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CHARACTERIZATION PROPERTIES FOR STARLIKENESS AND CONVEXITY OF SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A CLASS OF FRACTIONAL DERIVATIVE OPERATORS

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Abstract. This paper investigates the characterization properties exhibited by a class of fractional derivative operators of certain analytic functions in the open unit disk to be starlike or convex. Further characterization theorems associated with the Hadamard product (or convolution) are also studied.

1. Introduction and Preliminaries

Let E denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z) \in E$ is said to be starlike of order ρ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \tag{1.2}$$

form some ρ $(0 \le \rho < 1)$ and for all $z \in U$. Further, a function $f(z) \in E$ is said to be convex of order ρ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \tag{1.3}$$

for some ρ $(0 \le \rho < 1)$, and for all $z \in U$. We denote by $S^*(\rho)$ and $K(\rho)$ the subclasses of E consisting of starlike and convex functions of order ρ $(0 \le \rho < 1)$, respectively, in the unit disk U.

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It follows then that $f(z) \in K(\rho)$ if and only if

$$zf'(z) \in S^*(\rho). \tag{1.4}$$

For general reference to aforementioned definitions and statements, we refer to [**1**]. In the present paper we undertake to establish characterization properties satisfied by a class of fractional derivative operators (defined below by (2.1)) of certain analytic functions in the open unit disk for starlikeness and convexity. Characterization properties associated with the Hadamard product (or convolution) are also investigated, and some consequences of our main results briefly indicated.

2. A Class of Fractional Derivative Operators

Following Raina and Nahar [**3**] (see also [**5**]), we define the fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ of a function $f(z)$ (involving the familiar Gaussian hypergeometric function ${}_2F_1$) as follows:

$$\begin{aligned} J_{0,z}^{\lambda,\mu,\eta} f(z) &= \frac{d^m}{dz^m}\Big\{\frac{z^{\lambda-\mu}}{\Gamma(m-\lambda)} \\ &\times \int_0^z (z-t)^{m-\lambda-1} {}_2F_1\big(\mu-\lambda, m-\eta; m-\lambda; 1-\frac{t}{z}\big) f(t)\,dt\Big\}, \\ &(m-1 \le \lambda < m; m \in \mathbb{N} \text{ and } \mu, \eta \in \mathbb{R}) \end{aligned} \tag{2.1}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^r), \quad z \to 0 \tag{2.2}$$

for

$$r > \max\{o, \mu-\eta\} - 1. \tag{2.3}$$

It being understood that $(z-t)^{m-\lambda-1}$ $(m \in \mathbb{N})$ denotes the principal value for $0 \le \arg(z-t) < 2\pi$, and is well defined in the unit disk U. The operator $J_{0,z}^{\lambda,\mu,\eta}$ includes the well-known Riemann-Liouville and Erdélyi-Kober operators of fractional calculus (see [**4**] and [**7**]). Indeed, we have

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = {}_oD_z^\lambda f(z) \quad (\lambda \ge 0), \tag{2.4}$$

and

$$J_{0,z}^{\lambda,m,\eta} f(z) = \frac{d^m}{dz^m}\big(E_{o,z}^{m-\lambda,\eta-m}\big), \ (m-1 \le \lambda < m; m \in \mathbb{N}). \tag{2.5}$$

As usual, $(\lambda)_n$ stands for the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n=0 \\ \lambda(\lambda+1)\dots(\lambda+n-1), & n \in \mathbb{N} \end{cases} \tag{2.6}$$
$$(\lambda \neq 0, -1, -2, \dots).$$

It is convenient to introduce here the fractional operator $P_{o,z}^{\lambda,\mu,\eta}$ which is defined in terms of $J_{0,z}^{\lambda,\mu,\eta}$ as follows:

$$P_{o,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2-\mu+\eta)} z^{\mu} J_{o,z}^{\lambda,\mu,\eta} f(z), \tag{2.7}$$
$$(\lambda \geq 0; \mu < 2; \eta > \max\{\lambda, \mu\} - 2)$$

Application of definition (2.1) yields the following known formula giving the image of the power function z^k under the fractional derivative operator $P_{o,z}^{\lambda,\mu,\eta}$:

Lemma 1([**3**]). *Let* $\lambda \geq 0$*;* $k > \max\{0, \mu-\eta\} - 1$*, then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu}. \tag{2.8}$$

3. Characterization Properties

Before stating and proving our main results, we require the following lemmas:

Lemma 2([**8**]). *Let the function* $f(z)$ *defined by* (1.1) *satisfy*

$$\sum_{n=2}^{\infty} \frac{\eta-\rho}{1-\rho} |a_n| \leq 1 \qquad (0 \leq \rho < 1), \tag{3.1}$$

then $f(z) \in S^*(\rho)$*. The equality in* (3.1) *is attained for the function* $f(z)$ *given by*

$$f(z) = z + \frac{1-\rho}{n-\rho} z^n \quad (n \geq 2). \tag{3.2}$$

Lemma 3([**8**]). *Let the function* $f(z)$ *defined by* (1.1) *satisfy*

$$\sum_{n=2}^{\infty} \frac{n(n-\rho)}{1-\rho} |a_n| \leq 1 \qquad (0 \leq \rho < 1), \tag{3.3}$$

then $f(z) \in K(\rho)$*. The equality in* (3.3) *is attained for the function* $f(z)$ *given by*

$$f(z) = z + \frac{1-\rho}{n(n-\rho)} z^n \quad (n \geq 2). \tag{3.4}$$

Theorem 1. *Let* $\lambda,\ \mu,\ \eta \in \mathbb{R}$ *such that*

$$\lambda \geq 0,\ \ \mu < 2,\ \ \max\{\lambda, \mu\} - 2 < \eta \leq \frac{\lambda(\mu - 3)}{\mu}. \tag{3.5}$$

Also, let the function $f(z)$ *defined by* (1.1) *satisfy*

$$\sum_{n=2}^{\infty} \frac{n-\rho}{1-\rho}|a_n| \leq \frac{(2-\mu)(2+\eta-\lambda)}{2(2+\eta-\mu)}, \tag{3.6}$$

for $0 \leq \rho < 1$. *Then*

$$P_{o,z}^{\lambda,\mu,\eta} f(z) \in S^*(\rho).$$

Proof. Applying Lemma 1, we have from (1.1) and (2.7):

$$P_{o,z}^{\lambda,\mu,\eta} f(z) = z + \sum_{n=2}^{\infty} \delta(n) a_n z^n, \tag{3.7}$$

where

$$\delta(n) = \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}} \qquad (n \geq 2). \tag{3.8}$$

We observe that the function $\delta(n)$ defined by (3.8) satisfies the inequality $\delta(n+1) \leq \delta(n)$, $\forall n \geq 2$, provided that $\eta \leq \frac{\lambda(\mu-3)}{\mu}$, thereby, showing that $\delta(n)$ is non-increasing. Thus, under the conditions stated in (3.5), we have

$$0 < \delta(n) \leq \delta(2) = \frac{2(2+\eta-\mu)}{(2-\mu)(2+\eta-\lambda)}. \tag{3.9}$$

Therefore, (3.6) and (3.9) yield

$$\sum_{n=2}^{\infty} \frac{n-\rho}{1-\rho}\delta(n)|a_n| \leq \delta(2) \sum_{n=2}^{\infty} \frac{n-\rho}{1-\rho}|a_n| \leq 1. \tag{3.10}$$

Hence, by Lemma 2, we conclude that

$$P_{o,z}^{\lambda,\mu,\eta} f(z) \in S^*(\rho),$$

and the proof is complete. □

Remark 1. The equality in (3.6) is attained for the function $f(z)$ defined by

$$f(z) = z + \frac{(1-\rho)(2-\mu)(2+\eta-\lambda)}{2(2-\rho)(2+\eta-\mu)} z^2. \tag{3.11}$$

In an analogous manner, we can prove with the help of Lemma 3 the following result which characterizes the class $K(\rho)$.

Theorem 2. *Under the conditions stated in* (3.5)*, let the function* $f(z)$ *defined by* (1.1) *satisfy*

$$\sum_{n=2}^{\infty} \frac{n(n-\rho)}{1-\rho}|a_n| \leq \frac{(2-\mu)(2+\eta-\lambda)}{2(2+\eta-\mu)}, \tag{3.12}$$

for $0 \leq \rho < 1$*. Then* $P_{o,z}^{\lambda,\mu,\eta} f(z) \in K(\rho)$*.*

Remark 2. The equality in (3.12) is attained for the function $f(z)$ defined by

$$f(z) = z + \frac{(1-\rho)(2-\mu)(2+\eta-\lambda)}{4(2-\rho)(2+\eta-\mu)} z^2. \tag{3.13}$$

4. Further Characterization Properties

Let $f_i(z) \in E$ $(i = 1, 2)$ be given by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n. \tag{4.1}$$

Then, the Hadamard product (or convolution) $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{4.2}$$

We need the following result due to Ruscheweyh and Sheil-Small [**6**]:

Lemma 4. *Let* $g(z)$*,* $h(z)$ *be analytic in* U *and satisfy* $g(0) = h(0) = 0$*,* $g'(0) \neq 0$*,* $h'(0) \neq 0$*. Also, let*

$$g(z) * \left\{ \frac{1+abz}{1-bz} h(z) \right\} \neq 0 \quad (z \in U - \{0\}), \tag{4.3}$$

for a *and* b *on the unit circle. Then, a function* $\phi(z)$ *analytic in* U *such that* $\mathrm{Re}\{\phi(z)\} > 0$*, satisfies the inequality:*

$$\mathrm{Re}\left\{ \frac{(g * \phi h)(z)}{(g * h)(z)} \right\} > 0 \quad (z \in U). \tag{4.4}$$

Theorem 3. *Let the conditions stated in* (3.5) *hold true, and let the function* $f(z)$ *defined by* (1.1) *belong to the class* $S^*(\rho)$, *and satisfy:*

$$F(z) * \left\{ \frac{1+abz}{1-bz} f(z) \right\} \neq 0 \quad (z \in U - \{0\}), \tag{4.5}$$

for a *and* b *on the unit circle, where*

$$F(z) = z + \sum_{n=2}^{\infty} \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}} z^n. \tag{4.6}$$

Then

$$P_{o,z}^{\lambda,\mu,\eta} f(z) \in S^*(\rho).$$

Proof. Using (3.7) and (4.6), we have

$$\begin{aligned} P_{o,z}^{\lambda,\mu,\eta} f(z) &= z + \sum_{n=2}^{\infty} \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}} a_n z^n \\ &= (F * f)(z). \end{aligned} \tag{4.7}$$

By setting $g(z) = F(z)$, $h(z) = f(z)$, $\phi(z) = \frac{zf'(z)}{f(z)} - \rho$, in Lemma 4, we find with the help of (4.7) that

$$\begin{aligned} &\operatorname{Re}\left\{ \frac{(g * \phi h)(z)}{(g * h)(z)} \right\} > 0 \\ &\Rightarrow \operatorname{Re}\left\{ \frac{(F * zf')(z)}{(F * f)(z)} \right\} - \rho > 0 \\ &\Rightarrow \operatorname{Re}\left\{ \frac{z(F * f)'(z)}{(F * f)(z)} \right\} - \rho > 0 \\ &\Rightarrow \operatorname{Re}\left\{ \frac{z(P_{o,z}^{\lambda,\mu,\eta} f(z))'}{P_{o,z}^{\lambda,\mu,\eta} f(z)} \right\} - \rho > 0 \\ &\Rightarrow P_{o,z}^{\lambda,\mu,\eta} f(z) \in S^*(\rho), \end{aligned}$$

and the proof is complete.

Theorem 4. *Let the conditions stated in* (3.5) *hold true and, let the function* $f(z)$ *defined by* (1.1) *belong to the class* $K(\rho)$, *and satisfy:*

$$F(z) * \left\{ \frac{1+abz}{1-bz} zf'(z) \right\} \neq (z \in U - \{0\}), \tag{4.8}$$

for a and b on the unit circle, where $F(z)$ is given by (4.6)*. Then*

$$P_{o,z}^{\lambda,\mu,\eta} f(z) \in K(\rho).$$

Proof. From (1.4) and Theorem 3, we conclude that

$$\begin{aligned} & f(z) \in K(\rho) \\ & \quad \Leftrightarrow z f'(z) \in S^*(\rho) \\ & \quad \Rightarrow P_z^{\lambda,\mu,\eta}(z f'(z)) \in S^*(\rho) \\ & \quad \Leftrightarrow (F * z f')(z) \in S^*(\rho) \\ & \quad \Leftrightarrow z(F * f)'(z) \in S^*(\rho) \\ & \quad \Leftrightarrow (F * f)(z) \in K(\rho) \\ & \quad \Leftrightarrow P_{o,z}^{\lambda,\mu,\eta} f(z) \in K(\rho) \quad \text{(in view of (4.7))}, \end{aligned}$$

which proves Theorem 4. □

To establish the next characterization property, we need another result due to Ruscheweyh and Sheil-Small [**6**]:

Lemma 5. *Let $g(z)$ be convex and let $h(z)$ be starlike in U. Then, for each function $\phi(z)$ analytic in U such that* $\mathrm{Re}\{\phi(z)\} > 0$*, satisfies the inequality:*

$$\mathrm{Re}\left\{ \frac{(g * \phi h)(z)}{(g * h)(z)} \right\} > 0 \quad (z \in U). \tag{4.9}$$

Theorem 5. *Let the conditions stated in* (3.5) *hold true, and let $f(z) \in S^*(\rho)$ and $F(z) \in K(\rho)$. Then*

$$P_{o,z}^{\lambda,\mu,\eta} f(z) \in S^*(\rho),$$

where $F(z)$ is given by (4.6)*.*

Proof. Theorem 5 is based upon Lemma 5 above, and its proof is analogous to Theorem 3. □

Based upon Theorem 5, we can easily prove the following:

Theorem 6. *Let the conditions stated in* (3.5) *hold true, and let $f(z) \in K(\rho)$ and $F(z) \in K(\rho)$. Then*

$$P_{o,z}^{\lambda,\mu,\eta} f(z) \in K(\rho),$$

where $F(z)$ is given by (4.6)*.*

5. Concluding Remarks

In view of the relationship (2.4) and (2.5), several characterization properties can be derived, exhibiting the starlikeness and convexity properties of Riemann-Liouiville fractional derivative operator (and Erdélyi-Kober fractional derivative operator) of analytic functions belonging to the subclasses $S^*(\rho)$ and $K(\rho)$ of E. We note that for $\mu = \lambda$, (2.7) by virtue of (2.4) gives

$$P_{o,z}^{\lambda,\lambda,\eta} = B_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} {}_0D_z^{\lambda} f(z), \tag{5.1}$$

and taking into account the above specialization, we are lead to the results obtained recently by Owa and Shen [**2**].

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