We known that $f \in S^2_{F(\cdot, x(\cdot))}$, (cf. [**18**]). So $x \in S(\xi)$ and therefore we have $S(\xi) = \cap_{n \geq 1} S_n(\xi)$. Using a result of [**15**], we conclude that $S(\xi)$ is a R_δ-set in $C(T, H)$. □

An immediate consequence of Theorem 3 above is the following Kneser-type theorem for (1).

Corollary 4. *If hypotheses $H(\varphi)$, $H(F)$ hold and for every $t \in T$ $\varphi(t, \cdot)$ is of compact type, then, for every $t \in T$, the set $R(t) = S(\xi)(t) = \{x(t) : x \in S(\xi)\}$ (the reachable set at time $t \in T$) is compact and connected in H.*

Also a consequence of Lemma 2' is the following continuity result about multifunction $\xi \mapsto S(\xi)$. For this result the following weaker version of hypothesis $H(F)$ will suffice.

$H(F)_1$: $\quad F : T \times H \to P_{fc}(H)$ is a multifunction such that

(i) $\forall\, x \in H$, $t \mapsto F(t,x)$ is measurable;

(ii) $\forall\, t \in T$, Gr $F(t,\cdot)$ is sequentially closed in $H \times H_w$; (here H_w stands for the Hilbert space H equipped with the weaker topology);

(iii) $\exists\, a, c \in L^2(T, R^+)$:

$$\|F(t,x)\| = \sup\{\|z\| : z \in F(t,x)\} \le a(t) + c(t)\|x\|, \text{a.e. in } T, \quad \forall\, x \in H.$$

Proposition 5. *If hypotheses $H(\varphi)$, $H(F)_1$ hold and for every $t \in T$ $\varphi(t,\cdot)$ is of compact type, then $S\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \to P_k(C(T,H))$ is u.s.c..*

Proof. The set $S(\xi)$ is nonempty for $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$ (see [**20**] with the obvious modifications), while the compactness of $S(\xi)$ follows from Lemma 2' as in the proof the Theorem 3. Now we need to show that given $C \subseteq C(T,H)$ nonempty closed, the set $S^-(C) = \{\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot) : S(\xi) \cap C \neq \emptyset\}$ is closed in $\overline{\operatorname{dom}}\,\varphi(0,\cdot) \subseteq H$. To this end let $\xi_n \in S^-(C)$, $n \ge 1$ and assume that $\xi_n \to \xi$ in H, with $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$. Let $x_n \in S(\xi_n) \cap C$, $n \ge 1$. For each $n \ge 1$ let $x_n = p(f_n)$, $f_n \in S^2_{F(\cdot,x_n(\cdot))}$. Since $\{f_n\}_n$ is bounded in $L^2(T,H)$ (cf. hypothesis $H(F)_1$(iii) by passing to a subsequence if necessary we may assume that $f_n \to f$ weakly in $L^2(T,H)$. From Lemma 2' we have that $x_n \to x$ in $C(T,H)$ and from hypothesis $H(F)_1$(ii) and Theorem 3.1 of Papageorgiou (cf. [**18**]), we have that $f \in S^2_{F(\cdot,x(\cdot))}$. So $x \in S(\xi) \cap C$, i.e. $\xi \in S^-(C)$. Therefore $S(\cdot)$ is u.s.c.. □

Next we will generate a continuous selector for the multifunction $\xi \mapsto S(\xi)$. For this we will need the following hypothesis on the orientor field $F(t,x)$.

$H(F)_2$: $\quad F\colon T \times H \to P_f(H)$ is a multifunction such that

(i) $\forall\, x \in H$, $t \mapsto F(t,x)$ is measurable;

(ii) $\exists\, x \in L^1(T, \mathbb{R}_+)$, such that $h(F(t,x), F(t,y)) \le k(t)\|x-y\|$ a.e. on T, $\forall\, x, y \in H$;

(iii) $\exists\, a, c \in L^2(T)$:

$$\|F(t,x)\| = \sup\{\|z\| : z \in F(t,x)\| \le a(t) + c(t)\|x\|, \text{a.e. in } T, \quad \forall\, x \in H.\}$$

□

Theorem 6. *If hypotheses $H(\varphi)$, $H(F)_2$ hold, then there exists $u\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \to C(T,H)$ a continuous map such that $u(\xi) \in S(\xi)$ for every $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$.*

Proof. Let $x_0(\xi)(\cdot) \in C(T,H)$ be the unique solution of the evolution equation (cf. [**26**])

$$-\dot{x}(t) \in \partial\varphi(t, x(t)), \text{ a.e. on} T, \quad x(0) = \xi \in \operatorname{dom}\, \varphi(0,\cdot).$$

Let $R_0\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \to P_f(L^1(T,H))$ be defined $R_0(\xi) = S^1_{F(\cdot,x_0(\xi)(\cdot))}$. Then $R_0(\cdot)$ is h-continuous and so we can apply Theorem 3 of [**3**] and get $r_0\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot)$

$\to L^1(T,H)$ a continuous map such that $r_0(\xi) \in R_0(\xi)$ for every $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$. Let $x_1(\xi)(\cdot) \in C(T,H)$ be the unique solution of

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + r_0(\xi)(t), \text{ a.e.}, \quad x(0)=\xi.$$

We claim that by induction we can generate two sequence $\{x_n(\xi)(\cdot)\}_{n\ge 0} \subseteq C(T,H)$ and $\{r_n(\xi)(\cdot)\}_{n\ge 0} \subseteq L^2(T,H)$, with $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$, satisfying:

(a) $x_n(\xi)(\cdot) \in C(T,H)$ is the unique solution of

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + r_{n-1}(\xi)(t), \text{ a.e.}, \quad x(0)=\xi, \quad \forall\, n \ge 1;$$

(b) $\xi \mapsto r_n(\xi)$ is continuous from $\overline{\operatorname{dom}}\,\varphi(0,\cdot)$ into $L^1(T,H)$, $\forall\, n \ge 0$;

(c) $r_n(\xi)(t) \in F(t, x_n(\xi)(t))$ a.e. on T, for every $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$, $\forall\, n \ge 0$;

(d) $\|r_n(\xi)(t) - r_{n-1}(\xi)(t)\| \le k(t)\beta_n(\xi)(t)$ a.e. on T, $\forall\, n \ge 1$,

with $\beta_n(\xi)(t) = \int_0^t \lambda(\xi)(s)\frac{(\theta(t)-\theta(s))^{n-1}}{(n-1)!}\,ds + b\left(\sum_{k=0}^n \frac{\varepsilon}{2^{k+1}}\right)\frac{(\theta(t))^{n-1}}{(n-1)!}$ with $\varepsilon > 0$, $\lambda(\xi)(t) = a(t) + c(t)\|x_0(\xi)(t)\|$ and $\theta(t) = \int_o^t k(s)\,ds$.

Since $\|x_0(\xi)(t) - x_0(\xi')(t)\| \le \|\xi - \xi'\|$ for every $t \in T$, $\xi, \xi' \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$ we see that $\xi \mapsto \lambda(\xi)(\cdot)$ and $\xi \mapsto \beta_n(\xi)(\cdot)$ are continuous from $\overline{\operatorname{dom}}\,\varphi(0,\cdot)$ into $L^1(T,H)$.

Suppose we were able to produce $\{x_k(\xi)\}_{k=0}^n$ and $\{r_k(\xi)\}_{k=0}^n$ satisfying (a)–(d) above.

Let $x_{n+1}(\xi)(\cdot) \in C(T,H)$ be the unique solution of

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + r_n(\xi)(t), \quad \text{a.e. on } T, \quad x(0) = \xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot).$$

As before because of the monotonicity of the subdifferential operator and using Lemma 1.5 of [**4**], we get

$$(*) \quad \|x_{n+1}(\xi)(t) - x_n(\xi)(t)\| \le \int_0^t \|r_n(\xi)(s) - r_{n-1}(\xi)(s)\|\,ds$$

$$\le \int_0^t k(s)\beta_n(\xi)(s)$$

$$= \int_0^t k(s)\int_0^s \lambda(\xi)(\tau)\frac{(\theta(s)-\theta(\tau))^{n-1}}{(n-1)!}\,d\tau\,ds + b\left(\sum_{k=0}^n \frac{\varepsilon}{2^{k+1}}\right)\int_0^t k(s)\frac{(\theta(s))^{n-1}}{(n-1)!}\,ds$$

$$= \int_0^t \lambda(\xi)(s)\int_s^t k(\tau)\frac{(\theta(\tau)-\theta(s))^{n-1}}{(n-1)!}\,d\tau\,ds + b\left(\sum_{k=0}^n \frac{\varepsilon}{2^{k+1}}\right)\frac{(\theta(t))^n}{n!}$$

$$= \int_0^t \lambda(\xi)(s)\int_s^t \frac{d}{d\tau}\frac{(\theta(\tau)-\theta(s))^n}{n!}\,d\tau\,ds + b\left(\sum_{k=0}^n \frac{\varepsilon}{2^{k+1}}\right)\frac{(\theta(t))^n}{n!}$$

$$= \int_o^t \lambda(\xi)(s)\frac{(\theta(t)-\theta(s))^n}{n!}\,ds + b\left(\sum_{k=0}^n \frac{\varepsilon}{2^{k+1}}\right)\frac{(\theta(t))^n}{n!}$$

$$< \beta_{n+1}(\xi)(t) \quad \text{a.e.}$$

Using hypothesis $H(F)_2$(ii) we have

$$(**) \qquad d(r_n(\xi)(t), F(t, x_{n+1}(\xi)(t)) \le k(t)\|x_n(\xi)(t) - x_{n+1}(\xi)(t)\| \le k(t)\beta_{n+1}(\xi)(t) \ \text{ a.e. on } T$$

Let $R_{n+1}\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \to P(L^1(T,H))$ be the multifunction defined by

$$R_{n+1}(\xi) = \Big\{z \in S^1_{f(\cdot,x_{n+1}(\xi)(\cdot))} : \|z(t) - r_n(\xi)(t)\| \le k(t)\beta_{n+1}(\xi)(t) \ \text{ a.e.}\Big\}.$$

From $(**)$ above we know that the multifunction

$$\Gamma_{n+1}\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \to P_{wkc}(L^1(T,H)),$$

defined by $\Gamma_{n+1}(\xi)(t) = \{v \in F(t, x_{n+1}(\xi)(t)) : \|v - r_n(\xi)(t)\| < k(t)\beta_{n+1}(\xi)(t)\}$, is such that $\Gamma_{n+1}(\xi)(t) \ne \emptyset$ a.e. on T.

By modifying the above multifunction on a Lebesgue-null of T, we may assume without any loss of generality that $\Gamma_{n+1}(\xi)(t) \ne \emptyset$ for every $t \in T$. Also from Theorem 3.3 of [**19**] we know that $t \mapsto F(t, x_{n+1}(\xi)(t))$ is measurable (hence graph measurable), while $(t,v) \mapsto \|v - r_n(\xi)(t)\| - k(t)\beta_{n+1}(\xi)(t) = \gamma_{n+1}(\xi)(t,v)$ is clearly jointly measurable. So

$$\operatorname{Gr}\Gamma_{n+1}(\xi) = \Big\{(v,t) \in \operatorname{Gr} F(\cdot, x_{n+1}(\xi)(\cdot)) : \gamma_{n+1}(\xi)(t) < 0\Big\} \in \mathcal{L}(T) \times B(H)$$

with $\mathcal{L}(T)$ being the Lebesgue σ-field of T. Apply Aumann' selection theorem (cf. [**24**]), to get $z\colon T \to H$ measurable such that $z(t) \in \Gamma_{n+1}(\xi)(t)$, $t \in T$, so $z(\cdot) \in R_{n+1}(\xi)$. Therefore $\xi \mapsto R_{n+1}(\xi)$ is l.s.c. with decomposable values. Apply Theorem 3 of [**3**] to get $r_{n+1}\colon \overline{\operatorname{dom}}\,\varphi(0,\cdot) \to L^1(T,H)$ a continuous map such that $r_{n+1}(\xi) \in \overline{R_{n+1}(\xi)}$ for every $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$. Hence

$$r_{n+1}(\xi)(t) \in F(t, x_{n+1}(\xi)(t)) \text{ a.e. and } \|r_{n+1}(\xi)(t) - r_n(\xi)(t)\| \le k(t)\beta_{n+1}(\xi)(t),$$

a.e. on T. Thus by induction we have produced the two sequences $\{x_n(\xi)\}_{n\ge1} \subseteq C(T,H)$ and $\{r_n(\xi)\}_{n\ge1} \subseteq L^2(T,H)$ satisfying (a)–(d) above.

Then using $(*)$ we have

$$\int_0^b \|r_n(\xi)(t) - r_{n-1}(\xi)(t)\|\,dt \le \int_0^b \lambda(\xi)(s)\frac{(\theta(b)-\theta(s))^n}{n!}\,ds + b\varepsilon\frac{(\theta(b))^n}{n!} \le \frac{(\theta(b))^n}{n!}\,[\|\lambda(\xi)\|_1 + b\varepsilon]\,.$$

Since $\xi \mapsto \lambda(\xi)$ is continuous from H into $L^2(T,H)$ is locally bounded. So from the above inequality we deduce that $\{r_n(\xi)\}_{n\ge1}$ is a $L^1(T,H)$-Cauchy sequence, locally uniformly in $\xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot)$. Also from (a) we have

$$\|x_{n+1}(\xi) - x_n(\xi)\|_{C(T,H)} \le \|r_n(\xi) - r_{n-1}(\xi)\|_{L^1(T,H)}$$
$$\Rightarrow \quad \{x_n(\xi)\}_{n\ge1} \text{ is Cauchy in } C(T,H), \text{ locally uniformly in } \xi \in \overline{\operatorname{dom}}\,\varphi(0,\cdot).$$

let $n \to \infty$. We have $x_{n+1}(\xi) \to x(\xi)$ in $C(T,H)$, $r_n(\xi) \to r(\xi)$ in $L^1(T,H)$ and both limits are continuous in $\xi \in \overline{\mathrm{dom}}\,\varphi(0,\cdot)$. Let $y(\xi) \in C(T,H)$ be the unique solution of

$$-\dot{z}(t) \in \partial\varphi(t,z(t)) + r(\xi)(t) \quad \text{a.e. } z(0) = \xi \in \text{dom } \varphi(0,\cdot).$$

Because of hypothesis $H(F)_2$(ii) we have $r(\xi)(t) \in F(t,x(\xi)(t))$ a.e.. As before we have

$$\begin{aligned} & \|x_n(\xi)(t) - y(\xi)(t)\| \le \int_0^t \|r_{n-1}(\xi)(s) - r(\xi)(s)\|\,ds, \quad t \in T \\ \Rightarrow \quad & x_n(\xi) \to y(\xi) \quad \text{in } C(T,H) \\ \Rightarrow \quad & x(\xi) = y(\xi), \xi \in \overline{\mathrm{dom}}\,\varphi(0,\cdot). \end{aligned}$$

Therefore $u\colon \xi \mapsto x(\xi)$ is the desired selector of $\xi \mapsto S(\xi)$. □

Remark. Note that Theorem 4 gives as an existence result for (1) without assuming that $\varphi(t,\cdot)$ is of compact type. However on the other hand $F(t,\cdot)$ is h-Lipschitz. So there is a trade off of hypotheses between $\varphi(t,x)$ and $F(t,x)$.

An immediate consequence of this theorem is the following corollary

Corollary 7. *If hypotheses $H(\varphi)$, $H(F)_2$ hold, if there is $K \in P_{fc}(H)$ bounded such that $S(K)(b) \subseteq K$, if* dom $\varphi(b,\cdot) \subseteq$ dom $\varphi(0,\cdot)$ *and if* $\overline{\mathrm{dom}}\,\varphi(0,\cdot) \cap K$ *is compact in H, then there exists a solution $x(\cdot) \in C(T,H)$ for the problem*

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + F(t,x(t)), \textit{a.e. on } T, \quad x(0) = x(b).$$

Proof. Let $u\colon \overline{\mathrm{dom}}\,\varphi(0,\cdot) \to C(T,H)$ be the continuous selector of the multifunction $\xi \mapsto S(\xi)$ guaranteed by Theorem 4. Let $e_b\colon C(T,H) \to H$ be the evaluation map, i.e. $e_b(x) = x(b)$. Let $\hat{u} = e_{b^0} u\colon \overline{\mathrm{dom}}\,\varphi(0,\cdot) \cap K \to \overline{\mathrm{dom}}\,\varphi(0,\cdot) \cap K$. This is a continuous and compact map. So Schauder's fixed point theorem gives us $\xi \in \overline{\mathrm{dom}}\,\varphi(0,\cdot) \cap K$ such that $\xi = \hat{u}(\xi)$. Then $u(\xi)(\cdot) \in C(T,H)$ is the desired periodic trajectory. □

4. Examples

Now let as work out some examples of parabolic distributed parameter systems to illustrate general abstract results.

Let $T = [0,b]$ and $Z \subseteq R^n$ a bounded domain with smooth boundary Γ. Let $g \in W^{1,1}(T, W_0^{1,p}(Z,R))$ be given and define

$$K(t) = \left\{ x \in W_0^{1,p}(Z) : x(z) \ge g(t,z) \quad \text{a.e. on } Z \right\}.$$

We consider the following controlled obstacle problem:

$$(3)\quad \begin{cases} \frac{\partial x}{\partial t} - \sum_{k=1} D_k(|D_k x(t,z)|^{p-2} D_k x(t,z)) \geq u(t,z) \ \text{ a.e. on } Q = T \times Z \\ x(t,z) \geq g(t,z) \ \text{ a.e. on } Q \\ \left(\frac{\partial x}{\partial t} - \sum_{k=1} D_k(|D_k x(t,z)|^{p-2} D_k x(t,z)) - u(t,z)\right)(x(t,z) - g(t,z)) = 0 \\ \text{a.e. on } Q \\ x|_{T\times\Gamma} = 0, \quad x(0,z) = x_0(z) \ \text{ a.e. on } Z, \quad p \geq 2, \\ \|u(t,\cdot)\|_2 \leq r(t, x(t,\cdot)), \text{a.e. on } T. \end{cases}$$

We need the following hypothesis on the function r:

$H(r)$: $r\colon T \times L^2(Z,R) \to \mathbb{R}_+$ is a function such that

(i) $\forall\, x \in L^2(Z,\mathbb{R})$, $t \mapsto r(t,x)$ is measurable;

(ii) $\forall\, t \in T$, $x \mapsto r(t,x)$ is u.s.c.;

(iii) $\exists\, a, c \in L^2(T,\mathbb{R}) : r(t,x) \leq a(t) + c(t)\|x\|_2$, a.e. on t, $\forall\, x \in L^2(Z,\mathbb{R})$.

Let $H = L^2(Z,\mathbb{R})$ and let $\varphi\colon T \times H \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(t,x) = \begin{cases} \frac{1}{p}\int_Z \|D(x)(z)\|^p\, dz, & \text{if } x \in K(t), \\ +\infty, & \text{otherwise.} \end{cases}$$

We claim that $\varphi(t,x)$ defined above satisfies hypothesis $H(\varphi)$. Indeed let $t \leq s$ and let $y \in K(t)$. Define $x = y - g(t,\cdot) + g(s,\cdot)$. Evidently $x \in K(s)$. Also we have

$$\|x - y\|_2 = \|g(s,\cdot) - g(t,\cdot)\|_2 \leq c_1 \int_t^s \|\dot{g}(\tau,\cdot)\|_p\, d\tau.$$

In addition through some elementary calculation we have that

$$\begin{aligned} |\varphi(t,x) - \varphi(t,y)| &\leq c\|Dg(s,\cdot) - Dg(t,\cdot)\|_p(\|Dg(t,\cdot)\|_p^{p-1} \\ &\qquad + \|Dg(s,\cdot)\|_p^{p-1} + \|Dy\|_p^{p-1}) \\ &\leq c_1 \int_t^s \|\dot{g}(\tau,\cdot)\|_p\, d\tau(\varphi(t,y) + 1) \end{aligned}$$

and so hypothesis $H(\varphi)$ has been satisfied.

Let $F\colon T \times H \to P_{fc}(H)$ be defined by $F(t,x) = \{u \in H : \|u\|_2 \leq r(t,x)\}$. Then it is easy to see that $F(t,x)$ defined above satisfies hypothesis $H(F)$. Directly from the definition of subdifferential we have (3) is equivalent to (1) with $\varphi(t,x)$ and $F(t,x)$ as above. Note that $W_0^{1,p}(Z,R) \hookrightarrow L^2(Z,R)$ compactly and so $\varphi(t,\cdot)$is of compact type. Therefore we can apply Theorem 3 and get:

Theorem 8. *If hypothesis $H(R)$ holds and $x_0 \in W_0^{1,p}(Z)$, $x_0 \geq g(0,z)$ a.e. on Z, then the solution set of* (3) *is a R_δ-set in $C(T, L^2(Z,R))$.*

Next consider the following problem with a discontinuous nonlinearity $u\colon Z \times R \to R$.

$$(4) \qquad \begin{cases} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_j(a_{ij}(t,z)D_i x(t,z)) \in [\underline{y}(z,x(t,z)), \overline{u}(z,x(t,z))] \\ -\frac{\partial x}{\partial n} \in \beta(x(t,z)) \quad \text{a.e. on } T \times Z, \\ x(0,z) = x_0(z) \quad \text{a.e. on } Z. \end{cases}$$

We need the following hypotheses:

$H(a)$: for every $1 \leq i,j \leq N$, $a_{ij} \in L^\infty(T \times Z, R)$, $a_{ij} = a_{ji}$, there exists $c > 0$ such that $c\|\xi\|_N^2 \leq \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j$ for every $\xi = (\xi_k)_{k=1}^N$ and there exists a function $h\colon T \to R$ of bounded variation such that $|a_{ij}(t,z) - a_{ij}(s,z)| \leq |h(t) - h(s)|$, $\forall\, t,s \in T$, $\forall\, z \in Z$.

$H(\beta)$: $\beta\colon \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone multivalued operator.

Then from Brezis (cf. [**4**]) we know that there exists $j \in \Gamma_0(R)$ such that $\beta = \partial j$. Assume that $j \geq 0$.

Following Rauch (cf. [**22**]), we set $\underline{u}(z,x) = \liminf_{y \to x} u(z,y)$ and $\overline{u}(z,x) = \limsup_{y \to x} u(z,y)$.

We will make the following hypothesis concerning $\underline{u}$ and $\overline{u}$.

$H(U)$: 1) the function u, $\underline{u}$ and $\overline{u}$ are superpositionally measurable, i.e. for every $x\colon Z \to \mathbb{R}$ measurable, $z \mapsto u(z,x(z)), \underline{u}(z,x(z)), \overline{u}(z,x(z))$ are all measurable,
2) there exist $a \in L^2(Z,\mathbb{R})$ and $c > 0$: $|u(z,x)| \leq a(z) + c|x|$ a.e. on Z, $\forall\, x \in R$.

Let $\theta \in L^2(T,R)$, $\theta \geq 0$ and define $F_0(t,z,x) = \{v \in \mathbb{R} : \theta(t)\underline{u} \leq (z,x) \leq v\theta(t)u(z,x)\}$. Evidently for every $x\colon Z \to \mathbb{R}$ measurable, we have that $(t,z) \to F_0(t,z,x(z))$ is measurable. Moreover since for every $(t,z) \in T \times Z$, $x \mapsto F_0(t,z,x)$ is u.s.c.. Hence if we define $F\colon T \times L^2(Z,\mathbb{R}) \to P_{fc}(L^2(Z,\mathbb{R}))$ by $F(t,x) = S^2_{F_0(t,\cdot,x(\cdot))}$, we have that $F(\cdot,\cdot)$ satisfies hypothesis $H(F)$.

Also let $\varphi\colon T \times L^2(Z,\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(t,x) = \begin{cases} \frac{1}{2}\int_Z \sum a_{ij}(t,z)D_i x(z) D_j x(z)\, dz + \int_\Gamma j(x(v))\, d\sigma(v), \\ \qquad \text{if } x \in H^1(Z,\mathbb{R}), j(x(\cdot)) \in L^1(\Gamma,\mathbb{R}) \\ +\infty, \\ \qquad \text{otherwise.} \end{cases}$$

Let $0 \le t \le s \le b$. For every $x \in \operatorname{dom} \varphi(t,\cdot) = \operatorname{dom} \varphi(s,\cdot)$ we have

$$\begin{aligned} |\varphi(t,x) - \varphi(s,x)| &= \left| \sum \int_Z (a_{ij}(t,z) - a_{ij}(s,z)) D_i x(z) D_j x(z)\, dz \right| \\ &\le c|h(f) - h(s)|\, \|Dx\|_2^2 \le c_1 |h(t) - h(s)| \varphi(t,x) \end{aligned}$$

(check hypotheses $H(a)$ and $H(\varphi)$). So we see that $H(\varphi)$ holds. In addition note that $\{x \in L^2(Z,\mathbb{R}) : \|x\|_2^2 + \varphi(t,x) \le \lambda\}$ is bounded in $H^1(Z,\mathbb{R})$ and $H^1(Z,\mathbb{R})$ embeds compactly in $L^2(Z,\mathbb{R})$. So we deduce that for every $t \in T$, $\varphi(t,\cdot)$ is of compact type.

Rewriting (4) in the equivalent abstract from (1) and using Theorem 3, we get:

Theorem 9. *If hypotheses $H(a)$, $H(\beta)$, $H(u)$ hold and $x_0(\cdot) \in H^1(Z,\mathbb{R})$ with $j(x_0(\cdot)) \in L^1(\Gamma,\mathbb{R})$, then the set of solution* (4) *is compact and acyclic (thus connected) in $C(T, L^2(Z,\mathbb{R}))$.*

Similarly we can treat the problem:

$$(5) \qquad \begin{cases} \frac{\partial x}{\partial t} - \sum D_j(a_{ij}(t,z) D_i x(t,z)) + \beta(x,(t,z)) \\ \qquad \in [\underline{u}(z,x(t,z)), \overline{u}(z,x(t,z)], \quad \text{a.e. on } T \times Z \\ x|_{T\times\Gamma} = 0 \\ x(0,z) = x_0(z) \quad \text{a.e. on } Z. \end{cases}$$

In this case $\varphi \colon T \times L^2(Z,\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi(t,x) = \begin{cases} \frac{1}{2} \int_Z \sum a_{ij}(t,z) D_i x(z) D_j x(z)\, dz + \int_Z j(x(z))\, dz, \\ \qquad \text{if } \; x \in H_0^1(Z,\mathbb{R}), j(x(\cdot)) \in L^1(Z,\mathbb{R}) \\ +\infty, \\ \qquad \text{otherwise.} \end{cases}$$

As before we can establish the following theorem

Theorem 10. *If hypotheses $H(a)$, $H(\beta)$, $H(u)$ hold and $x_0(\cdot) \in H_0^1(Z,\mathbb{R})$ with $j(x_0(\cdot)) \in L^1(Z,\mathbb{R})$, then the set of solution of* (5) *is compact and acyclic (thus connected) in $C(T, L^2(Z,\mathbb{R}))$.*

Remark. The problem $\frac{\partial x}{\partial t} - \Delta x = \sqrt{x}$, $\frac{\partial x}{\partial n} = 0$, and $\frac{\partial x}{\partial t} - \Delta x = \sqrt{x}$, $x|_{T\times Z} = 0$ are particular cases of (4) and (5) respectively. Therefore our work extends that of Kikuchi (cf. [**17**]) and the example in Balloti (cf. [**1**]).

References

1. Ballotti M. E., *Aronszajn's theorem for a parabolic partial differential equation*, Nonl. Anal. T. M. A. **9** (1985), 1183–1188.

2. Barbu V., *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

3. Bressan A. and Colombo G., *Extension and selections of maps with decomposable values*, Studia Math. **90** (1988), 69–86.

4. Brezis H., *Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.

5. Cellina A., *On the set of solution to Lipschitzian differential inclusions*, Diff. and Integ. Equations **1** (1988), 495–500.

6. De Blasi F., *Characterizations of Certain Classes of Semicontinuous Multifunctions by Continuous Approximations*, J. Math. Anal. Appl. **106** (1985), 1–18.

7. De Blasi F. and Myjak J., *On the solution sets for differential inclusions*, Bull. Pol. Acad. Sci. **33** (1985), 17–23.

8. De Blasi F. and Pianigiani G., *Non-convex valued differential inclusions in Banach spaces*, J. Math. Anal. Appl. **157** (1991), 469–494.

9. ______, *Topological properties of nonconvex differential inclusions*, Nonl. Anal. T. M. A. **20** (1993), 871–894.

10. De Blasi F., Pianigiani G. and Staicu V., *Topological properties of nonconvex differential inclusions of evolution type*, Nonl. Anal. T. M. A. **23** (1995), 711–720.

11. Himmelberg C. J., *Measurable relations*, Fundamenta Math. **87** (1975), 53–72.

12. Himmelberg C. J. and Van Vleck F. S., *A note on solution sets of differential inclusions*, Rocky Mountain J. Math. **12** (1982), 621–625.

13. Hu S., Lakshmikantham V. and Papageorgiou N. S., *On the properties of the solution set of semilinear evolution inclusions*, (to appear).

14. Hu S. and Papageorgiou N. S., *On the topological regularity of the solution set of differential inclusions with constraints*, J. Diff. Equations **107** (1994), 280–289.

15. Hyman D. M., *On decreasing sequence of compact absolute retracts*, Fundamenta Math. **64** (1969), 91–97.

16. Kemnochi N., *Some nonlinear parabolic variational inequalities*, Israel J. Math. **22** (1975), 305–331.

17. Kikuchi N., *Kneser's property for* $du/dt = \Lambda u + sqrtu$, Keio Univ. Math. Sem. Rep. **3** (1978), 45–48.

18. Papageorgiou N. S., *Convergence theorems for Banach space valued integrable multifunctions*, Inter. J. Math. and Math. Sci. **10** (1987), 433–442.

19. ______, *On measurable multifunctions with applications to random multivalued equations*, Math Japonica **32** (1987), 427–464.

20. ______, *On the solution set of evolution inclusions driven by time-dependent subdifferentials*, Math Japonica **37** (1992), 1087–1099.

21. ______, *On the topological property of the solution set of evolution inclusions involving time-dependent subdifferential operators*, B. U. M. I. **8B** (1994).

22. Rauch J., *Discontinuous semilinear differential equations and multiple valued maps*, Proc. Amer. Math. Soc. **64** (1977).

23. Staicu V., Sisa report, **42M**, 1990.

24. Wagner D. H., *Survey of measurable selection theorems*, SIAM J. Control Optim. **15** (1977), 859–903.

25. Yamada Y., *On evolution equations generated by subdifferential operators*, J. Fac. Sci. Univ. Tokyo **23** (1976), 491–515.

26. Yotsutani S., *Evolution equations associated with subdifferentials*, J. Math. Soc. Japan **31** (1978), 623–646.

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