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THE NUMERICAL VALUATION OF OPTIONS WITH UNDERLYING JUMPS

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Abstract. A Black-Scholes type model for American options will be considered where the underlying asset price experiences Brownian motion with random jumps. The mathematical problems is an obstacle problem for a linear one-dimensional diffusion equation with a functional source term. The problem is time discretized and solved at each time level iteratively with a Riccati method. Some numerical experiments for a call and put with multiple jumps are presented. Convergence of the iteration at a given time level will be discussed for the simpler problem of a European put where there is no free boundary.

1. Introduction

Mathematical modeling and simulation have become indispensable tools in the financial industry. Banks, brokerage houses and investors and their consulting services spend countless hours every day and night to simulate and predict the movement of prices for financial assets like stocks, options and bonds. Much of the mathematics employed in this field, particularly in academic research, is highly sophisticated and spans the fields of analysis, probability and stochastics, statistics, differential equations and, last but not least, numerical analysis which is the topic of this presentation.

In general, the numerical problems solved in the field of financial mathematics have tended to be relatively simple compared to those routinely faced in science and engineering. Only recently have models and algorithms been proposed which reach the state of the art in numerical analysis (see, e.g., the multigrid algorithm for a two-factor option [**2**]). Evidently, numerical analysts are becoming familiar with financial applications and the terminology associated with them.

While the numerical problems generally have been simple their algorithms have been subject to constraints not usually found in full scale simulations in science and engineering. Much of the numerical work in finance is carried out on small computers which do not have access to sophisticated program libraries. The computer more likely is a PC or a laptop rather than a Cray, although modern workstations are rapidly replacing the smaller machines. A second constraint is that a large

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volume of very repetitive calculations for slightly differing financial parameters must be carried out rapidly to help financial traders in real time. Hence there is a requirement for fast and fine-tuned standard financial codes. And finally, there is the ever present constraint on any algorithm that it be transparent, at least in outline, to the user who has to interpret the numerical results.

It is the purpose of this exposition to discuss the application of the method of lines to some mathematical models of finance which lead to diffusion problems. Properly implemented, the method of lines places minimum demand on computer resources other than clock speed, and since it solves the problem in terms of financial variables, rather than some transform of them, it is straightforward to understand and modify for the next generation of problems coming to the fore.

2. An Option with Jump Diffusion

An option in the financial world is the right, but not the obligation, to buy or sell an asset (e.g. a share of stock in a corporation, a commodity like grain, foreign currency, etc.) at a fixed price by a certain date in the future. The option to buy is named a "call", the option to sell is named a "put". Historically, options were traded as insurance rather than as an investment. A farmer with a put for his harvest or a manufacturer with a call for his raw material could plan for the financial future with some certainty regardless of the price of the commodity on the open market at the time the option expired. On the other hand, the seller of the option incurs the risk of having to buy or sell an asset at a loss and must be compensated for this risk through the sales price of the option. Determining the value of an option is a major concern of financial engineering.

The core of a widely accepted mathematical model for the value $V(S,t)$ of on option is the so-called Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 V_{ss} + (r-\rho)SV_s - rV - V_t = 0 \tag{2.1}$$

which is based on a stochastic model for the behavior of the price S of the underlying asset. In equation (2.1) t denotes the time to expiry of the option, the coefficient σ describes the so-called volatility of the market price of the asset, r is the risk-free interest rate available to the buyer for an alternative investment and ρ is the rate of a continuous dividend payment. The derivation of the Black-Scholes equation is described in detail in many financial textbooks (see, e.g. [**4**], [**5**], [**9**]), but the most accessible account for a mathematician with limited exposure to the financial world is the differential equations based approach of [**11**] on which most of our comments are based.

Equation (2.1) and some generalizations are used to model a bewildering variety of options such as European, American, Asian, barrier, compound etc. options. All of these names denote specific financial instruments which are reflected in the initial and boundary conditions for (2.1) and, possibly, in changes of the equation (2.1) itself. Here we shall concentrate on an American put which models the price of an option to sell an asset at a fixed price K on or before the time of expiry of the option. This is an optimal stopping problem where the diffusion equation (2.1) is subject to the boundary and initial conditions

$$\begin{aligned} &V(S_f(t)) = K - S_f(t), \qquad V_s(S_f(t)) = -1 \\ &\lim_{S\to\infty} V(S,t) = 0 \\ &V(S,0) = \max\{K - S, 0\}, \qquad S_f(0) = K. \end{aligned} \tag{2.2}$$

The objective is to find the current price $V(S,T)$ for an option which expires at T when the current asset price is S. Time to expiry may be months to years depending on the type of option.

Problem (2.1), (2.2) is a classical free boundary problem known as an obstacle problem where the surface $V(S,t)$ joins the obstacle $\phi(S) = \max\{K - S, 0\}$ at location $S(t)$. The free boundary $S = S(t)$ denotes the early exercise boundary. When the asset price S at time t falls to the value $S(t)$ then the option should be exercised to maximize the expected income. The initial condition $V(x,0)$ is particularly transparent. If at time of expiry ($t = 0$) the asset price S exceeds K then the option will not be exercised since the asset could be sold for more on the open market. Thus, the option has no value. On the other hand, if $S < K$ then there is a gain of $K - S$ for the buyer of the option. In either case the buyer had to pay $V(S,T)$ for the option which enters into the traders' profit and loss calculations.

The analysis of the obstacle problem (2.1), (2.2) and its numerical solution have been the subject of numerous investigations and many effective solution algorithms have been proposed. In particular, it has been argued [**7**] that the method of lines is a competitive numerical method for American puts following the Black-Scholes model and its generalization. Building on the discussion of [**7**] we would like to illustrate here the application of the method of lines to a modification of the Black-Scholes model where the value of the underlying asset experiences random jumps. This situation is said to arise particularly in currency markets [**1**]. The resulting numerical problem is of interest from an algorithmic view because as outlined next it leads to a diffusion equation with a functional term.

The derivation of a Black-Scholes type equation for a diffusion model with jumps dates back to [**6**] and is described, for example, in [**5**]. Here we shall follow the detailed exposition of [**8**] and the forthcoming notes of [**10**]. The essential component is the stochastic model

$$\frac{dS}{S} = (\mu - \rho)dt + \sigma dW_t + \int_{\mathbb{R}} \gamma(y) v(dt, dy)$$

for the value S of the underlying asset which incorporates deterministic growth at a rate μ, a Brownian motion drift depending on the volatility σ and random jumps of relative size $\gamma(y)$ occurring at random times with assigned measure v. When a portfolio is hedged under appropriate assumptions on the market and on the occurrence and size of jumps then the non-local parabolic problem arises

$$\begin{aligned}(2.3)\qquad &\frac{1}{2}\sigma^2 S^2 V_{ss} + (r-\rho)SV_s - rV - V_t \\ &\qquad = -\lambda \int_R [V((1+\gamma(y))S,t) - V(S,t) - \gamma(y)SV_s(S,t)]p(y)m(dy)\end{aligned}$$

where λ is the intensity of a Poisson process describing the arrival of jumps and where m describes the distribution of jumps of size $\gamma(y)$. As in [**7**] all market parameters in (2.3) may depend on t and S. At this time, only the American put is considered so that (2.3) must be solved subject to the boundary conditions (2.2).

A detailed derivation of the valuation equation (2.3) for an American put under appropriate financial hypotheses, its equivalence to a variational inequality, and a characterization of its solution may be found in [**8**]. While the analysis of [**8**] pertains to a very general jump diffusion model, the numerical simulation of options with jumps appears quite limited at this time and is usually based on the semi-analytic solution formulas of the Black-Scholes equation [**5**] (which fail in general when the volatility depends on S). We contend that the numerical methods available for Stefan type free boundary conditions provide useful alternate solution methods which, moreover, are no longer restricted to the Black-Scholes model with constant interest rates and volatilities. One such method is the method of lines coupled with a sweep method for the integration of a time discrete analog of the free boundary value problem [**7**]. Its application to a specific jump diffusion process will be described where $m(dy)$ is a counting measure so that (2.3) becomes

$$\begin{aligned}(2.4)\qquad &\frac{1}{2}\sigma^2 S^2 V_{ss} + \left(r - \rho - \lambda \sum_{i=1}^{m} \pi_i k_i\right) SV_s - (r+\lambda)V - V_t \\ &\qquad = -\lambda \sum_{i=1}^{m} \pi_i V((1+k_i)S,t)\end{aligned}$$

where π_i is the probability of a jump in the asset price S of relative size k_i occurring per unit time with probability λ. The special case of $n = 1$ is subject of a recent paper [**1**] on American calls where an approximate analytic solution is derived with a separation of variables technique applied to a simplified valuation equation. Some additional comments on this case may be found in [**10**]. Here we shall solve (2.4) numerically without any simplification of the model.

3. The Method of Lines Algorithm

Since the valuation equation is linear it is convenient to scale V and S according to

$$x = S/K, \qquad u = V/K$$

so that (2.4) is replaced by the free boundary problem

$$\frac{1}{2}\sigma^2 x^2 u_{xx} + \left(r - \rho - \lambda \sum_{i=1}^{m} \pi_i k_i\right) x u_x - (r+\lambda)u - u_t = -\lambda \sum_{i=1}^{m} \pi_i u((1+k_i)x, t).$$

Consistent with our earlier work on pricing options we shall approximate the time-continuous problem with a sequence of time discrete free boundary problems for the solution $\{u_n, s_n\}$ at time level t_n. A first order time approximation leads to

$$\frac{1}{2}\sigma^2 x^2 u'' + \left(r - \rho - \lambda \sum_{i=1}^{m} \pi_i k_i\right) x u' - \left(r + \lambda + \frac{1}{\Delta t}\right) u = -\lambda \sum_{i=1}^{m} \pi_i u((1+k_i)x) - \frac{1}{\Delta t} u_{n-1}(x) \tag{3.1}$$

while the recommended second order method uses

$$\frac{1}{2}\sigma^2 x^2 u'' + \left(r - \rho - \lambda \sum_{i=1}^{m} \pi_i k_i\right) x u' - \left(r + \lambda + \frac{3}{2\Delta t}\right) u = -\lambda \sum_{i=1}^{m} \pi_i u((1+k_i)x) - \frac{3}{2\Delta t} u_{n-1}(x) - \frac{1}{2\Delta t} (u_{n-1}(x) - u_{n-2}(x)) \tag{3.2}$$

both subject to

$$\begin{aligned} u(s) &= 1 - s, \qquad u'(s) = -1 \\ \lim_{x\to\infty} u(x) &= 0, \\ u_0(x) &= \max\{1-x, 0\}, \end{aligned} \tag{3.3}$$

where for convenience a constant time step has been assumed and the subscript n has been suppressed.

It is well known that an American put without jumps has a continuous free boundary $s(t)$ on $[0, T]$ with

$$\lim_{t\to 0} s(t) = 1.$$

However, as discussed below, in the presence of jumps even an American put may satisfy the condition known for an American call with continuous dividend payment

$$\lim_{t\to 0} s(t) \neq 1.$$

Our numerical method is fully time implicit and does not require the location of $s(0+)$.

Both approximations of the differential equation are written as

$$Lu \equiv a(x)u'' + b(x)u' - c(x)u = H(x,u) + \gamma(x) \tag{3.4}$$

where the coefficients and the right hand side are read off by comparing (3.1) or (3.2) with (3.4). We note that $a(x) > 0$ and $c(x) = O(1/\Delta t)$ on $(0,\infty)$. γ incorporates the history of the process.

To solve the free boundary problem for (3.4) it is, of course, possible to replace the differential equation with a typical finite difference approximation as suggested in [**11**] for the Black-Scholes equation where the jump terms are obtained from an interpolation between bracketing mesh points. However, the sweep method advocated in [**7**] does not seem to be able to account for the functional terms. It does remain applicable if we use an iteration of the form

$$Lu^{k+1} = H(x,u^k) + \gamma \tag{3.5}$$

where u^0 is extrapolated from the solution at preceding time levels. The solution u^k in each iteration is found from the Riccati transformation method described in [**7**] and summarized below.

For a put the infinite interval $[0,\infty)$ is truncated to $[0,X]$ and the boundary condition at infinity is approximated by the so-called up-and-out barrier condition

$$u(X) = 0.$$

In order to compute functional terms the solution is continued as

$$u(x) = 0 \qquad \text{for } x \geq X$$

and

$$u(x) = 1 - x \qquad \text{for } x \leq s.$$

The solution $u(x)$ of (3.5) is then written as

$$u(x) = R(x)v(x) + w(x) \tag{3.6}$$

where R, w and v are found from the Riccati equation

$$R' = 1 + \frac{b}{a}R - \frac{c}{a}R^2, \qquad R(X) = 0 \tag{3.7}$$

and the linear equations

$$w' = -\frac{c}{a}Rw + \frac{R(H+\gamma)}{a}, \qquad w(X) = 0 \tag{3.8}$$

$$v' = \left(\frac{c}{a}R - \frac{b}{a}\right)v + \frac{c}{a}w + \frac{(H+\gamma)}{a}, \qquad v(s) = -1 \tag{3.9}$$

The free boundary s is the first positive root of

$$\phi(x) = 1 - x - [R(x)(-1) + w(x)] = 0 \tag{3.10}$$

below the free boundary s_{n-1} at the preceding time level. The iteration at time level t_n terminates when $\max\{|u^{k+1} - u^k|, |s^{k+1} - s^k|\} < 10^{-6}$. The integration of (3.7), (3.8) is called the forward sweep, the integration of (3.9) is the backward sweep.

For the numerical integration of the sweep equations a fixed but not necessarily constant mesh is imposed on $[0, X]$. The Riccati equation (3.7) for a time independent volatility and interest rate and constant time step is time and iteration independent. Hence it is integrated only once and stored. Similarly, all time and iteration independent information needed for the evaluation of the right hand side, such as interpolation weights for the functional terms, is precomputed and stored. In each iteration at a given time level the linear equation for w is integrated and the algebraic sign of ϕ is monitored on $(0, s_{n-1})$. If it changes sign then s is the zero of the Lagrange interpolant through the closest four nodal values of ϕ. The first step of the integration of v goes from s to the next largest fixed mesh point and then proceeds along the regular mesh. All integrations are carried out with the trapezoidal rule. The resulting algebraic equations are quadratic for the Riccati equation and linear for w and v and hence can be solved analytically. The root of the cubic interpolant of ϕ is found with Newton's method.

4. A numerical Example

To illustrate the performance of the numerical method we shall recompute some data of Table 1 in [**7**] in the presence of jumps. Specifically, let us consider the Black-Scholes model with two jumps

$$\begin{aligned}\frac{1}{2}\sigma^2 x^2 u_{xx} + (r - \rho - \lambda(\pi_1 k_1 + \pi_2 k_2))x u_x - (r+\lambda)u - u_t \\ = -\lambda(\pi_1 u((1+k_1)x, t) + \pi_2 u((1+k_2)x, t))\end{aligned} \tag{4.1}$$

where $k_1 k_2 < 0$. Fig. 4.1 shows the free boundaries obtained with the implicit Euler method (3.1) for a case without jumps ($\lambda = 0$) and with jumps ($\lambda = .1$). Fig. 4.2 shows u, $u' \equiv v$ and $u'' \equiv v'$ (given by the right-hand-side of equation (3.9)). The plotted curves correspond to a linear interpolation of the nodal values. No further smoothing was employed.

We remark that very exaggerated jump conditions were chosen simply to obtain clearly different early exercise boundaries. Even so, only two iterations per time

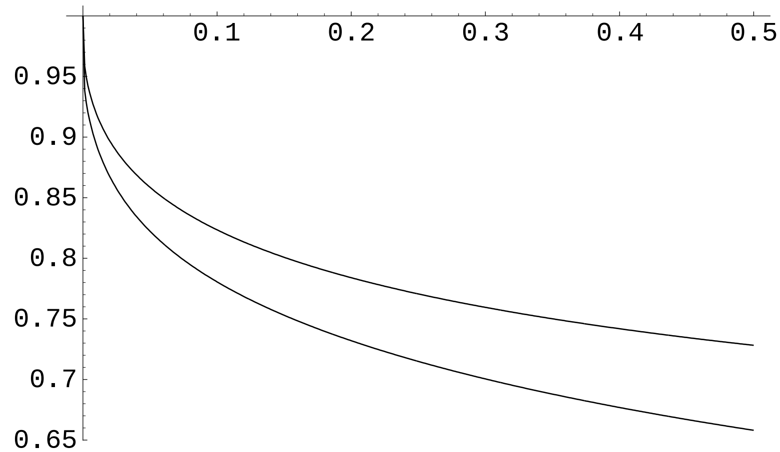

Figure 4.1. Early exercise boundary for an American put with $r = .12$, $\rho = 0$, $\sigma = .4$, $T = .5$. Upper curve: no jumps. Lower curve: $\lambda = .5$, $k_1 = .5$, $\pi_1 = .5$, $k_2 = -.5$, $\pi_2 = .5$, $\Delta x = .001$, $\Delta t = .5/400$, $X_1 = 2.5$.

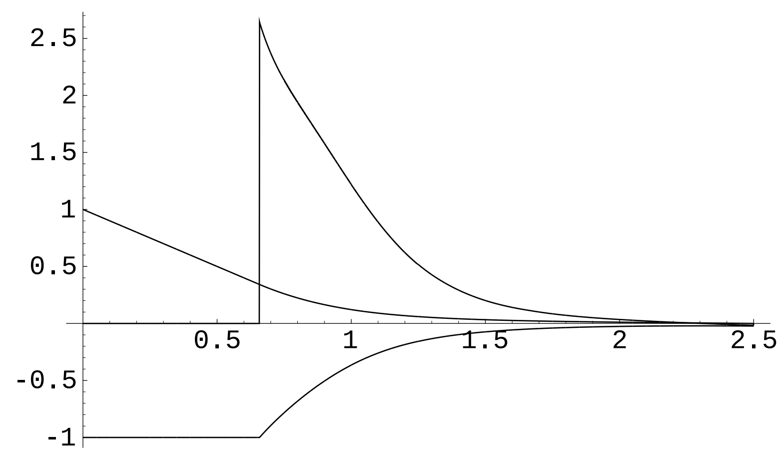

Figure 4.2. Value and two "Greeks" for the put with jumps of Fig. 4.1. Upper curve: "gamma" $\equiv u''(x)$. Middle curve: value of the option $\equiv u(x)$. Lower curve: "delta" $\equiv u'(x)$.

step were required to meet our convergence criterion of $|s^{k+1} - s^k|$, $|u^{k+1} - u^k| \leq 10^{-6}$.

A necessary condition for the existence of an early exercise boundary, both for a put and call is

$$u''(s) \geq 0$$

since otherwise u cannot lie above the obstacle $\max\{0, 1 - x\}$ (for a put) or $\max\{0, x - 1\}$ (for a call). We find from (3.1) that at the free boundary s

$$\frac{1}{2}\sigma^2 s^2 u''(1) = \left(r - \rho - \lambda \sum_{i=1}^{n_p} \pi_i (k_i s - u(1 + k_i)s)\right) - \lambda \sum_{j=1}^{n_m} \pi_j \left(k_j s - u((1 + k_j)s) + (r + \lambda)(1 - s)\right)$$

where the first sum on the right collects all positive jumps $\{k_i > 0\}$ and the second sum accounts for negative jumps $\{k_j \leq 0\}$. This expression can be rewritten algebraically as

$$\frac{1}{2}\sigma^2 s^2 u''(s) = r - \rho s - \sum_{i=1}^{n_p} \pi_i [u((1 + k_i)s) - (1 - (1 + k_i)s)]$$

Since for a put $u(x) > \max\{0, 1 - x\}$ for $x > s$ we cannot conclude that $u''(s) > 0$ under all circumstances. On the other hand, since $u((1+k)s) - (1-(1+k)s) \to 0$ as $s \to 0$ it follows that $\lim_{s\to 0} u''(s) > 0$. Hence there would appear to be a free boundary $s(t)$ such that $\lim_{t\to 0} s(t) \neq 1$ as in an American call with a small dividend. Numerical experiments with $r = .0001$, $\rho = 0$, $\sigma = .4$ and a single positive jump of size $k_1 = .5$ with probability $\lambda = .1$ consistently lead to the estimate $\lim_{t\to 0} s(t) = .666$ and a smooth subsequent evolution of $s(t)$.

When the method of lines is applied to an American call with jumps then the boundary data for (4.1) become

$$u(0, t) = 0, \qquad u(s(t), t) = s - 1, \qquad u_x(s(t), t) = 1.$$

The sweeps are now carried out in reversed directions. (3.7), (3.8) are integrated from 0 to the free boundary $s(t)$, the function (3.10) becomes

$$\phi(x) \equiv x - 1 - [R(x)(+1) + w(x)] = 0$$

and the equation (3.9) is integrated backward from s to 0. To avoid the (numerical) degeneracy at $x = 0$ the coefficient of u_{xx} is approximated by

$$\sigma(x) = \max\{10^{-6}, 1/2\sigma^2 x^2\}.$$

Alternatively, a down-and-out barrier condition can be imposed. No further changes are necessary.

For comparison a method of lines code for calls with jumps was applied to the simulation reported in [**1**]. The following data are representative

We know little about the data cited in [**1**]. However, the method of lines data are unchanged when the space and time step are varied.

case 1:	$r = .06$,	$\rho = .1$	S	Finite Difference	Approx Method	MOL
case 1:	$r = .06$,	$\rho = .1$	80	1.15	1.16	1.15
			90	3.45	3.47	3.46
			100	7.65	7.65	7.67
			110	13.77	13.75	13.80
			120	21.48	21.42	21.52
case 2:	$r = .06$,	$\rho = .02$	80	1.40	1.40	1.41
			90	4.02	4.02	4.04
			100	8.60	8.60	8.64
			110	15.05	15.05	15.12
			120	22.93	22.92	23.03

MOL: $\Delta x = .001$, $\Delta t = .25/400$.

Table 4.1. Comparison of MOL call values with data reported in [**1**]. Strike price $K = 100$, $\sigma = .4$, $T = .25$, $\lambda = 1$, $k = -.1$.

5. Comments on Convergence

Since at every time level we propose to solve the functional differential equation (3.4) iteratively the question of convergence of the iteration arises. We remark that such iteration is commonly applied to functional differential equations (see, e.g. the collection of papers on boundary value problems for functional differential equations [3]) for which convergence is established by fixed point methods common in the theory of ordinary differential equations. However, for our elliptic operator a technique of proof borrowed from finite elements appears simpler to apply.

We cannot yet treat the free boundary problem for the American put with jumps but we can consider the simpler problem of the European put with a single jump. This problem does not have a free boundary. It has the following structure

$$Lu \equiv a(x)u'' + b(x)u' - c(x)u = H(x, u(\phi(x)) + \gamma(x), \qquad x \in I \tag{5.1}$$

$$u(x) = \psi(x), \qquad x \in I^c$$

where I is the open interval (x_0, x_1) contained in $[X_0, X_1]$ and I^c denotes its complement in $[X_0, X_1]$. We assume that

$$\phi\colon I \to [X_0, X_1]$$

describes a positive jump ($\phi(x) \ge x$) or a negative jump ($\phi(x) \le x$).

Throughout this discussion we shall assume that all data functions are smooth. In addition we impose the hypotheses:

$$a(x) > 0, \qquad x \in \bar{I}$$

$$c(x) \ge c_0 > 0, \qquad x \in I$$

$$|H(x,u) - H(x,v)| \le L|u - v|, \qquad x \in I$$

and

$$\psi \in H^1[X_0, X_1].$$

For a given $f \in H^1[X_0, X_1]$ let T denote the mapping

$$u = Tf$$

where u is the solution of

$$\begin{gathered} Lu = H(x, f(\phi(x)) + \gamma(x), \qquad x \in I \\ u(x_0) = \psi(x_0), \qquad u(x_1) = \psi(x_1) \\ u = \psi, \qquad x \in I^c. \end{gathered}$$

Since H is continuous in x, and a and c are strictly positive on $\bar{I}$ it follows that this two point boundary value problem has a unique classical solution and that $u \in H^1[X_0, X_1]$. In particular, if we define

$$S = \left\{ f \in H^1[X_0, X_1],\ f = \psi,\ x \in I^c \right\}$$

then T maps S into S.

Convergence of our numerical method for the European put follows if T is contractive in $L_2[X_0, X_1]$. To handle the convection term bu' in (5.1) let w denote a solution of

$$(a(x)w)'' - (b(x)w)' - 2\alpha c(x)w = 0, \qquad x \in I \tag{5.2}$$

where $\alpha \in (0, 1)$ is a constant. We shall make the assumption that boundary or initial data for (5.2) can be chosen such that

(5.3) w is positive on I and $w' > 0$ on I for positive jumps ($\phi(x) \geq x$).

(5.4) w is positive on I and $w' < 0$ on I for negative jumps ($\phi(x) \leq x$).

In either case it follows that $w(x)/w(\phi(x)) \leq 1$. If both positive and negative jumps occur then (5.3), (5.4) must be replaced by a bound on $w(x)/w(\phi(x))$ which would enter into the estimates given below.

We define an equivalent norm on $L_2[X_0, X_1]$ induced by the inner product

$$\langle f, g \rangle = \int_{X_0}^{X_1} f(x)g(x)c(x)w(x)\, dx$$

where c and w are continued as constants over I^c. Then for $f, g \in S$ and $u = Tf$ and $v = Tg$ it follows from

$$\int_{X_0}^{X_1} (Lu - Lv)(u - v)w\, dx = \int_{x_0}^{x_1} (H(x, f(\phi(x)) - H(x, g(\phi(x))(u - v)w\, dx$$

and $u = v$ on I^c that

$$\int_{X_0}^{X_1} (u-v)'^2 (aw)\, dx + (1-\alpha)\|u-v\|^2 = -\int_{x_0}^{x_1} H(x, f(\phi(x)) - H(x, g(\phi(x))(u-v)w\, dx.$$

Standard estimates now yield

$$(1-\alpha+\lambda_1)\|u-v\|^2 \le L \int_{X_0}^{X_1} |f(\phi(x)) - g(\phi(x))|\, |u-v| w\, dx$$

where λ_1 is the smallest eigenvalue of the Sturm-Liouville problem

$$\begin{aligned} (awu')' + \lambda cwu &= 0, \\ u(x_0) = u(x_1) &= 0\,. \end{aligned}$$

Since $0 < w(x)/w(y) \le 1$ for $y = \phi(x)$ and $dy = \phi'(x)\, dx$ we find that

$$\begin{aligned} L \int_{X_0}^{X_1} |f(y) - g(y)|\, |u-v| w\, dx &\le \frac{L}{c_m} \int_{X_0}^{X_1} |f(y) - g(y)| c(y) w(x) dx \|u-v\| \\ &\le \frac{L}{c_m \sqrt{|\phi'|_m}} \|f-g\| \|u-v\| \end{aligned}$$

where $|\phi'|_m = \min |\phi'|$. We can conclude that T is a contraction whenever

$$\delta \equiv \frac{L}{c_m \sqrt{|\phi'|_m}(1-\alpha+\lambda_1)} < 1. \tag{5.5}$$

For any continuous initial guess u^0 we observe that $Tu^0 \in S$ so that convergence is global. Moreover, it follows from our discussion that the sequence $\{u^k\}$ is in fact a sequence of equicontinuous functions on $[X_0, X_1]$ so that u^k converges to a classical solution of (5.1) as $k \to \infty$.

When the above considerations are applied to the Black-Scholes model for a European put with a negative jump then the following specific data arise

$$\begin{aligned} a(x) &= \frac{1}{2}\sigma^2 x^2 \\ b(x) &= r - \rho - \lambda k \\ c(x) &= \left(r + \lambda + \frac{1}{\Delta t}\right) \\ \gamma(x) &= -\frac{1}{\Delta t} u_{n-1}(x) \end{aligned}$$

with a jump term of the form

$$H(x, u(\phi(x)) = -\lambda u((1+k)x) \qquad \text{where } k > -1.$$

The problem is defined on the interval $[0, \infty)$ which for the computation is truncated to $[\epsilon, X_1]$ with $0 < \epsilon \ll 1$ and $X_1 \gg 1$. The boundary data for the European put (see [11])

$$u(\epsilon) = e^{-rt_n} \quad \text{and} \quad u(X_1) = 0$$

are continued as constants over I^c in $[0, \infty)$. The existence of an appropriate weight function w follows for Δt so small that

$$2\alpha\left(r + \lambda + \frac{1}{\Delta t}\right) - (r - \lambda k) > 0$$

because then initial conditions $w(\epsilon) = 1$, $w'(\epsilon) > 0$ or $w(X_1) = 1$, $w'(X_1) < 0$ both guarantee a monotone w. Hence the contraction constant (5.5) is valid which in this case is

$$\delta \leq \frac{\Delta t}{(1-\alpha)(1 + \Delta t(r + \lambda))}\,.$$

We see that the contraction constant is of order Δt uniformly with respect to ϵ and X_1. It seems reasonable to expect that the free boundary of the American put does not greatly influence this conclusion and hence that the rapid convergence observed in numerical experiments was to be expected. We remark that in this setting the Riccati equation (3.6) has a non-positive solution $R(x)$ which is bounded below by $-m\sqrt{\Delta t}x$ for sufficiently large $m > 0$ because $R'(x) < -m\sqrt{\Delta t}$ whenever $R(x) = -m\sqrt{\Delta t}x$ implies that $R(x)$ cannot cross the line $y = -m\sqrt{\Delta t}x$. Moreover, if $0 \leq u_{n-1} \leq 1$ and $0 \leq u^{k-1} < 1$ then $0 \leq w \leq 1$ and $0 \leq u^k \leq 1$ uniformly with respect to ϵ. Hence neither the analysis nor the algorithm break down as $\epsilon \to 0$ nor does the functional term influence the stability of the time implicit discretization of the European put problem.

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