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MAXIMAL ELEMENTS OF SUPPORT

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Abstract. We introduce a set that is tightly close to the set of the Jacobson radical of module (the intersection of all maximal elements in support). In the last section, it is proved that the set of zero divisors of a module is equal to the union of the maximal elements of the support of module if the module is finitely generated and injective.

0. Introduction

Throughout this note the ring R is commutative (not necessarily Noetherian) with non-zero identity. The notion of prime ideals is central in the commutative ring theory. The set $\operatorname{Spec}(R)$ of prime ideals of a ring R is a topological space, and the localization of rings with respect to this topology is an important technique for studying them. In addition, the maximal element of this set is very useful. There is a similar notion for modules that is support of modules. The set of prime ideals $\mathfrak{p}$ such that there exists a cyclic submodule M, and is written $\operatorname{Supp}(M)$.

Let $\mathrm{J}_R(M)$ be the Jacobson radical of the R-module M (the intersection of all maximal elements of the support of M). Let $\mathrm{N}_R(M)$ be the union of all maximal elements of the support of M. Then it is easy to see that $\mathrm{J}_R(M) \subseteq \mathrm{N}_R(M)$ and the equality holds if and only if the support of M has only one element. The set $\mathrm{N}'_R(M)$ is defined by

$$\mathrm{N}'_R(M) = \big\{x \in \mathrm{N}_R(M) \mid x + \mathrm{N}_R(M) \subseteq \mathrm{N}_R(M)\big\}.$$

We show that $\mathrm{J}_R(M) \subseteq \mathrm{N}'_R(M)$ and there is equality if the support of M has only finite elements. By an example we show that the inequality may be strict. In the last section we prove that for any finitely generated and injective R-module M, the set of zero divisors of M is equal to the set $\mathrm{N}_R(M)$. As a corollary of this result we have that "if R is a self-injective ring then each non-unit element in R is a zero divisor in R".

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1. Support of Modules

In this section we study the intersection and the union of the maximal elements of the support of a module.

Definition 1.1. Let M be an R-module. The **support** of M is denoted by $\mathrm{Supp}(M)$ and it is defined by

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathrm{Ann}(N) \text{ for some cyclic submodule } N \text{ of } M\}.$$

Note that this definition is equivalent with the classical definition of support (cf. [**M**, pp. 26]) that is

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

The next lemma is a well-known result, cf. [**M**, pp. 25].

Lemma 1.2. *The following hold:*

a) $M \neq 0$ *if and only if* $\mathrm{Supp}(M) \neq \emptyset$.
b) $\mathrm{Supp}(M) \subseteq \mathrm{Spec}(R/\mathrm{Ann}(M))$.
c) $\mathrm{MaxSupp}(M) \subseteq \mathrm{MaxSpec}(R/\mathrm{Ann}(M))$, *where* $\mathrm{MaxSupp}(M)$ *is the set of all maximal elements in* $\mathrm{Supp}(M)$.
d) If M is a finitely generated R-module then we have equality in (b) and (c).

Remark 1.3. The inequality of (1.2b) may be strict, for example, if (R, m) is a local ring and $M = \mathrm{E}(R/\mathfrak{m})$, injective envelope of the field $R/\mathfrak{m}$, then $\mathrm{Ann}(M) = 0$ and so $\mathrm{Spec}(R/\mathrm{Ann}(M)) = \mathrm{Spec}(R)$. On the other hand $\mathrm{Supp}(M) = \{\mathfrak{m}\}$.

Also the inequality in (1.2c) may be strict. For example let R be an integral domain and $\{\mathfrak{m}, \mathfrak{n}\} \subseteq \mathrm{MaxSpec}(R)$. Let $M = \mathrm{E}(R/\mathfrak{m})$. Then $\mathrm{MaxSupp}(M) = \{\mathfrak{m}\}$ but $\mathfrak{n} \in \mathrm{MaxSpec}(R/\mathrm{Ann}(M)) = \mathrm{MaxSpec}(R)$.

Definition 1.4. Let M be an R-module. The **Jacobson radical** of M is denoted by $\mathrm{J}_R(M)$ and it is the intersection of all elements in $\mathrm{MaxSupp}(M)$. Also the union of all elements in $\mathrm{MaxSupp}(M)$ is denoted by $\mathrm{N}_R(M)$.

Lemma 1.5. *Let M be an R-module. Then $r \in \mathrm{J}_R(M)$ if and only if $1 + tr \notin \mathrm{N}_R(M)$ for any $t \in R$.*

Proof. "if" Let $\mathfrak{m} \in \mathrm{MaxSupp}(M)$ such that $r \notin \mathfrak{m}$. Then $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ and hence $\mathfrak{m} + rR = R$. Therefore, there exist $x \in \mathfrak{m}$ and $t \in R$ such that $x + tr = 1$ and hence $1 - rt \in \mathrm{N}_R(M)$, which is a contradiction.

"only if" Let $t \in R$ such that $1 + tr \in \mathrm{N}_R(M)$. Then there exists a maximal ideal $\mathfrak{m} \in \mathrm{MaxSupp}(M)$ such that $1 + tr \in \mathfrak{m}$. On the other hand $tr \in \mathfrak{m}$. Therefore $1 \in \mathfrak{m}$, which is a contradiction. $\square$

Definition 1.6. The R-module M is said to be **local module** if $|\mathrm{MaxSupp}(M)| = 1$. Also the R-module M is said to be **semi-local module** if $|\mathrm{MaxSupp}(M)| < \infty$. Clearly, all non-zero modules over a semi-local (resp. local) ring is a semi-local (resp. local) module.

Theorem 1.7. *The following are equivalent:*

i) M *is a local module.*
ii) $\mathrm{J}_R(M) = \mathrm{N}_R(M)$.
iii) $\mathrm{N}_R(M)$ *is an ideal of* R.

Proof. "(i$\Rightarrow$ ii)" and "(ii$\Rightarrow$ iii)" are obvious.
(iii$\Rightarrow$ i)" Since $1 \notin \mathrm{N}_R(M)$ we have $\mathrm{N}_R(M) \neq R$ and hence there exists $\mathfrak{m} \in \mathrm{MaxSpec}R$ such that $\mathrm{N}_R(M) \subseteq \mathfrak{m}$. On the other hand $\mathfrak{m} \subseteq \mathrm{N}_R(M)$. Therefore $\mathrm{N}_R(M) = \mathfrak{m}$ and hence $\mathrm{MaxSupp}(M) = \{\mathfrak{m}\}$. $\square$

Definition 1.8 (see [**C**]). Let M be an R-module. We define $\mathrm{N}'_R(M)$ by

$$\mathrm{N}'_R(M) = \{x \in \mathrm{N}_R(M) \mid x + \mathrm{N}_R(M) \subseteq \mathrm{N}_R(M)\}.$$

Theorem 1.9. *Let* M *be an* R*-module. Then the following hold:*

a) $\mathrm{J}_R(M) \subseteq \mathrm{N}'_R(M) \subseteq \mathrm{N}_R(M)$
b) $\mathrm{J}_R(M) = \mathrm{N}'_R(M)$ *if and only if* $\mathrm{N}'_R(M)$ *is an ideal of* R.
c) If M *is a semi-local then* $\mathrm{J}_R(M) = \mathrm{N}'_R(M)$.

Proof. "(a)" Set $x \in \mathrm{J}_R(M)$ and $t \in \mathrm{N}_R(M)$. Then there exists $\mathfrak{m} \in \mathrm{MaxSupp}(M)$ such that $t \in \mathfrak{m}$. Since $x \in \mathfrak{m}$ we have $x + t \in \mathfrak{m}$ and hence $x + t \in \mathrm{N}_R(M)$. Thus $\mathrm{J}_R(M) \subseteq \mathrm{N}'_R(M)$.

"(b)" The 'Only if' part is obvious. For the 'If' part, set $x \in \mathrm{N}'_R(M)$ and $t \in R$. Since $\mathrm{N}'_R(M)$ is an ideal of R we have $tx \in \mathrm{N}'_R(M)$. We claim that $1+tx \notin \mathrm{N}_R(M)$. In the other case if $1 + tx \in \mathrm{N}_R(M)$ then $1 \in \mathrm{N}_R(M)$, which is a contradiction. Therefore $x \in \mathrm{J}_R(M)$.

"(c)" Let $\mathrm{MaxSupp}(M) = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_t\}$. Suppose $x \in \mathrm{N}'_R(M)$. Then there exists $1 \leq r \leq t$ such that $x \in \bigcap_{i=1}^{r} \mathfrak{m}_i$ and $x \notin \bigcup_{i=r+1}^{t} \mathfrak{m}_i$. We claim that $r = t$. In the other case by the prime avoidence theorem we have $\bigcap_{i=r+1}^{t} \mathfrak{m}_i \nsubseteq \bigcup_{i=1}^{r} \mathfrak{m}_i$ and hence there exists $y \in \bigcap_{i=r+1}^{t} \mathfrak{m}_i \backslash \bigcup_{i=1}^{r} \mathfrak{m}_i$. Since $y \in \mathrm{N}_R(M)$ we have $x + y \in \mathrm{N}_R(M)$. On the other hand $x + y \notin \mathfrak{m}_i$ for each $1 \leq i \leq t$, which is a contradiction. $\square$

Remark 1.10. The inequalities in 1.9(a) may be strict. For the inequality in the right-hand side let M be a semi-local module but not local then $\mathrm{J}_R(M) = \mathrm{N}'_R(M) \nsubseteq \mathrm{N}_R(M)$. For the inequality in the left-hand side, let $(D, \mathfrak{m})$ be a local regular ring that is not a field. Then $\mathrm{J}_D(D) = \mathfrak{m} \neq 0$. It is easy to see that $\mathrm{J}_{D[x]}(D[x]) = 0$ and

$$\mathrm{N}_{D[x]}(D[x]) = \mathrm{N}_D(D) \cup \{g \in D[x] \mid \deg(g) \geq 1\}.$$

Now we show that $\mathrm{N}'_D(D) \subseteq \mathrm{N}'_{D[x]}(D[x])$. Assume that $a \in \mathrm{N}'_D(D)$ then $a \in \mathrm{N}_D(D)$ and hence $a \in \mathrm{N}_{D[x]}(D[x])$. Let $f \in \mathrm{N}_{D[x]}(D[x])$. Then we have two cases:

(i) "$f \in \mathrm{N}_D(D)$" In this case we have $a + f \in \mathrm{N}_D(D)$ and hence $a + f \in \mathrm{N}_{D[x]}(D[x])$.

(ii) "$f \in D[x]$ with $\deg f \geq 1$" Let $f = \Sigma_{i=0}^{n} a_i x^i$ and let $a_n \neq 0$. Then $\deg a + f \geq 1$ and hence $a + f \in \mathrm{N}_{D[x]}(D[x])$. Therefore $a \in \mathrm{N}'_{D[x]}(D[x])$ and so $\mathrm{N}'_D(D) \subseteq \mathrm{N}'_{D[x]}(D[x])$.

Since $0 \neq \mathrm{J}_D(D) \subseteq \mathrm{N}'_D(D) \subseteq \mathrm{N}'_{D[x]}(D[x])$ we have that $\mathrm{N}'_{D[x]}(D[x]) \neq 0$. On the other hand $\mathrm{J}_{D[x]}(D[x]) = 0$.

By using the next lemma we can put $\mathrm{N}'_R(M)$ instead of $\mathrm{J}_R(M)$ in the Nakayama lemma and in the Krull's intersection theorem, cf. [**M**, 2.2 and 8.9].

Lemma 1.11. *Let M be an R-module and let $\mathfrak{a}$ be an ideal of R. Then $\mathfrak{a} \subseteq N'_R(M)$ if and only if $\mathfrak{a} \subseteq \mathrm{J}_R(M)$.*

Proof. Let $x \in \mathfrak{a}$ and $r \in R$. Then $rx \in \mathfrak{a}$ and hence $1+rx \notin \mathrm{N}'_R(M)$. Therefore $x \in \mathrm{J}_R(M)$ by (1.5). □

2. Injective and Flat Modules

Let M be an R-module. The prime ideal $\mathfrak{p}$ is said weakly associated to M if there exists an element $x \in M$ such that $\mathfrak{p}$ is a minimal among the prime ideals containing the annihilator $\mathrm{Ann}(x)$, see [**B**, Chapt. 4, Sect. 1, Exercise 17]. The set of weakly associated prime ideals of the R-module M is denoted by $\mathrm{A\widetilde{ss}}_R(M)$.

Recall that the set of zero divisors of M, $\mathrm{Z}_R(M)$, is defined by

$$\mathrm{Z}_R(M) = \{a \in R \mid M \xrightarrow{a\cdot} M \ \text{ is not injective}\}$$

Theorem 2.1 (see [**B**, Chapt. 4, Sect. 1, Exercise 17]). *Let M be an R-module. Then the following hold:*

a) $\mathrm{A\widetilde{ss}}_R(M) \subseteq \mathrm{Supp}(M)$,

b) $\mathrm{Z}_R(M) = \bigcup_{\mathfrak{p} \in \mathrm{A\widetilde{ss}}_R M} \mathfrak{p}$.

Now we bring the dual notion of $\mathrm{Z}_R(M)$.

Definition 2.2 (see [**Y1**]). For the R-module M the subset $\mathrm{W}_R(M)$ of R is defined by

$$\mathrm{W}_R(M) = \{a \in R \mid M \xrightarrow{a\cdot} M \ \text{ is not surjective}\}.$$

Lemma 2.3. *Let M be an R-module. Then the following hold;*

a) $\mathrm{W}_R(M) \subseteq \mathrm{N}_R(R)$

b) $\mathrm{J}_R(R) \subseteq \mathrm{W}_R(M)$ *if M is a finitely generated R-module.*

Proof. "(a)" Set $x \in \mathrm{W}_R(M)$. Then $xM \neq M$ and hence x is a non-unit element of R. Therefore $x \in \mathrm{N}_R(R)$.

"(b)" Set $x \in \mathrm{J}_R(R)$. Then $Rx \subseteq \mathrm{J}_R(R)$ and hence by the Nakayama lemma we have that $xM = (Rx)M \neq M$. □

Theorem 2.4. *Let M be an R-module. Then the following hold;*

a) $\mathrm{W}_R(M) \subseteq \mathrm{N}_R(M)$

b) We have equality in (a) if M is a finitely generated R-module.

Proof. "(a)" If $M = 0$ then there is nothing to prove. Let $M \neq 0$ and let $x \in \mathrm{W}_R(M)$. Then $xM \neq M$ and hence $M/xM \neq 0$. Let $\mathfrak{m} \in \mathrm{MaxSupp}(M/xM)$. Then $(M/xM)_\mathfrak{m} \neq 0$ and hence $M_\mathfrak{m}/(x/1)M_\mathfrak{m} \neq 0$. Therefore by (2.3a) we have $x/1 \in \mathrm{W}_{R_\mathfrak{m}}(M_\mathfrak{m}) \subseteq \mathrm{N}_{R_\mathfrak{m}}(R_\mathfrak{m}) = \mathfrak{m}R_\mathfrak{m}$ and hence $x \in \mathfrak{m}$. Now the assertion follows from the fact that $\mathfrak{m} \in \mathrm{MaxSupp}(M)$.

"(b)" If $M = 0$ then there is nothing to prove. Let $M \neq 0$ and let $x \in \mathrm{N}_R(M)$. Then there exists $\mathfrak{m} \in \mathrm{MaxSupp}(M)$ such that $x \in \mathfrak{m}$. Thus $x/1 \in \mathfrak{m}R_\mathfrak{m} = \mathrm{J}_R(R_\mathfrak{m})$. Since M is a finitely generated R-module we have $M_\mathfrak{m}$ is a non-zero finitely generated $R_\mathfrak{m}$-module. Therefore by (2.3b), we have that $\mathrm{J}_{R_\mathfrak{m}}(R_\mathfrak{m}) \subseteq \mathrm{W}_{R_\mathfrak{m}}(M_\mathfrak{m})$ and hence $x/1 \in \mathrm{W}_{R_\mathfrak{m}}(M_\mathfrak{m})$. Thus $M_\mathfrak{m}/(x/1)M_\mathfrak{m} \neq 0$ and hence $(M/xM)_\mathfrak{m} \neq 0$. Therefore $M/xM \neq 0$ and so $x \in \mathrm{W}_R(M)$. □

Definition 2.5 (see [**SV**]). An R-module M is said to be *finitely cogenerated* (the dual notion of finitely generated) if $\mathrm{E}(M)$ is isomorphic to a direct sum of finitely many injective envelope of simple modules.

Theorem 2.6. *Let M be an R-module. Then the following hold;*

a) $\mathrm{Z}_R(M) \subseteq \mathrm{N}_R(M)$

b) We have equality in (a) if M is a finitely cogenerated R-module.

Proof. "(a)" Use (2.1).

"(b)" Since M is finitely cogenerated we have that $\mathrm{Supp}(M) \subseteq \mathrm{MaxSpec}(R)$ and hence $\mathrm{Z}_R(M) = \mathrm{N}_R(M)$ by (2.1). □

Theorem 2.7. *Let M be an injective R-module. Then the following hold;*

a) $\mathrm{W}_R(M) \subseteq \mathrm{Z}_R(M)$

b) We have equality in (a) if M is a finitely generated R-module.

Proof. "(a)" Set $x \in \mathrm{W}_R(M)$. If $x \notin \mathrm{Z}_R(M)$ we have the map $\varphi : xM \to M$ with $\varphi(xt) = t$ for any $t \in M$. Since M is an injective R-module, the map φ induces the map $\psi : M \to M$ such that for all $t \in M$ we have that $t = \varphi(xt) = \psi(xt) = x\psi(t) \in xM$. Thus $xM = M$, which is a contradiction.

"(b)" We have $x \in \mathrm{Z}_R(M) \subseteq \mathrm{N}_R(M)$ by (2.1). Now the assertion follows from (2.4b). □

Corollary 2.8. *If R is a self-injective ring then the set of zero divisors of R is equal to the set of non-units in R.*

Theorem 2.9. *Let M be a flat R-module. Then the following hold;*

a) $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$
b) We have equality in (a) if M is a finitely cogenerated R-module.

Proof. "(a)" Set $x \in \mathrm{Z}_R(M)$. Then there exists a non-zero element $t \in M$ such that $xt = 0$. Thus we have the non-zero map $\varphi : R/(x) \longrightarrow M$ with $\varphi(r+(x)) = rt$ for any $r \in R$. Therefore $\mathrm{Hom}(R/(x), M) \neq 0$ and hence there exists an injective module E such that $\mathrm{Hom}(\mathrm{Hom}(R/(x), M), E) \neq 0$. Since

$$\mathrm{Hom}(\mathrm{Hom}(R/(x), M), E) \cong R/(x) \otimes \mathrm{Hom}(M, E),$$

we have that $x \in \mathrm{W}_R(\mathrm{Hom}(M, E))$. Since $\mathrm{Hom}(M, E)$ is an injective module we have $x \in \mathrm{Z}_R(\mathrm{Hom}(M, E)$ by (2.6), and hence $\mathrm{Hom}(R/(x), \mathrm{Hom}(M, E))$ is non-zero. Therefore $\mathrm{Hom}(R/(x) \otimes M, E) \neq 0$ and hence $R/(x) \otimes M \neq 0$. Thus $x \in \mathrm{W}_R(M)$

"(b)" By (2.4) and (2.6b) we have

$$\mathrm{W}_R(M) \subseteq \mathrm{N}_R(M) = \mathrm{Z}_R(M).$$

Now the assertion follows from (a). □

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