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THE STABILITY OF THE EQUATION $f(xy) - f(x) - f(y) = 0$ ON GROUPS

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Abstract. Let G be a group and let E be a Banach space. Suppose that a mapping $f\colon G \to E$ satisfies the relation $||f(xy) - f(x) - f(y)|| \le c$ for some $c > 0$ and any $x, y \in G$. The problem of existence of mappings $l\colon G \to E$ such that the following relations hold 1) $l(xy) - l(x) - l(y) = 0$ for any $x, y \in G$; 2) the set $\{\, ||l(x) - f(x)||\, ; \forall x, y \in G\}$ is bounded is considered.

The question 'if we replace a given functional equation by a functional inequality, then under what conditions we can state that the solutions of the inequality are close to the solutions of the equation' was posed for the functional equation $f(xy) = f(x) \cdot f(y)$ for x, y in the group G in [**11**] in connection with the results of the papers [**6–8**].

For a mapping f of the group G into a semigroup of linear transformations of a vector space,in the papers [**1**], [**2**], [**10**] the problem on sufficient conditions of the coincidence of the solution of a functional inequality $\|f(xy) - f(x) \cdot f(y)\| < c$ with a solution of the corresponding functional equation $f(xy) - f(x) \cdot f(y) = 0$ was studied. In the papers [**4**], [**5**], [**9**] it was independently shown that if a continuous mapping f of a compact group G into the algebra of endomorphisms of a Banach space satisfies the relation $\|f(xy) - f(x) \cdot f(y)\| \leqslant \delta$ for all $x, y \in G$ with a sufficiently small $\delta > 0$, then it is ε-close to a continuous representation g of the same group in the same Banach space (i.e., we have $\|f(x) - g(x)\| < \varepsilon$ for all $x \in G$).

In the paper [**6**] D. H. Hayers proved that if $f\colon E \to E'$ is a mapping between Banach spaces such that

$$||f(x + y) - f(x) - f(y)|| \le \varepsilon$$

for some $\varepsilon > 0$ and any x, $y \in E$, then there is unique mapping $l\colon E \to E'$ such that the following relations hold

$$||f(x) - l(x)|| \le \varepsilon, \quad l(x + y) - l(x) - l(y) = 0, \quad \forall x, y \in E.$$

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This property Hyers in [**6**] and Ulam in [**11**] called the stability of the equation

$$f(x+y) - f(x) - f(y) = 0.$$

Suppose that G is an arbitrary group and E is an arbitrary Banach space. Consider the following equation

$$f(xy) - f(x) - f(y) = 0\ , \quad x, y \in G, \quad f(x) \in E. \tag{1}$$

Definition 1. We say that the equation (1) is (G,E)–**stable** if for any $\psi\colon G \to E$ satisfying the functional inequality

$$||\psi(xy) - \psi(x) - \psi(y)|| \leq \varepsilon \tag{2}$$

for some $\varepsilon > 0$ and any x, $y \in G$ there is $l\colon G \to E$ such that for some $\delta > 0$ the following relations hold

$$||\psi(x) - l(x)|| \leq \delta, \quad l(xy) - l(x) - l(y) = 0, \quad \forall x, y \in G. \tag{3}$$

Let T be nonempty set, by $B(T)$ we denote the Banach space of real bounded functions on T.

In this paper we consider the problem of $(G, B(T))$ stability.

Definition 2. By a **quasicharacter** of group G we mean real-valued function f such that the set $\{f(xy) - f(x) - f(y);\ x, y \in G\}$ is bounded.

The set of quasicharacters of group G is real linear space (with respect to the usual operations of addition of functions and their multiplication by numbers) which we denote by $KX(G)$.

Definition 3. We say that a quasicharacter φ is a **pseudocharacter** if for any $x \in G$ and any $n \in Z$ the relation $f(x^n) = nf(x)$ holds.

The subspace of $KX(G)$ consisting of pseudocharacters denote by $PX(G)$. By $X(G)$ denote the subspace of $PX(G)$ consisting of real additive characters of the group G.

We say that a pseudocharacter φ of a group G is **nontrivial** if $\varphi \notin X(G)$.

Proposition 1. *If a group G has nontrivial pseudocharacter, then the equation* (1) *is not (G,E) stable for any Banach space E.*

Proof. Letφ be a nontrivial pseudocharacter of group G and let e be an element of E such that $||e|| = 1$. Define a map $f\colon G \to E$ as follows: for any $x \in G$ we set $f(x) = \varphi(x)e$. Then for some $\varepsilon > 0$ the function f satisfies to (2). Suppose that there is $l\colon G \to E$ such that for some $\delta > 0$ the relation (3) holds. Then for any $x \in G$ and any $n \in Z$ we have $n||f(x) - l(x)|| = ||f(x^n) - l(x^n)|| \leq \delta$. Hence, $f \equiv l$. This contradicts to the relation $\varphi \notin X(G)$.

The Proposition is proved. □

Lemma 1. *Let G be a group and $\varphi \in PX(G)$. Suppose that for any $x, y \in G$ we have $|\varphi(x \cdot y) - \varphi(x) - \varphi(y)| < \varepsilon$; then*

1) *the inequality $|\varphi(x_1 \cdot x_2 \cdot \ldots \cdot x_{n+1}) - \sum_{i=1}^{n+1} \varphi(x_i)| < n \cdot \varepsilon$ holds for any positive integer n and any $x_1, x_2, \ldots, x_n \in G$;*
2) *if φ is a bounded function, then $\varphi \equiv 0$;*
3) *the set $\{ \varphi(a^{-1}b^{-1}ab);\ \forall a, b \in G\}$ is bounded;*
4) *$\varphi(a^{-1}ba) = \varphi(b)$ for any $a, b \in G$.*

Proof. Assertion 1) is easily proved by induction on n. Let us prove 2). If δ is a positive number such that $|\varphi(x)| < \delta$ for any $x \in G$, then for any positive integer n we have $n|\varphi(x)| = |\varphi(x^n)| < \delta$, therefore, $\varphi(x) = 0$. Assertion 3)immediately follows from 1). Let us prove 4).

From assertion 1) it follows that $|\varphi((a^{-1}ba)^n) - \varphi(a^{-1}) - \varphi(b^n) - \varphi(a)| < 2\varepsilon$. Hence, $|\varphi(a^{-1}b^n a) - \varphi(b^n)| < 2 \cdot \varepsilon$, or $n|\varphi(a^{-1}ba) - \varphi(b)| < 2 \cdot \varepsilon$. Since the latter inequality holds for all $n > 1$, we obtain $\varphi(a^{-1}ba) = \varphi(b)$.

The Lemma is proved. □

Proposition 2. *Let G be a group and $\varphi \in KX(G)$. Then:*

1) *for each $x \in G$ there exist a $\lim_{n\to\infty} \frac{1}{n}\varphi(x^n)$;*
2) *a function $\widehat{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n}\varphi(x^n)$ is a pseudocharacter of G and the following decomposition*

$$KX(G) = PX(G) \dot{+} B(G)$$

holds;

3) *the kernel of the mapping $\varphi \to \widehat{\varphi}$ coincide with the space $B(G)$.*

Proof. 1) Let c be a positive number such that

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \le c, \quad \forall x, y \in G. \tag{4}$$

The assertion 1) from the Lemma 1 implies that for any $x \in G$ and any $n \in N$ the following relations hold

$$\begin{gathered} |\varphi(x^n) - n\varphi(x)| \le (n-1) \cdot c, \\ \left| \frac{1}{n}\varphi(x^n) - \varphi(x) \right| \le c. \end{gathered} \tag{5}$$

Let us fix $x \in G$. Denote by $S(x)$ the subsemigroup of G generated by element x. From (5) it follows that there exist a sequence of positive integers $M = \{m_i;\ i \in N\}$, such that there exist a limit

$$a(x) = \lim_{\substack{m\to\infty \\ m\in M}} \frac{1}{m}\varphi(x^m).$$

Let us show that for each $y \in S(x)$ there exist a limit

$$a(y) = \lim_{\substack{m\to\infty\\ m\in M}} \frac{1}{m}\varphi(y^m)$$

and if $y = x^k \quad k \in N$, then $a(y) = ka(x)$. Indeed, from (5) it follows

$$\left|\frac{1}{n}\varphi(x^{k\cdot n}) - \varphi(x^k)\right| \le c. \tag{5-a}$$

Hence,

$$\left|\frac{1}{k\cdot n}\varphi(x^{k\cdot n}) - \frac{1}{k}\varphi(x^k)\right| \le \frac{1}{k}c.$$

The latter estimation implies

$$\frac{1}{n}\lim_{\substack{k\to\infty\\ k\in M}} \frac{1}{k}\varphi(x^{kn}) = a(x),$$

and

$$\lim_{\substack{m\to\infty\\ m\in M}} \frac{1}{m}\varphi(x^{m\cdot n}) = n\cdot a(x), \quad \forall n \in N. \tag{6}$$

From (6) it follows that $\forall y \in S(x)$ there exists

$$\lim_{\substack{m\to\infty\\ m\in M}} \frac{1}{m}\varphi(y^m).$$

Denote this limit by $a(y)$. Let $y = x^n$, from (6) we get $a(y) = na(x)$. For $l \in N$ we have $y^l = x^{nl}$, therefore $a(y^l) = l\cdot n\cdot a(x) = l\cdot a(y)$. And we obtain the following equality

$$a(y^l) = l\cdot a(y), \quad \forall y \in S(x),\ \forall l \in N. \tag{7}$$

From (5-a) it follows that $\forall y \in S(x)$ we have the inequality

$$|a(y) - \varphi(y)| \le c. \tag{8}$$

Now suppose that K is a sequence of positive integers different from M such that there exists a limit

$$b(x) = \lim_{\substack{k\to\infty\\ k\in K}} \frac{1}{k}\varphi(y^k).$$

Then as above we obtain that for each $l \in N$ and each $y \in S(x)$ the following relations hold

$$b(y^l) = l \cdot b(y), \tag{7-a}$$

$$|b(y) - \varphi(y)| \leq c. \tag{8-a}$$

From (8) and (8-a) it follows

$$|a(y) - b(y)| \leq 2c, \quad \forall y \in S(x).$$

Taking into account (7), (7-a), we get that for each $l \in N$ the relation

$$2 \cdot c \geq |a(y^l) - b(y^l)| = l|a(y) - b(y)|$$

holds. This means that $a(y) = b(y)$, $\forall y \in S(x)$. From the latter we obtain that for each $x \in G$ there exist $\lim_{n\to\infty} \frac{1}{n}\varphi(x^n)$.

2) Denote by $\widehat{\varphi}$ which is given by the formula

$$\widehat{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n}\varphi(x^n), \quad x \in G.$$

From (7), (7–a) we obtain

$$\widehat{\varphi}(x^l) = l\widehat{\varphi}(x), \quad \forall x \in G, \quad \forall l \in N. \tag{9}$$

From (5) it follows that

$$|\widehat{\varphi}(x) - \varphi(x)| \leq c, \quad \forall x \in G.$$

Hence, taking into account (4), we get

$$|\widehat{\varphi}(xz) - \widehat{\varphi}(x) - \widehat{\varphi}(z)| \leq 4c, \quad \forall x, z \in G. \tag{10}$$

From (9) it follows that $\widehat{\varphi}(1) = 0$. Further, $|\widehat{\varphi}(xx^{-1}) - \widehat{\varphi}(x) - \widehat{\varphi}(x^{-1})| \leq 4c$. Hence, for each $x \in G$ the following inequality $|\widehat{\varphi}(x) + \widehat{\varphi}(x^{-1})| \leq 4c$ holds. Therefore, $\forall m \in N$ we obtain

$$4c \geq |\widehat{\varphi}(x^m) + \widehat{\varphi}((x^{-1})^m)| = m|\widehat{\varphi}(x^m) + \widehat{\varphi}(x^{-1})|.$$

From the latter we have

$$\widehat{\varphi}(x^{-1}) = -\widehat{\varphi}(x), \quad \forall x \in G. \tag{11}$$

Now let $m \in N$, then from (11) and (9) it follows $\widehat{\varphi}(x^{-m}) = \widehat{\varphi}((x^m)^{-1}) = -\widehat{\varphi}(x^m) = -m\widehat{\varphi}(x)$.

Thus, for any $x \in G$ and any $m \in Z$ the following relation $\widehat{\varphi}(x^m) = m\widehat{\varphi}(x)$ holds. Hence, taking into account (10), we get $\widehat{\varphi} \in PX(G)$.

Now let us verify the following decomposition $KX(G) = PX(G) \dot{+} B(G)$.

It is clear that subspace of $KX(G)$ generated by $PX(G)$ and $B(G)$ is their direct sum. Let us show that the latter is coincide with $KX(G)$.

Suppose that $\varphi \in KX(G)$, $c > 0$ and

$$|\varphi(xy) - \varphi(x) - \varphi(y)| < c, \quad \forall x, y \in G.$$

From the assertion 1) we have $\widehat{\varphi} \in PX(G)$ and $|\varphi(x) - \widehat{\varphi}(x)| \le c$. Hence, $\varphi(x) = \widehat{\varphi}(x) + \delta(x)$, where $\delta \in B(G)$.

3) It is evidently that the mapping $\varphi \to \widehat{\varphi}$ is linear. Now 3) follows from 2) and the relation

$$\widehat{\varphi} = \varphi, \quad \forall \varphi \in PX(G).$$

The Proposition is proved. □

Proposition 3. *Suppose $\varphi \in KX(G)$; then the following conditions are equivalent:*

1) *$\varphi \in PX(G)$;*
2) *φ is a real additive character on each abelian subgroup in G;*
3) *$\varphi(x^n) = n\varphi(x)$, $\forall x \in G$, $\forall n \in N$;*
4) *there exists integer m such that $|m| > 1$ and $\varphi(x^m) = m\varphi(x)$, $\forall x \in G$.*

Proof.

1) $\to$ 2). Let $c > 0$ such that for any x, y from G the following inequality $|\varphi(xy) - \varphi(x) - \varphi(y)| \le c$ holds. Suppose that A is an abelian subgroup of G and $a, b \in A$. Then for each $n \in N$ we get

$$\begin{aligned} c &\ge |\varphi(a^n b^n) - \varphi(a^n) - \varphi(b^n)| \\ &= |\varphi((ab)^n) - \varphi(a^n) - \varphi(b^n)| = n|\varphi(ab) - \varphi(a) - \varphi(b)|. \end{aligned}$$

Hence, $\varphi(ab) = \varphi(a) - \varphi(b)$.

2) $\to$ 3) Obviously.

3) $\to$ 1) Obviously.

1) $\to$ 4) Obviously.

4) $\to$ 3) Let $m > 1$. Let

$$\varphi = \widehat{\varphi} + \delta, \quad \text{where } \widehat{\varphi} \in PX(G), \quad \delta \in B(G). \tag{12}$$

From the condition it follows that for each $k \in N$ and each $x \in G$ the equality $\varphi(x^{m^k}) = m^k \varphi(x)$ holds. Taking into account (12), we obtain $\delta(x^{m^k}) = m^k(\varphi(x) - \widehat{\varphi}(x))$. Therefore, $\delta \equiv 0$ and $\varphi \equiv \widehat{\varphi}$.

Now assume that $m < -1$. Then $m^2 > 1$ and $\varphi(x^{m^2}) = m^2\varphi(x)$, $\forall x \in G$. Hence, $\varphi \equiv \widehat{\varphi}$ and $\varphi \in PX(G)$.

The Proposition is proved. □

Denote by **NPX** the class of groups consistion of all groups G such that $PX(G) = X(G)$. And denote by **PX** the class of groups consisting of all groups G such that $PX(G) \neq X(G)$.

The class **PX** is not empty because any noncyclic free group F belongs to **PX**. In the paper [**3**] a description of the space $PX(F)$ is given.

Suppose that E is finite dimensional Banach space. It is easy to see that the equation (1) is (G, E) stable if and only if $G \in$ **NPX**.

Now consider the problem of (G, E) stability for $E = B(T)$.

Let $\psi \in B(T)$ and let $||\psi||$ denote a usual norm of element ψ in $B(T)$. Let $f\colon G \to E$ and $\varepsilon > 0$ such that the relation (2) holds. The mapping $f\colon G \to E$ may by considered as a function on two arguments (x, t); $x \in G$, $t \in T$. For any $t_0 \in T$ the function $x \to f(x, t_0)$ is an element from $KX(G)$ such that the relation

$$||f(xy, t_0) - f(x, t_0) - f(y, t_0)|| \leq \varepsilon, \quad \forall x, y \in G \tag{13}$$

holds.

Now for every $t_0 \in T$ let us fix some quasicharacter $x \to f(x, t_0)$ such that the relation (13) holds. Then it is clear that the function $x \to f(x, t) \in B(T)$ satisfies to (2).

Now from Proposition 2 we obtain the following fact.

Proposition 4. *The equation* (1) *is* $(G, B(T))$ *stable if and only if* $G \in$ **NPX**.

Lemma 2. *Let G be an arbitrary group and let G' be its commutator subgroup. Suppose that φ is a pseudocharacter of G, such that $\varphi\big|_{G'} \equiv 0$. Then $\varphi \in X(G)$.*

Proof. First consider the case when $G = F$ is a free group of rank two with free generators x, y. Suppose that $\varphi(x) = \alpha$, $\varphi(y) = \beta$ and ξ is real additive character of group F such that $\xi(x) = \alpha$, $\xi(y) = \beta$. Then $\psi = \varphi - \xi \in PX(F)$ and $\psi(x) = \psi(y) = 0$. Each element z of group F is uniquely representable in the form

$$z = x^i y^j \cdot c,$$

where $c \in F'$. Let $\varepsilon > 0$ such that the relation

$$|\varphi(uv) - \varphi(u) - \varphi(v)| \leq \varepsilon, \qquad \forall u, v \in F$$

holds.

Then from Lemma 1 we get $|\psi(z) - \psi(x^i) - \psi(y^j) - \psi(c)| \leq 2\varepsilon$. The latter implies $|\psi(z)| \leq 2\varepsilon$, $\forall z \in F$. Hence, $\psi \equiv 0$.

Thus we have $\varphi \equiv \xi$ on F and $\varphi \in X(F)$.

Now assume that F is a free group of arbitrary rank, $\varphi \in PX(F)$ and $\varphi\big|_{F'} \equiv 0$. Let a, b be an arbitrary elements from F. Then the subgroup generated by elements $\{a,b\}$ is either cyclic or free group of rank two. Hence, $\varphi(ab) = \varphi(a) + \varphi(b)$ and $\varphi \in X(F)$.

Now let G be an arbitrary group and $\tau\colon F \to G$ an epimorphism of some free group F onto G. Suppose that φ is an element from $PX(G)$ such that $\varphi\big|_{G'} \equiv 0$. Consider the mapping $\psi = \varphi \circ \tau \in PX(F)$. Since $\tau\colon F' \to G'$ then $\psi\big|_{F'} \equiv 0$. Therefore $\psi \in X(F)$.

Let us check that $\varphi \in X(G)$. Indeed, suppose that there are $a, b \in G$ such that $\varphi(ab) \neq \varphi(a) + \varphi(b)$. Let u, v be elements from F such that $u^\tau = a$, $v^\tau = b$. Then $\psi(u) = \varphi(a)$, $\psi(v) = \varphi(b)$, $\psi(uv) = \varphi(ab)$.

Hence, $\psi(uv) \neq \psi(u) + \psi(v)$ and we come to contradiction to the relation $\psi \in X(F)$. Thus, for any u, v from G we have $\varphi(uv) = \varphi(u) + \varphi(v)$.

The Lemma 2 is proved. □

Suppose that G is an arbitrary group and a, b, α, $\beta \in G$. It is easily to verify that the following relations hold:

$$[a,b]^\alpha = [a\alpha, b] \cdot [\alpha, b]^{-1}, \quad [a,b]^\beta = [a,\beta]^{-1} \cdot [a, b\beta], \tag{14}$$

Theorem 1. *Let G be an arbitrary group and let $G^{(k)}$ be k-th member of its derivative series. If $\varphi \in PX(G)$ and $\varphi\big|_{G^{(k)}}$ is a character of $G^{(k)}$, then $\varphi \in X(G)$.*

Proof. First consider the case when $G = F$ is a free group. Suppose that $\varphi\big|_{F'} \in X(F')$, then from (14) we obtain the following equalities:

$$\varphi([a\alpha, b]) = \varphi([a,b]) + \varphi([\alpha,b])\ , \quad \varphi([a, b\beta]) = \varphi([a,b]) + \varphi([a,\beta]).$$

Therefore,

$$\varphi([a^n, b^m]) = n \cdot m\varphi([a,b]) \quad \forall n, m \in N.$$

Now from Lemma 1 assertions 2) and 3) it follows $\varphi\big|_{F'} \equiv 0$.

Therefore, by Lemma 2 we get $\varphi \in X(F)$.

Suppose that for numbers $1, \dots, k$ the theorem in the case $G = F$ is true. Let us prove for $k+1$. Let $\varphi\big|_{F^{(k+1)}}$ be a character. Let us check that in this case the function $\varphi\big|_{F^{(k)}}$ is a character too. Suppose that there are a, $b \in F^{(k)}$ such that $\varphi(ab) \neq \varphi(a) + \varphi(b)$. Then subgroup H generated by elements a, b is free group of rank two. Since $H' \subset F^{(k+1)}$, then by the induction hypothesis we obtain that $\varphi\big|_{H'}$ is a character. From Lemma 2 it follows that the function $\varphi\big|_{H}$ is character. Thus, $\varphi\big|_{F^{(k)}}$ is a character. By the induction hypothesis the function φ is a character of group F.

Now let G be an arbitrary group and let τ be an epimorphism of free group F onto G. Then for each $k \in N$ we have $\tau(F^{(k)}) = G^{(k)}$. Let $\varphi \in PX(G)$ and

$\varphi\big|_{G^{(k)}} \in X(G^{(k)})$. Then the function $\psi = \varphi \circ \tau$ is a pseudocharacter of group F such that $\psi\big|_{F^{(k)}} \in X(F^{(k)})$. Hence, $\psi \in X(F)$. Thus, $\varphi \in X(G)$.

The Theorem is proved. □

It is evidently that any periodical group belongs to **NPX**. Theorem implies that if a group G is solvable, then G is an **NPX**-group.

Corollary 1. *Let G be an arbitrary group. Suppose that H is an invariant subgroup of G, G/H is a solvable group and $PX(H) = X(H)$. Then $PX(G) = X(G)$,i.e., the class* **NPX** *relatively solvable extensions is closed.*

Proof. There exists $k \in N$ such that $G^{(k)} \subset H$. Hence, $\forall \varphi \in PX(G)$ the relation $\varphi\big|_{G^{(k)}} \in X(G^{(k)})$ holds. Now from Theorem 1 we obtain $\varphi \in X(G)$. □

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