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ON THE ROTATION SETS FOR NON–CONTINUOUS CIRCLE MAPS

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Abstract. We give some examples of non-continuous circle maps whose rotation sets lack the good properties they have in the case of continuous maps.

1. Introduction

Let $e\colon \mathbf{R} \longrightarrow \mathbf{S}^1$ be the natural projection defined by $e(x) = \exp(2\pi i x)$. A map $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ is called a **lifting** of a map $f\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ if $e \circ F = f \circ e$ and there is a $d \in \mathbf{Z}$ such that $F(x+1) = F(x) + d$ for all $x \in \mathbf{R}$. This d is called the **degree** of the lifting F. Note that since we do not impose continuity for F, every f has liftings of all degrees.

We shall only consider maps (liftings) of degree one. Following [**4**], a map $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ is called an **old** map (old is a mnemonic for 'degree one lifting') if $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$. Clearly, F is an old map if and only if there exists $f\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ such that F is a lifting of f of degree one. It is easy to see that in this case $F(x+k) = F(x) + k$ for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$, and that iterates of an old map are also old maps.

A point $x \in \mathbf{R}$ is called **periodic** mod 1 of **period** $q \in \mathbf{N}$ and **rotation number** p/q for an old map F if $F^q(x) - x = p \in \mathbf{Z}$ and $F^i(x) - x \notin \mathbf{Z}$ for $i = 1, 2, \dots, q-1$. Similarly, for any $x \in \mathbf{R}$ we can define its **orbit** mod 1 under F as the set $e^{-1}(\mathrm{Orb}_f(e(x))) = \mathrm{Orb}_F(x) + \mathbf{Z} = \cup_{n \geq 0}(F^n(x) + \mathbf{Z})$. The orbit mod 1 of a periodic mod 1 point is also called a **lifted cycle** of F.

For an old map F and a point $x \in \mathbf{R}$ we define $\overline{\rho}_F(x)$ and $\underline{\rho}_F(x)$ (or $\overline{\rho}(x)$ and $\underline{\rho}(x)$ if no confusion seems possible) as

$$\overline{\rho}_F(x) = \limsup_{m\to\infty} \frac{F^m(x) - x}{m} \quad \text{and} \quad \underline{\rho}_F(x) = \liminf_{m\to\infty} \frac{F^m(x) - x}{m} .$$

When $\overline{\rho}_F(x) = \underline{\rho}_F(x)$ we write $\rho_F(x)$ or $\rho(x)$ to denote both $\overline{\rho}_F(x)$ and $\underline{\rho}_F(x)$. The number $\rho_F(x)$ (if it exists) is called the **rotation number of x with respect**

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to F. It is well known [**1**, Lemma 3.7.2] that if $x \in \mathbf{R}$ is a periodic mod 1 point of period q and rotation number p/q for an old map F, then $\rho_F(x) = p/q$. Thus we can talk about the **rotation number** $\rho(P)$ **of a lifted cycle** P of an old map. For non-decreasing old maps $\rho_F(x)$ always exists and is independent on x [**5**, Theorem 1]. So, we denote by $\rho(F)$ the **rotation number of** F in this case. In the general case, we define the **rotation set of** F as

$$\mathrm{Rot}(F) = \{\rho_F(x) : x \in \mathbf{R} \text{ and } \overline{\rho}_F(x) = \underline{\rho}_F(x)\}\ .$$

Let $\mathcal{B}_1$ be the class of old maps which are bounded on $[0,1]$ endowed with the topology of the uniform convergence. That is, $\mathcal{B}_1$ is a metric space with the distance

$$d(F,G) = \sup_{x \in \mathbf{R}} |F(x) - G(x)| = \sup_{x \in [0,1]} |F(x) - G(x)|\ .$$

For $F \in \mathcal{B}_1$ we can define the maps F_l and F_u as follows:

$$F_l(x) = \inf\{F(y) : y \geq x\} \quad \text{ and } \quad F_u(x) = \sup\{F(y) : y \leq x\}\ .$$

These maps are old and non-decreasing (see [**3**, Lemma 3]). Hence there exist $\rho(F_l)$ and $\rho(F_u)$.

It is well known [**1**] that, in the class $\mathcal{L}_1$ of old continuous maps, the following important properties of the rotation set hold:

1. If $p/q \in \mathrm{Int}(\mathrm{Rot}(F))$, then there exists a lifted cycle of F with period q and rotation number p/q.
2. $\mathrm{Rot}(F) = [\rho(F_l), \rho(F_u)]$.
3. In particular, $\mathrm{Rot}(F)$ is a connected set.
4. $\mathrm{Bd}(\mathrm{Rot}(F))$ depends continuously on F.

These properties play a crucial role in the study of the dynamics of the circle maps of degree one (see for instance [**1**]). When dealing with similar problems for non continuous old maps it is helpful to know whether and to which extent the above properties hold. Misiurewicz [**4**] has shown that the situation for the class of old heavy maps is similar to the continuous case. Indeed, a map $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ is called **heavy** if for every $x \in \mathbf{R}$ the finite limits

$$F(x+) = \lim_{y \searrow x} F(y) \quad \text{ and } \quad F(x-) = \lim_{y \nearrow x} F(y)$$

exist and $F(x-) \geq F(x) \geq F(x+)$. From [**4**] it follows easily that 1–4 also hold for old heavy maps (see also [**3**]). In a similar way, Esquembre [**3**] shows that these properties still hold on a class of maps slightly broader than the old heavy ones.

The aim of this paper is to show by means of some examples that, if we consider a larger class of maps, then properties 1–4 are no longer true. Namely, we will consider the class of old maps with a finite number of jump discontinuities in the interval $[0,1]$. In fact there is a family of trivial examples which show that the above properties can fail in that class. To describe these examples we will introduce the notion of a Q**-map** as follows. Let $Q \subset \mathbf{R}$ be such that $Q + \mathbf{Z} = Q$ and $Q \cap [0,1]$ is finite. Any piecewise constant old map g such that $g(\mathbf{R}) = Q$ will be called a Q**-map**. Note that any Q-map G where Q is the union of two lifted cycles of G having different rotation number gives an example of a map in our class such that Property 3 fails. On the other hand, if we take the Q-map F where Q consists of a single badly ordered lifted cycle of rotation number p/q (see [**2**] for a precise definition) of a continuous old map and F coincides with this map on Q, then Property 2 does not hold for F (this follows from the proof of Proposition 2.1 of [**2**]; see also [**1**, Section 3.8]).

Despite of the fact that the above family of maps already show that Properties 2 and 3 can fail in the class of old maps with a finite number of jump discontinuities in the interval $[0,1]$, we are interested in finding maps whose rotation set has nonempty interior and still display these phenomena. These examples and examples showing that Properties 1 and 4 also can fail in that class are given below.

Let a, e_i $(i = 0,1,2)$ and x_i $(i = 0,1,2,3)$ be points in $[0,1)$ such that

$$0 = e_0 < x_0 < a < x_3 < e_1 < x_1 < e_2 < x_2 < 1,$$

and let F be an old map such that:

$$\begin{aligned}
&F(a) = 1,\ F(e_0) = e_1,\ F(e_1) = e_2,\ F(e_2) = 1,\\
&F(x_0) = x_1,\ F(x_1) = x_2,\ F(x_2) = x_3 + 1,\ F(x_3) = x_0 + 1,\\
&F \text{ is affine on } [e_0,x_0],\ [a,x_3],\ [e_1,x_1] \text{ and } [e_2,x_2];\ \text{and}\\
&F \text{ is constant on } [x_0,a),\ [x_3,e_1),\ [x_1,e_2) \text{ and } [x_2,1).
\end{aligned}$$

Proposition 1.1. *We have* $\mathrm{Rot}(F) = [1/3, 1/2]$ *but there is no lifted cycle of* F *with period* 5 *and rotation number* $2/5$. *That is, Property* 1 *does not hold for this map.*

Let b, u_i $(i = 0,1,2)$ and y_i $(i = 0,1,2,3,4)$ be points in $[0,1)$ such that

$$0 = u_0 < y_0 < y_2 < u_1 < y_1 < b < y_3 < u_2 < y_4 < 1,$$

and let G be an old map such that:

$$\begin{aligned}
&G(b) = u_2,\ G(u_0) = u_1,\ G(u_1) = G(u_2) = 1,\\
&G(y_0) = y_1,\ G(y_1) = y_2 + 1,\ G(y_2) = y_3,\ G(y_3) = y_4,\ G(y_4) = y_0 + 1,\\
&G \text{ is affine on } [u_0,y_0],\ [y_0,y_2],\ [u_1,y_1],\ [b,y_3] \text{ and } [u_2,y_4];\ \text{and}\\
&G \text{ is constant on } [y_2,u_1),\ [y_1,b),\ [y_3,u_2) \text{ and } [y_4,1).
\end{aligned}$$

Proposition 1.2. *$\rho(G_l) = 1/3$ but $\mathrm{Rot}(G) = [2/5, 1/2]$. That is, Property 2 does not hold for this map.*

Remark 1.3. Let us also consider the old map H defined by

$$H(t) = \begin{cases} G(t) & \text{for all } t \in [0, u_1) \cup [b, 1) \\ G(t) - 1 & \text{for all } t \in [u_1, b). \end{cases}$$

In a similar way to Proposition 1.2 one can show that $\rho(H_u) = 1/4$ but $\mathrm{Rot}(H) = [0, 1/5]$.

Next we present an example whose rotation set is the union of two disjoint intervals. This map is obtained as follows from maps G and H defined above.

$$\widetilde{G}(t) = \begin{cases} H(2t)/2 & \text{for all } t \in [0, u_2/2) \\ (H(2t)+1)/2 & \text{for all } t \in [u_2/2, 1/2) \\ (G(2t-1)+1)/2 & \text{for all } t \in [\frac{1}{2}, \frac{u_1+1}{2}) \cup [\frac{b+1}{2}, \frac{u_2+1}{2}) \\ (G(2t-1)+2)/2 & \text{for all } t \in [\frac{u_1+1}{2}, \frac{b+1}{2}) \cup [\frac{u_2+1}{2}, 1). \end{cases}$$

Proposition 1.4. *$\mathrm{Rot}(\widetilde{G}) = [0, 1/5] \cup [2/5, 1/2]$. That is, Property 3 does not hold for this map.*

We also use the map G to see how Property 4 can fail. To this end we introduce the family

$$G_\mu = (\min(G, G_l + \mu))_u,$$

where $0 \le \mu \le \mu(G) = \sup_{x \in \mathbf{R}}(G - G_l)(x) = y_2 + 1 - u_2$. Note that all these maps are old and non-decreasing (see the proof of [**1**, Proposition 3.7.17(a)]). Hence $\mathrm{Rot}(G_\mu) = \{\rho(G_\mu)\}$.

Proposition 1.5. *If $0 \le \mu < 1 - u_2$, then $\rho(G_\mu) = 1/3$. If $1 - u_2 \le \mu \le \mu(G)$, then $\rho(G_\mu) = 1/2$. So, Property 4 does not hold for the family G_μ.*

In fact, Proposition 1.5 tells us that, unlike the continuous case, the property (see [**1**, Lemma 3.7.12])

> *The function $F \mapsto \rho(F)$ on the space of all nondecreasing old maps is continuous,*

is no longer true for maps from $\mathcal{B}_1$.

This paper is organized as follows. In Section 2 we state the notation and some preliminary results and, in Section 3, we prove Propositions 1.1, 1.2, 1.4 and 1.5.

2. Notation and Preliminary Results

To study the possible lifted cycles and their rotation numbers in our examples we shall use the notions of f-covering and f-graph (see [**1**, Chapter 1]) slightly modified for old maps with a finite number of jump discontinuities in the interval $[0, 1]$. From now on X will denote $\mathbf{R}$ or a closed interval of the real line.

Let $K, L \subset X$ be two intervals and let f be a map from X to itself. We say that K **f-covers** L if $f|_K$ is continuous and there exists a subinterval J of K such that $f(J) = L$. Note that, if L is closed, then we can take J also closed.

With this definition all the basic results from [**1**] about f-covering are still true and we shall use them freely. We will also need the following simple lemma about an infinite sequence of intervals each of them covering the next one.

Lemma 2.1. *Let $I_0, I_1, \dots, I_i, \dots \subset X$ be a sequence of proper closed intervals and let $f_i \colon I_i \longrightarrow X$ $(i \geq 0)$ be continuous maps such that $f_i(I_i) \supset I_{i+1}$. Then there exist points $x_i \in I_i$ such that $f_i(x_i) = x_{i+1}$ for all $i \geq 0$.*

Proof. For each $n \in \mathbf{N}$ we get a finite sequence of closed intervals $K_i^{(n)} \subset I_i$ $(i = 0, 1, \dots, n)$ such that $K_n^{(n)} = I_n$, $K_i^{(n)} \subset K_i^{(n-1)}$ (with $K_0^{(0)} = I_0$) and $f_i(K_i^{(n)}) = K_{i+1}^{(n)}$ for $i = 0, 1, \dots, n-1$. Indeed, this is obvious for $n = 1$ by [**1**, Lemma 1.2.1], and the inductive step is given by [**1**, Lemma 1.2.6] applied to the intervals $K_0^{(n)}, K_1^{(n)}, \dots, K_n^{(n)}, I_{n+1}$ (and the maps $f_i|_{K_i^{(n)}}$ for $i = 0, 1, \dots, n$).

The sequence of nested closed intervals $\{K_0^{(n)}\}_{n \in \mathbf{N}}$ has nonempty intersection. Let x_0 be any point in this intersection and define inductively $x_{i+1} = f_i(x_i)$ for all $i \geq 0$. Clearly, we have that $x_i \in K_i^{(n)} \subset I_i$ for all $n \in \mathbf{N}$ and $i = 0, 1, \dots, n$.$\square$

Since an old map F is completely determined by its definition on $[0, 1]$, we shall consider Markov graphs whose vertices will be subintervals of $[0, 1]$. More precisely, if P is a finite subset of $[0, 1]$, the closure of each connected component of $[0, 1] \setminus P$ will be called a **P-basic** interval. Since the image of $[0, 1]$ by an old map is not necessarily contained in $[0, 1]$, the P-basic intervals can F-cover P-basic intervals as well as their translations by integers. Let I and J be P-basic intervals and let $k \in \mathbf{Z}$. We will write

$$I \xrightarrow{k} J$$

to denote that I F-covers $J + k$. Each of these arrows will be called a **labeled arrow**.

Remark 2.2. Let F be an old map, let P be a finite subset of $[0, 1]$, let I and J be P-basic intervals and let $k, l \in \mathbf{Z}$. Then the following statements are equivalent.

a) $I \xrightarrow{k} J$,
b) I $(F - k)$-covers J,
c) $I + l$ F-covers $J + k + l$.

The oriented graph whose vertices are the P-basic intervals and whose arrows are all the possible labeled arrows, will be called **the lifted F-graph of P**. The notions of **path** and **loop** are the same as for any graph. If $\alpha = I_0 \xrightarrow{k_0} I_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{n-2}} I_{n-1} \xrightarrow{k_{n-1}} I_n$ is a path in the lifted F-graph of P, then the **jump** of α is defined by $\sum_{i=0}^{n-1} k_i$. We shall also consider **infinite paths** in the lifted F-graph of P, that is, infinite sequences of labeled arrows such that the end of each arrow is the beginning of the next one.

Let F be an old map and let P be a finite subset of $[0,1]$. If Q is a lifted cycle of F of period q and rotation number p/q and $\alpha = I_0 \xrightarrow{k_0} I_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{q-2}} I_{q-1} \xrightarrow{k_{q-1}} I_0$ is a loop with jump p in the lifted F-graph of P, then we say that α and Q are **associated** to each other if there exists $x \in Q$ such that $F^i(x) \in I_i + \sum_{j=0}^{i-1} k_j$ for $i = 0, 1, \dots, q-1$.

The next lemma shows how the F-graph can be used to compute the rotation number of a point x with respect to F.

Lemma 2.3. *Let F be an old map and let P be a finite subset of $[0,1]$. Assume that, for $x \in [0,1]$, there exists an infinite path $I_0 \xrightarrow{k_0} I_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{n-2}} I_{n-1} \xrightarrow{k_{n-1}} I_n \xrightarrow{k_n} \cdots$ in the lifted F-graph of P such that $F^i(x) \in I_i + p_i$ for all $i \geq 0$, where $p_i = \sum_{j=0}^{i-1} k_j$. Then*

$$\liminf_{n\to\infty} \frac{p_n}{n} \leq \underline{\rho}_F(x) \leq \overline{\rho}_F(x) \leq \limsup_{n\to\infty} \frac{p_n}{n}.$$

Proof. Since $0 \leq x \leq 1$ and $p_n \leq F^n(x) \leq p_n + 1$, we have that $p_n - 1 \leq F^n(x) - x \leq p_n + 1$ and, hence, $(p_n - 1)/n \leq (F^n(x) - x)/n \leq (p_n + 1)/n$ for all $n \in \mathbf{N}$. Taking limits, the proof is complete. $\square$

The main technical tool we shall use in our examples is given by the following lemma.

Lemma 2.4. *Let F be an old map and let P be a finite subset of $[0,1]$. Assume that there are two loops α and β in the lifted F-graph of P that can be concatenated. Let p_α and p_β be the jumps and let q_α and q_β be the lengths of α and β, respectively. Then $\langle p_\alpha/q_\alpha, p_\beta/q_\beta\rangle \subset \mathrm{Rot}(F)$.*

Proof. Without loss of generality we may assume that $p_\alpha/q_\alpha \leq p_\beta/q_\beta$. Take $a \in [p_\alpha/q_\alpha, p_\beta/q_\beta]$ and let $\{p_n/q_n\}_{n\in\mathbf{N}} \subset \mathbf{Q} \cap [p_\alpha/q_\alpha, p_\beta/q_\beta]$ be a sequence such that $p_n/q_n \to a$. For each $n \in \mathbf{N}$ we take, for instance, $u_n = q_n p_\beta - p_n q_\beta$ and

$$v_n = \begin{cases} p_n q_\alpha - q_n p_\alpha & \text{if } u_n \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then the loop $\gamma_n = \alpha^{u_n}\beta^{v_n}$ has length $l_n = u_n q_\alpha + v_n q_\beta$ and jump $j_n = u_n p_\alpha + v_n p_\beta$ and we have that $j_n/l_n = p_n/q_n$. We now construct an infinite path $\omega =$

$\gamma_1^{t_1}\gamma_2^{t_2}\cdots\gamma_n^{t_n}\cdots$ in the lifted F-graph of P, where the number of times t_n we repeat each loop γ_n will be determined later.

If such a path is $\omega = I_0 \xrightarrow{k_0} I_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{m-2}} I_{m-1} \xrightarrow{k_{m-1}} I_m \xrightarrow{k_m} \cdots$, then we put $p_i = \sum_{j=0}^{i-1} k_j$ for all $i \geq 0$. Since $p_{i+1} = p_i + k_i$, by Remark 2.2 we have that $I_i + p_i$ F-covers $I_{i+1} + p_{i+1}$ for all $i \geq 0$. Hence, by [**1**, Lemma 1.2.1] and Lemma 2.1, there exist points $x_i \in I_i + p_i$ such that $F(x_i) = x_{i+1}$ for all $i \geq 0$. That is, $x_i = F^i(x_0) \in I_i + p_i$ for all $i \geq 0$.

Now we will show how to choose the numbers t_n in order that $a = \rho(x_0) = \lim_{m\to\infty} p_m/m$. For each $n \in \mathbf{N}$ let $k_i^{(n)}$ $(i = 0, 1, \ldots, l_n - 1)$ be the labels of the arrows of the loop γ_n. Let us set

$$s_n = \max\left\{\left|\sum_{i=0}^{j} k_i^{(n)}\right| : j = 0, 1, \ldots, l_n - 1\right\}$$

and recursively take $t_n \in \mathbf{N}$ such that

$$t_1 l_1 + t_2 l_2 + \cdots + t_n l_n > n \max(s_{n+1}, l_{n+1})\,.$$

Note that for each $m \geq t_1 l_1$ there exist $n \in \mathbf{N}$, $t \in \{0, 1, \ldots, t_{n+1} - 1\}$ and $l \in \{0, 1, \ldots, l_{n+1} - 1\}$ such that

$$m = \sum_{i=1}^{n} t_i l_i + t l_{n+1} + l\,.$$

Then, clearly,

$$p_m = \sum_{i=1}^{n} t_i j_i + t j_{n+1} + \sum_{i=0}^{l-1} k_i^{(n+1)}$$

and, from the choice of the numbers t_n,

$$\frac{l}{\sum_{i=1}^{n} t_i l_i + t l_{n+1}} < \frac{l_{n+1}}{\sum_{i=1}^{n} t_i l_i} < \frac{1}{n} \quad \text{and} \quad \frac{\left|\sum_{i=0}^{l-1} k_i^{(n+1)}\right|}{m} \leq \frac{s_{n+1}}{\sum_{i=1}^{n} t_i l_i} < \frac{1}{n}\,.$$

When $m \to \infty$ it also happens that $n \to \infty$ and

$$\frac{\sum_{i=1}^{n} t_i j_i + t j_{n+1}}{\sum_{i=1}^{n} t_i l_i + t l_{n+1}} \longrightarrow a$$

by the Stolz criterion. Hence

$$\frac{p_m}{m} = \frac{\sum_{i=1}^{n} t_i j_i + t j_{n+1}}{\sum_{i=1}^{n} t_i l_i + t l_{n+1} + l} + \frac{\sum_{i=0}^{l-1} k_i^{(n+1)}}{m} \xrightarrow{m\to\infty} a\,.$$

By Lemma 2.3 we have that $\rho(x_0) = a$ and, therefore, $a \in \mathrm{Rot}(F)$. □

3. Proofs of the Results

Consider the map F defined in page 117. Note that $\{e_0, e_1, e_2\}+\mathbf{Z}$ is a lifted cycle of F of period 3 and rotation number $1/3$, and $\{x_0, x_1, x_2, x_3\}+\mathbf{Z}$ is a lifted cycle of F of period 4 and rotation number $2/4$. If $P_F = \{a, e_0, e_1, e_2, x_0, x_1, x_2, x_3\}$, then $P_F + \mathbf{Z}$ is F-invariant. For this map we have

$$F_l(t) = \begin{cases} F(t) & \text{for all } t \in [0, a) \cup [e_1, 1) \\ e_2 & \text{for all } t \in [a, e_1). \end{cases}$$

Analogously, if z denotes the only point in $[e_2, x_2]$ such that $F(z) = x_0 + 1$, then

$$F_u(t) = \begin{cases} F(t) & \text{for all } t \in [0, e_1) \cup [z, 1) \\ x_0 + 1 & \text{for all } t \in [e_1, z). \end{cases}$$

Lemma 3.1. *$\rho(F_l) = 1/3$ and $\rho(F_u) = 1/2$.*

Proof. Note that e_0 is a periodic mod 1 point of period 3 and rotation number $1/3$ for F_l, and that x_0 is a periodic mod 1 point of period 2 and rotation number $1/2$ for F_u. Then, the lemma follows from the fact that these maps are old and non-decreasing and [**5**, Theorem 1]. □

Let us consider the intervals $I_0 = [e_0, x_0]$, $I_1 = [e_1, x_1]$, $I_2 = [e_2, x_2]$ and $I_3 = [a, x_3]$. The lifted F-graph of P_F contains the loop $\alpha = I_0 \xrightarrow{0} I_1 \xrightarrow{0} I_2 \xrightarrow{1} I_3 \xrightarrow{1} I_0$ and the arrow $I_2 \xrightarrow{1} I_0$, which also gives the loop $\beta = I_0 \xrightarrow{0} I_1 \xrightarrow{0} I_2 \xrightarrow{1} I_0$. We have also the arrow $I_2 \xrightarrow{1} [x_0, a]$. However, $[x_0, a]$ is not the beginning of any arrow, but $F([x_0, a]) = \{x_1, 1\}$. And there are no more arrows in this graph.

Proof of Proposition 1.1. By [**3**, Lemma 3.i] and Lemma 3.1, we have that $\text{Rot}(F) \subset [1/3, 1/2]$. Since the loops α and β can clearly be concatenated, it follows from Lemma 2.4 that $[1/3, 1/2] \subset \text{Rot}(F)$. Hence $\text{Rot}(F) = [1/3, 1/2]$.

Now we claim that for each lifted cycle of F there exists a unique (modulo shifts) loop in the lifted F-graph of P_F associated to it. Let $x \in [0, 1)$ be a point of the lifted cycle. If $x \in P_F$ the claim holds trivially (then the loop will be either α or β or one of their shifts). So let us assume that $x \notin P_F$, and therefore $\text{Orb}_F(x) \cap (P_F + \mathbf{Z}) = \emptyset$. That is, we have that for all $j \geq 0$, $F^j(x) \in \text{Int}(J_j) + p_j$, where J_j is some P_F-basic interval and p_j is some integer. Clearly, for every j these J_j and p_j are uniquely determined. Note that if I and J are P_F-basic intervals and $l, k \in \mathbf{Z}$, then $F(\text{Int}(I) + l) \cap (\text{Int}(J) + l + k) \neq \emptyset$ is possible only if $I \xrightarrow{k} J$. Since $x \in [0, 1)$, $p_0 = 0$. Let q be the period of x. For $j = 1, \ldots, q$, we find arrows $J_{j-1} \xrightarrow{k_{j-1}} J_j$ such that $p_j = \sum_{s=0}^{j-1} k_s$. Since $F^q(x) = x + p$ for some integer p, we have $J_q = J_0$ and $p_q = p$. Thus, the claim is also proved in the case $x \notin P_F$, with the loop

$$J_0 \xrightarrow{k_0} J_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{q-2}} J_{q-1} \xrightarrow{k_{q-1}} J_0.$$

Clearly, the lifted F-graph of P_F contains no loop of length 5, hence there is no lifted cycle of F of period 5. □

Consider now the map G defined in page 117. Note that $\{u_0, u_1\} + \mathbf{Z}$ is a lifted cycle of G of period 2 and rotation number 1/2, and $\{y_0, y_1, y_2, y_3, y_4\} + \mathbf{Z}$ is a lifted cycle of G of period 5 and rotation number 2/5. From now on let P_G be the set $\{b, u_0, u_1, u_2, y_0, y_1, y_2, y_3, y_4\}$. Then $P_G + \mathbf{Z}$ is G-invariant. We have

$$G_l(t) = \begin{cases} G(t) & \text{for all } t \in [0, u_1) \cup [b, 1) \\ u_2 & \text{for all } t \in [u_1, b) \end{cases}$$

and

$$G_u(t) = \begin{cases} G(t) & \text{for all } t \in [0, b) \\ y_2 + 1 & \text{for all } t \in [b, 1). \end{cases}$$

Lemma 3.2. *$\rho(G_l) = 1/3$ and $\rho(G_u) = 1/2$.*

Proof. Clearly u_0 is a periodic mod 1 point of period 3 and rotation number 1/3 for G_l, and it is a periodic mod 1 point of period 2 and rotation number 1/2 for G_u. Then the lemma follows from the fact that these maps are old and non-decreasing and [**5**, Theorem 1]. □

In what follows we will use the following notation. Set $L_0 = [u_0, y_0]$, $L_1 = [u_1, y_1]$, $L_2 = [y_0, y_2]$, $L_3 = [b, y_3]$ and $L_4 = [u_2, y_4]$. The lifted G-graph of P_G contains the loop $\lambda = L_0 \xrightarrow{0} L_1 \xrightarrow{1} L_2 \xrightarrow{0} L_3 \xrightarrow{0} L_4 \xrightarrow{1} L_0$ and the arrow $L_1 \xrightarrow{1} L_0$, which also gives the loop $\eta = L_0 \xrightarrow{0} L_1 \xrightarrow{1} L_0$. We have also the arrow $L_2 \xrightarrow{0} [y_1, b]$. However, $[y_1, b]$ is not the beginning of any arrow, but $G([y_1, b]) = \{y_2 + 1, u_2\}$. And there are no more arrows in this graph.

Proof of Proposition 1.2. In view of Lemma 3.2, it only remains to show that $\mathrm{Rot}(G) = [2/5, 1/2]$. Since the loops λ and η can be concatenated, it follows from Lemma 2.4 that $[2/5, 1/2] \subset \mathrm{Rot}(G)$.

Now we prove that $\mathrm{Rot}(G) \subset [2/5, 1/2]$. Let $x \in [0, 1)$ be such that $\overline{\rho}_G(x) = \underline{\rho}_G(x)$. If $\mathrm{Orb}_G(x)$ meets $P_G + \mathbf{Z}$, then obviously $\rho(x) \in \{2/5, 1/2\}$. So let us assume that $\mathrm{Orb}_G(x) \cap (P_G + \mathbf{Z}) = \emptyset$. That is, we have that for all $j \geq 0$, $G^j(x) \in \mathrm{Int}(J_j) + p_j$, where J_j is some P_G-basic interval and p_j is some integer. In a similar way as above, for every j these J_j and p_j are uniquely determined. If I and J are P_G-basic intervals and $l, k \in \mathbf{Z}$, then $G(\mathrm{Int}(I) + l) \cap (\mathrm{Int}(J) + l + k) \neq \emptyset$ is possible only if $I \xrightarrow{k} J$. Since $x \in [0, 1)$, $p_0 = 0$ and there exist arrows $J_{j-1} \xrightarrow{k_{j-1}} J_j$ such that $p_j = \sum_{s=0}^{j-1} k_s$ for all $j \geq 0$. Each of these arrows belongs to the subgraph of the lifted G-graph of P_G formed by the loops λ and η. Thus there exists $m_0 \leq 4$ such that $G^{m_0}(x) \in \mathrm{Int}(L_0) + p_{m_0}$. Moreover, $\mathrm{Orb}_G(G^{m_0}(x))$ follows a path of the form $\lambda^{n_0}\eta^{n_1}\lambda^{n_2}\eta^{n_3}\ldots$ with $n_i \geq 0$. For each $m \in \mathbf{N}$, denote by γ_m the path of length m described by $G^j(x)$ for $j = 0, 1, \ldots, m$ and let m_λ (respectively m_η)

denote the number of times that the loop λ (respectively η) is contained in γ_m. Clearly, there exist $m_1 \le 4$ and $s_0, s_1 \le 2$ such that $m = m_0 + 5m_\lambda + 2m_\eta + m_1$ and $p_m = s_0 + 2m_\lambda + m_\eta + s_1$. Moreover,

$$\lim_{m\to\infty}\left(\frac{p_m}{m} - \frac{2m_\lambda + m_\eta}{5m_\lambda + 2m_\eta}\right) = 0.$$

Since

$$\frac{2}{5} \le \frac{2m_\lambda + m_\eta}{5m_\lambda + 2m_\eta} \le \frac{1}{2},$$

it follows that $2/5 \le \liminf_{m\to\infty} p_m/m \le \limsup_{m\to\infty} p_m/m \le 1/2$. Hence, by Lemma 2.3, we have that $\rho(x) \in [2/5, 1/2]$. This ends the proof of the proposition. □

Proof of Proposition 1.4. Note that the sets $[0, 1/2) + \mathbf{Z}$ and $[1/2, 1) + \mathbf{Z}$ are $\widetilde{G}$-invariant. Denote by φ a continuous nondecreasing old map such that the interval $[0, 1/2]$ is mapped affinely to $[0, 1]$ by φ, and let $\varphi'(x) = \varphi(x - 1/2)$ for each $x \in \mathbf{R}$. Observe that $H \circ \varphi = \varphi \circ \widetilde{G}$ and $G \circ \varphi' = \varphi' \circ \widetilde{G}$. Consequently, $\mathrm{Rot}(\widetilde{G}) = \mathrm{Rot}(G) \cup \mathrm{Rot}(H)$ because φ and φ' are old maps such that $\varphi(0) = \varphi'(0) = 0$. So, the proposition follows from Proposition 1.2 and Remark 1.3. □

Proof of Proposition 1.5. Since $G(t) \ge 1$ for all $t \in [u_1, b)$, for $0 \le \mu < 1 - u_2$ we have that

$$\min(G, G_l + \mu)(t) = \begin{cases} G(t) & \text{for all } t \in [0, u_1) \cup [b, 1) \\ (G_l + \mu)(t) = u_2 + \mu & \text{for all } t \in [u_1, b). \end{cases}$$

Let $\sigma = \sup\{x \in [b, u_2) : G(x) \le u_2 + \mu\}$. Since G is continuous and nondecreasing on $[b, u_2)$, we have that $G(x) \le u_2 + \mu$ for all $x \in [b, \sigma)$ (if $b < \sigma$) and $G(\sigma) \ge u_2 + \mu$. Thus,

$$G_\mu(t) = \begin{cases} G(t) & \text{for all } t \in [0, u_1) \cup [\sigma, 1) \\ u_2 + \mu & \text{for all } t \in [u_1, \sigma). \end{cases}$$

Now, note that $G_\mu(u_2 + \mu) \in L_0 + 1$, $G_\mu(L_0) = L_1$ and $G_\mu(L_1) = \{u_2 + \mu\}$. Hence $u_2 + \mu$ is a periodic mod 1 point of G_μ of period 3 and rotation number 1/3. Thus, $\rho(G_\mu) = 1/3$.

In the other case, that is $1 - u_2 \le \mu \le \mu(G)$, there exists a point $y \in L_1$ such that $G(y) = u_2 + \mu$. Therefore, since G is nondecreasing on $[u_1, b)$,

$$\min(G, G_l + \mu)(t) = \begin{cases} G(t) & \text{for all } t \in [0, y] \cup [b, 1) \\ (G_l + \mu)(t) = u_2 + \mu & \text{for all } t \in [y, b). \end{cases}$$

Let $\tau = \sup\{x \in [u_2, 1) : G(x) \le u_2 + \mu\}$. Since G is continuous and nondecreasing on $[u_2, 1)$, we have that $G(x) \le u_2 + \mu$ for all $x \in [u_2, \tau)$ (if $u_2 < \tau$) and $G(\tau) \ge u_2 + \mu$. Thus,

$$G_\mu(t) = \begin{cases} G(t) & \text{for all } t \in [0, y] \cup [\tau, 1) \\ u_2 + \mu & \text{for all } t \in [y, \tau). \end{cases}$$

Now it happens that 0 is a periodic mod 1 point of G_μ of period 2 and rotation number 1/2. Hence $\rho(G_\mu) = 1/2$.

Recall that, since G_μ is nondecreasing, $\mathrm{Rot}(G_\mu) = \{\rho(G_\mu)\}$ for each μ. $\square$

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