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SOME CONSTRUCTIONS RELATED TO REES MATRIX RINGS

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Dedicated to the memory of L. M. Gluskin

Abstract. Simple rings with a one-sided minimal ideal may be represented as Rees matrix rings, and conversely. The latter are defined as $I \times \Lambda$ - matrices over a division ring with only a finite number of nonzero entries with certain addition and multiplication.

For Rees matrix rings we construct here their isomorphisms, their translational hulls and isomorphisms of the translational hulls, all this in terms of certain type of matrices of arbitrary size over division rings. We also study r-maximal Rees matrix rings. This theory runs parallel to that of Rees matrix semigroups.

1. Introduction and summary

Simple rings with a one-sided minimal ideal admit a faithful representation as Rees matrix rings, and conversely. The latter are constructed by means of two nonempty sets I and Λ, a division ring Δ and a function $P : \Lambda \times I \to \Delta$ satisfying certain conditions. The elements are $I \times \Lambda$ - matrices over Δ with a finite number of nonzero entries with multiplication using the function P, called a sandwich matrix since the product is of the form $X * Y = XPY$, and addition is by entries. This ring is denoted by $M(I, \Delta, \Lambda; P)$; a precise definition will be given in Section 3.

The construction of Rees matrix rings is closely similar to the construction of Rees matrix semigroups where a group with a zero adjoined stands instead of a division ring and the only condition on the sandwich matrix P is that it contains a nonzero entry in each row and each column. The elements are $I \times \Lambda$ - matrices with at most one nonzero entry and their multiplication is the same as in the case of rings. In both of these cases, we multiply matrices of arbitrary size by ignoring the sums of an arbitrary number of zeros. Completely 0-simple semigroups admit a faithful representation as Rees matrix semigroups and conversely. The former may be characterized as simple semigroups having a 0-minimal left and a 0-minimal right ideal, which is quite close to the definition of rings mentioned at the outset.

For Rees matrix semigroups, we studied in [**12**] the following subjects: isomorphisms, the translational hull and isomorphisms of the translational hulls, all this

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in matrix notation, as well as r-maximal completely 0-simple semigroups. The purpose of this work is to study these concepts for Rees matrix rings. For a general treatment of these subjects, we refer the reader to the monograph [**11**], where the interplay of semigroups and rings discussed above is of central interest and importance. In particular, the translational hull of a Rees matrix ring is its maximal essential extension, or viewed differently, the translational hull is a maximal primitive ring with the socle which is a simple ring with a one-sided minimal ideal.

Khalezov [**8**] constructed all automorphisms of the multiplicative semigroup of the ring of $n \times n$ - matrices over a division ring; except for the case $n = 1$, they are ring automorphisms. This was generalized by Gluskin [**2**] to matrices over Euclidean rings. In [**3**] Gluskin considered isomorphisms of certain other subrings of the full matrix ring and in [**4**] to linear transformations on a vector space of arbitrary dimension. This was later systematized in [**11**].

In Section 2 we summarize briefly the needed background on semigroups mainly concerning Rees matrix semigroups and their isomorphisms. Section 3 consists of a short compendium of notation and an outline of the needed background on Rees matrix rings. For these rings, we construct isomorphisms in Section 4, the translational hull in Section 5 and isomorphisms of the latter in Section 6. We conclude by a study of r-maximal Rees matrix rings in Section 7.

2. Background on semigroups

First let I and Λ be nonempty sets, G be a group, G^o be the group G with a zero adjoined, $P : \Lambda \times I \to G^o$ be a function, considered as a matrix $P = (p_{\lambda i})$, such that every row and every column contains a nonzero entry. On the set $S = (I \times G \times \Lambda) \cup \{0\}$ define a multiplication by

$$(i, g, \lambda)\ (j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda i} \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then the Rees factor semigroup S/J modulo the ideal $J = I \times 0 \times \Lambda$ denoted by $\mathcal{M}^o(I, \Delta, \Lambda; P)$ is called a **Rees matrix semigroup**.

A semigroup is completely 0-simple if it has no proper ideals and a 0-minimal left and a 0-minimal right ideal. Then we have the following version of the celebrated Rees theorem.

Theorem 2.1. *A semigroup S is completely 0-simple if and only if S is isomorphic to a Rees matrix semigroup.*

For a proof, consult [**1**, Theorem 3.5]. The above version is adjusted for a comparison with rings. The element (i, g, λ) may be thought of as a $I \times \Lambda$ - matrix over G^o with the sole nonzero entry in the (i, λ) - position equal to g. Writing 0 for the zero $I \times \Lambda$ - matrix, we arrive at the following multiplication

$$X * Y = XPY \qquad (X, Y \in S)$$

where the product on the right is that of usual matrices where we ignore arbitrary sums of zeros; P is called a **sandwich matrix**.

For any isomorphism $\omega : G \to G'$ of groups, we set $0\omega = 0$ and use the notation ω also for the resulting isomorphism $G^o \to G'^o$. For any matrix $(x_{i\lambda})$ over G^o and $c \in G^o$, we write

$$(x_{i\lambda})\omega = (x_{i\lambda}\omega), \quad c(x_{i\lambda}) = (cx_{i\lambda}), \quad (x_{i\lambda})c = (x_{i\lambda}c).$$

A $X \times Y$ - matrix A over G^o is **permutational** if every row and every column of A contains exactly one nonzero entry. Let

$$S = \mathcal{M}^o(I, \Delta, \Lambda; P), \qquad S' = \mathcal{M}^o(I', \Delta', \Lambda'; P')$$

be Rees matrix semigroups in their matrix form, let
U be a permutational $I \times I'$ - matrix over G'^o,
$\omega : G \to G'$ be an isomorphism,
V be a permutational $\Lambda \times \Lambda'$ - matrix over G'^o
satisfying $(P\omega)U = VP'$, and define a mapping $\theta = \theta(U, \omega, V)$ by

$$\theta : X \to U^{-1}(X\omega)V \qquad (X \in S).$$

(Note that for a permutational U, the matrix U^{-1} is well defined.)

Theorem 2.2. *The mapping $\theta = \theta(U, \omega, V)$ is an isomorphism of S onto S'. Conversely, every isomorphism of S onto S' can be so constructed.*

For a proof, see [**1**, Corollary 3.12], and for a full discussion of the above results, consult [**1**] and [**12**].

3. Background on rings

We summarize here very briefly the needed terminology and notation concerning rings; for the rest, we refer the reader to the monograph [**11**]. We first recall the most frequently used symbolism:
Δ - division ring,
V - left vector space over Δ,
U - right vector spaces over Δ,
(U, V) - pair of dual vector spaces over Δ,
$\mathcal{L}(V)$ - ring of linear transformations on V (written on the right),
$\mathcal{F}(V)$ - ring of linear transformations on V of finite rank,
$\mathcal{L}_U(V)$ - ring of linear transformations on V with an adjoint in U,
$\mathcal{F}_U(V) = \mathcal{L}_U(V) \cap \mathcal{F}(V)$
and for linear transformations on U, written on the left, we use the same notation with a prime affixed,
$\mathfrak{M}\Delta^\star$ - the multiplicative group of $\Delta^\star = \Delta \backslash \{0\}$.
For a ring R, R^+ is its additive group, $\mathfrak{M}R$ its multiplicative semigroup and $\mathcal{A}(R)$ its group of automorphisms. If R has an identity and a is contained in its group of units, define a mapping ε_a by

$$\varepsilon_a : x \to a^{-1}xa \qquad (x \in R)$$

and let

$$\mathcal{I}(R) = \{\varepsilon_a \,|\, a \in R \text{ is invertible}\}.$$

The following is an excerpt from [**11**].

Definition 3.1. Let I and Λ be nonempty sets and P be a $\Lambda \times I$ - matrix over a division ring Δ. Then P is **left row independent** if for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ and $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$, the equation $\sum\limits_{j=1}^{n} \delta_j p_{\lambda_j i} = 0$ for all $i \in I$ implies that $\delta_1 = \delta_2 = \cdots = \delta_n = 0$. Also P is **right column independent** if for any $i_1, i_2, \ldots, i_n \in I$ and $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$, the equation $\sum\limits_{j=1}^{n} p_{\lambda i_j} \delta_j = 0$ for all $\lambda \in \Lambda$ implies that $\delta_1 = \delta_2 = \cdots = \delta_n = 0$.

We denote by ι_I the $I \times I$ identity matrix and by $|I|$ the cardinality of I.
We are now able to construct the main device.

Notation 3.2. Let I and Λ be nonempty sets and P be a left row independent and right column independent $\Lambda \times I$ - matrix over a division ring Δ, say $P = (p_{\lambda i})$. Let R be the set of all $I \times \Lambda$ - matrices over Δ with only a finite number of nonzero entries with addition

$$(a_{i\lambda}) + (b_{i\lambda}) = (a_{i\lambda} + b_{i\lambda})$$

and multiplication

$$(a_{i\lambda}) * (b_{i\lambda}) = (a_{i\lambda}) P (b_{i\lambda}),$$

where the last product is the usual product of matrices with arbitrary sums of zeros equal to zero.

The above definition of multiplication makes sense since $(a_{i\lambda})P$ is a product of a $I \times \Lambda$ - matrix by a $\Lambda \times I$ - matrix with a resulting $I \times I$ - matrix with only a finite number of nonzero rows and $(a_{i\lambda})P(b_{i\lambda})$ is a product of a $I \times I$ - matrix by a $I \times \Lambda$ - matrix resulting in a matrix with only a finite number of nonzero entries. It follows that R is closed under both operations and routine calculation shows that it forms a ring.

Definition 3.3. The ring R in Notation 3.2. is a **Rees matrix ring**, denoted by $\mathcal{M}(I, \Delta, \Lambda; P)$.

Notice that this construction, due to E. Hotzel, is very close to that of a Rees matrix semigroup in Section 3. In fact, the following construction, given by Gluskin for the finite dimensional case in [**3**] and the general case in [**4**], brings us back to Rees matrix semigroups.

Let (U, V) be a pair dual vector spaces over a division ring Δ and denote by (v, u) the values of the associated bilinear form. In each 1-dimensional subspace of U fix a nonzero vector $u_i, i \in I_o$, and in each 1-dimensional subspace of V fix a nonzero vector $v_\lambda, \lambda \in \Lambda_o$. For any $\lambda \in \Lambda_o$ and $i \in I_o$, let $p_{\lambda i} = (v_\lambda, u_i)$ and $P_o = (p_{\lambda i})$. This matrix has a nonzero entry in every row and every column, so we may define a Rees matrix semigroup

$$M = \mathcal{M}^o(I_o, \mathfrak{M}\Delta^\star, \Lambda_o; P_o).$$

Its elements can be interpreted as linear transformations by defining

$$v(i, \gamma, \lambda) = (v, u_i)\gamma v_\lambda \qquad (v \in V)$$

and the zero of M be the zero linear transformation. The resulting mapping $M \to \mathcal{L}(V)$ is an isomorphism of M onto the semigroup of all linear transformations of rank ≤ 1 with an adjoint in U, denoted by $\mathcal{F}_{2,U}(V)$. We can obtain a dual result by defining

$$(i, \gamma, \lambda)u = u_i\gamma(v_\lambda, u) \qquad (u \in U).$$

The additive closure of $\mathcal{F}_{2,U}(V)$ equals $\mathcal{F}_U(V)$. Hence every element of $\mathcal{F}_U(V)$ can be written as

$$b = \sum_{k=1}^{n} (i_k, \gamma_k, \lambda_k). \tag{1}$$

This representation is not unique but nonzero elements of $\mathcal{F}_{2,U}(V)$ can be uniquely written as (i, γ, λ) with $\gamma \in \Delta^\star$.

As in the case of general Rees matrix semigroups, we may interpret the element (i, γ, λ) as the $I_o \times \Lambda_o$ - matrix over Δ with the sole nonzero entry equal to γ in the (i, λ) position to be denoted by $[i, \gamma, \lambda]$. The zero of M is interpreted as the zero $I \times \Lambda$ - matrix.

We now fix a basis $Z = \{z_\lambda\}_{\lambda \in \Lambda}$ of V and a basis $W = \{w_i\}_{i \in I}$ of U. Given (1), we may write

$$u_{i_k} = \sum_j w_j \tau_{j i_k}, \qquad v_{\lambda_k} = \sum_j \sigma_{\lambda_k \mu} z_\mu \qquad (k = 1, 2, \ldots, n). \tag{2}$$

With this notation, we introduce a mapping η by

$$\eta : b \to \left(\sum_{k=1}^{n} \tau_{j i_k} \gamma_k \sigma_{\lambda_k \mu} \right) \qquad (b \in \mathcal{F}_U(V)) \tag{3}$$

where on the right we have a $I \times \Lambda$ - matrix with entries $b_{j\mu} = \sum_{k=1}^{n} \tau_{j i_k} \gamma_k \sigma_{\lambda_k \mu}$, and the 0-transformation maps onto the 0-matrix. For every $\lambda \in \Lambda$ and $i \in I$, we let $p_{\lambda i} = (v_\lambda, u_i)$ and set $P = (p_{\lambda i})$. From the proof of [**11**, Theorem II.2.8], (iv) $\Rightarrow$ (vi)), we deduce the following result.

Theorem 3.4. *The Rees matrix ring $R = \mathcal{M}(I, \Delta, \Lambda; P)$ is defined and η is an isomorphism of $\mathcal{F}_U(V)$ onto R.*

This is the key part of the proof of the next theorem. On [**11**, p.16] it was attributed to the author through an inconspicuous and unfortunate typographical error. It and its proof, both due to E. Hotzel, were published in [**11**, Theorem II.2.8]. A ring containing a minimal one-sided ideal is termed **atomic**.

Theorem 3.5. *A ring R is simple and atomic if and only if R is isomorphic to a Rees matrix ring.*

For the following special case, we introduce the notation

$$\Delta_I = \mathcal{M}(I, \Delta, I; \iota_I).$$

Hence Δ_I consists of all $I \times I$ - matrices over Δ with only a finite number of nonzero entries with the usual addition and multiplication of matrices (again ignoring arbitrary sums of zeros). Clearly, see [**11**, Corollary II.2.7],

$$\Delta_I \cong \Gamma_J \Leftrightarrow |I| = |J|, \ \Delta \cong \Gamma.$$

Hence we may write $\Delta_{|I|}$, as in [**11**], instead of Δ_I. Call Δ_I, a **symmetric** Rees matrix ring.

For details and proof of the above assertions as well as a general treatment of this subject, we refer the reader to [**11**].

A ring R with identity 1 is said to be **directly finite** if for any $a, b \in R, ab = 1$ implies that $ba = 1$. Following Munn [**10**], we say that a ring R, not necessarily having an identity, is **quasi directly finite** if for any $a, b \in R, ab = a + b$ implies that $ab = ba$. In the case R has an identity element, simple argument shows that the two definitions are equivalent. For a ring R with identity 1, in view of [**9**, VI.3, Corollary 2.4] direct finiteness is equivalent to the absence of a bicyclic subsemigroup of $\mathfrak{M}R$ with identity 1. For information on direct finiteness, consult [**5**, Chapter 5].

Proposition 3.6. *Every simple atomic ring is quasi directly finite.*

Proof. In view of the above results, we may consider $R = \mathcal{F}_U(V)$ for a dual pair (U, V) of vector spaces. By [**11**, Theorem I.3.20], R is a locally matrix ring. For any $a, b \in R$, there thus exists a subring R' of R such that $a, b \in R'$ and $R' \cong \Delta_n$ for some division ring Δ and some $n > 0$. The last ring is directly finite by [**5**, Proposition 5.2]. It follows that $ab = a + b$ implies that $ab = ba$, as required. □

Recall from [**7**] that a ring R has **unique addition** if for the given multiplication, the addition of R is the only one which makes it a ring.

We shall often encounter the hypothesis that $R = \mathcal{M}(I, \Delta, \Lambda; P)$ is not a division ring. The following lemma contains a clasification of this assumption.

Lemma 3.7. *Let* $R = \mathcal{M}(I, \Delta, \Lambda; P) \cong \mathcal{F}_U(V)$.

(i) $\dim U = |I|$ *and* $\dim V = |\Lambda|$.

(ii) R has an identity if and only if $|\Lambda| = n < \infty$. In such a case $|I| = n$ and R is isomorphic to the ring of $n \times n$ - matrices over Δ.

(iii) R is a division ring if and only if $|\Lambda| = 1$.

Proof. (i) This follows from Theorem 3.4 and its preamble.

(ii) This is a consequence of part (i) and results contained in [**11**]

(iii) This follows directly from part (ii). □

4. Isomorphisms of Rees matrix rings

We shall express these in terms of matrices. Again we need some more concepts.

Definition 4.1. Let X and Y be nonempty sets and A be a $X \times Y$ - matrix over a division ring Δ. The matrix A is **row finite** if every row of A has only a finite number of nonzero entries. If this is the case, A is **invertible** if there exists a row finite $Y \times X$ - matrix B over Δ such that $AB = \iota_X$ and $BA = \iota_Y$. The uniqueness of B follows without difficulty, hence we may write $A^{-1} = B$. We have the corresponding definitions for columns instead of rows. If $A = (a_{xy})$, ω is an isomorphism of Δ and $c \in \Delta$, then

$$A\omega = (a_{xy}\omega), \qquad cA = (ca_{xy}), \qquad Ac = (a_{xy}c).$$

The above concepts will now be used for the basic construction which yields the desired isomorphisms. This is an analogue for rings of the construction of $\theta(U, \omega, V)$ in Section 2 for Rees matrix semigroups.

Notation 4.2. Let

(4) $R = \mathcal{M}(I, \Delta, \Lambda; P)$ and $R' = \mathcal{M}'(I', \Delta', \Lambda'; P')$ be Rees matrix rings,
U be an invertible column finite $I \times I'$ - matrix over Δ',
$\omega : \mathfrak{M}\Delta \to \mathfrak{M}\Delta'$ be an isomorphism,
V be an invertible row finite $\Lambda \times \Lambda'$ - matrix over Δ'

such that $(P\omega)U = VP'$. Define a mapping $\chi = \chi(U, \omega, V)$ by

$$\chi : X \to U^{-1}(X\omega)V \qquad (X \in R).$$

Compare the next result with Theorem 2.2 for which the additional information here is also valid.

Theorem 4.3. *Let (4) be given where R is not a division ring. The mapping $\chi(U, \omega, V)$ is an isomorphism R onto R' with inverse $\chi(U^{-1}\omega^{-1}, \omega^{-1}, V^{-1}\omega^{-1})$. Moreover,*

$$\chi(U, \omega, V) = \chi(U', \omega', V') \Leftrightarrow U' = cU, \quad \omega' = \omega\varepsilon_{c^{-1}}, \quad V' = cV$$

for some $c \in \Delta'^{\star}$. Conversely, every isomorphism of R onto R' can be so constructed.

Proof. That $\chi = \chi(U, \omega, V)$ is a homomorphism of $\mathfrak{M}R$ into $\mathfrak{M}R'$ is straightforward to verify (see the proof of [**11**, Proposition II.7.6]). Also

$$(P'\omega^{-1})(U^{-1}\omega^{-1}) = (P'U^{-1})\omega^{-1} = (V^{-1}(P\omega))\omega^{-1} = (V^{-1}\omega^{-1})P$$

so that $\chi' = \chi(U^{-1}\omega^{-1}, \omega^{-1}, V^{-1}\omega^{-1})$ is defined. It follows easily that $\chi\chi'$ and $\chi'\chi$ are identity mappings on their respective domains. Hence $\chi' = \chi^{-1}$. Therefore χ is an isomorphism of $\mathfrak{M}R$ onto $\mathfrak{M}R'$. Since R is not a division ring, if $\mathfrak{M}R \cong \mathfrak{M}R'$, then also R' is not a division ring and [**11**, Corollary II.7.5] implies that they are rings with unique addition. Hence every isomorphism of $\mathfrak{M}R$ onto $\mathfrak{M}R'$ is additive, see [**7**], and hence χ is an isomorphism of R onto R'.

Suppose that $\chi(U, \omega, V) = \chi(U', \omega', V')$. Then for all $[i, g, \lambda] \in R$, we have

$$U^{-1}([i, g, \lambda]\omega)V = {U'}^{-1}([i, g, \lambda]\omega')V'.$$

From here on the argument is quite similar to that in the proof of the corresponding part of [**12**, Theorem 3.4] and may be omitted. The converse implication follows at once.

The converse part of the theorem was established in [**11**, Proposition II.7.6], which incidentally yields that ω is additive. $\square$

We now prove directly that ω figuring in Theorem 4.3 is additive.

Corollary 4.4. *Let the notation and hypotheses be as in Theorem 4.3. Then the mapping ω is additive and is thus an isomorphism of Δ onto Δ'.*

Proof. By Theorem 4.3, $\chi = \chi(U, \omega, V)$ is additive. Hence for any $[i, g, \lambda]$, $[i, h, \lambda] \in R$, we get

$$\begin{aligned} U^{-1}[i, g\omega + h\omega, \lambda]V &= U^{-1}([i, g\omega, \lambda] + [i, h\omega, \lambda])V \\ &= U^{-1}[i, g, \lambda]\omega V + U^{-1}[i, h, \lambda]\omega V = [i, g, \lambda]\chi + [i, h, \lambda]\chi \\ &= ([i, g, \lambda] + [i, h, \lambda])\chi = [i, g + h, \lambda]\chi = U^{-1}[i, g + h, \lambda]\omega V \\ &= U^{-1}[i, (g + h)\omega, \lambda]V \end{aligned}$$

and thus $g\omega + h\omega = (g + h)\omega$, as required. □

We now consider briefly the automorphism group of a Rees matrix ring.

Proposition 4.5. *Let $R = \mathcal{M}(I, \Delta, \Lambda; P)$ and assume that R is not a division ring. Let Γ be the set of all (U, ω, V) where*
U is an invertible column finite $I \times I$ - matrix over Δ,
ω is an automorphism of $\mathfrak{M}\Delta$,
V is an invertible row finite $\Lambda \times \Lambda$ - matrix over Δ
satisfying the condition $(P\omega)U = VP$, with multiplication

$$(U, \omega, V)(U', \omega', V') = ((U\omega')U', \omega\omega', (V\omega')V').$$

Then Γ is a group and, with $R' = R$ in Notation 4.2, the mapping

$$\chi : (U, \omega, V) \to \chi(U, \omega, V) \qquad ((U, \omega, V) \in \Gamma)$$

is a homomorphism of Γ onto the automorphism group of R with kernel

$$K = \{(c\iota_I, \varepsilon_{c^{-1}}, c\iota_\Lambda) \mid c \in \Delta^\star\}.$$

The mapping

$$\kappa : c \to (c\iota_I, \varepsilon_{c^{-1}}, c\iota_\Lambda) \qquad (c \in \Delta^\star)$$

is an antihomomorphism of $\mathfrak{M}\Delta^\star$ onto K whose kernel is the center of $\mathfrak{M}\Delta^\star$.

Proof. The argument is entirely multiplicative so that the proof of [**12**, Proposition 3.11] carries over verbatim to this case. □

We conclude this section by considering a special type of isomorphism. Toward this end, we first introduce the following symbolism.

Notation 4.6. Let $R = \mathcal{M}(I, \Delta, \Lambda; P)$ and denote by R_0 the set of all matrices in R which have at most one nonzero entry.

Clearly R_0 is closed under multiplication and

$$R_0 = \{[i, g, \lambda] \in R \mid i \in I, g \in \Delta^\star, \lambda \in \Lambda\} \cup \{0\}$$

so that an easy verification shows that $R_0 \cong \mathcal{M}^o(I, \mathfrak{M}\Delta^\star, \Lambda; P)$ and R_0 is a completely 0-simple semigroup. For the notation below consult Section 2.

Lemma 4.7. *Let R and R' be as in (4), assume that R is not a division ring and let $\theta = \theta(U, \omega, V)$ be an isomorphism of R_0 onto R'_0. Then $\chi = \chi(U, \omega, V)$ is defined, it is an isomorphism of R onto R' and is the unique extension of θ to a homomorphism of R^+ onto R'^+.*

Proof. First note that R not being a division ring implies the same for R'. Since U is column monomial and V is row monomial, the condition of definability of $\theta(V, \omega, V)$ and $\chi(U, \omega, V)$ is the same. Hence χ is an isomorphism of R onto R'.

Let τ be an extension of θ to a homomorphism of R^+ onto R'^+. Then

$$\begin{aligned}\left(\sum_{k=1}^{n}[i_k, g_k, \lambda_k]\right)\tau &= \sum_{k=1}^{n}[i_k, g_k, \lambda_k]\tau = \sum_{k=1}^{n}[i_k, g_k, \lambda_k]\theta \\ &= \sum_{k=1}^{n}[i_k, g_k, \lambda_k]\chi = \left(\sum_{k=1}^{n}[i_k, g_k, \lambda_k]\right)\chi.\end{aligned}$$

Since all elements of R are of this form, we conclude that $\tau = \chi$. □

Corollary 4.8. *Let $R = \mathcal{M}(I, \Delta, \Lambda; P)$ and*

$$\mathcal{A}_0(R) = \{\chi \in \mathcal{A}(R) \mid R_0\chi = R_0\}.$$

Then $\mathcal{A}_0(R)$ is a subgroup of $\mathcal{A}(R)$ isomorphic to $\mathcal{A}(R_0)$.

Proof. The first assertion is obvious while the second follows from Lemma 4.7. □

Theorem 4.9. *Let*

$$\chi = \chi(U, \omega, V) : R = \mathcal{M}(I, \Delta, \Lambda; P) \to R' = \mathcal{M}(I', \Delta', \Lambda'; P')$$

be an isomorphism and assume that R is not a division ring. Then χ maps R_0 onto R'_0 if and only if both U and V are permutational.

Proof. Necessity. The hypothesis implies that $\theta = \chi|_{R_0}$ is an isomorphism of R_0 onto R'_0. By Theorem 2.2, we have that $\theta = \theta(U', \omega', V')$ for suitable parameters. By Lemma 4.7, $\chi(U', \omega', V')$ is the unique extension of θ to an isomorphism of R onto R'. It follows that $\chi(U, \omega, V) = \chi(U', \omega', V')$ which by Theorem 4.3 implies the existence of $c \in \Delta'^{\star}$ such that $U' = cU, \omega' = \omega\varepsilon_{c^{-1}}, V' = cV$. Hence $U = c^{-1}U'$ and $V = c^{-1}V'$ and since both U' and V' are permutational, so are both U and V.

Sufficiency. Let $U = (u_{ij'})$ and $V = (v_{\lambda\mu'})$. By hypothesis, there exist bijections $\xi : I \to I'$ and $\eta : \Lambda \to \Lambda'$, and functions $u : I \to \Delta'^{\star}$, $v : \Lambda \to \Delta'^{\star}$, say $i \to u_i$ and $\lambda \to v_\lambda$ such that

$$u_{ij'} = \begin{cases} u_i & \text{if } j' = i\xi \\ 0 & \text{otherwise} \end{cases}, \qquad v_{\lambda\mu'} = \begin{cases} v_\lambda & \text{if } \mu' = \lambda\eta \\ 0 & \text{otherwise} \end{cases}.$$

Then $U^{-1} = (u'_{j'i})$ where

$$u'_{j'i} = \begin{cases} u_i^{-1} & \text{if } j' = i\xi \\ 0 & \text{otherwise} \end{cases}.$$

Let $[k,g,\nu]=(x_{i\lambda})$ and $U^{-1}((x_{i\lambda})\omega)V=(y_{j'\mu'})$. Then

$$
\begin{aligned}
y_{j'\mu'} &= \sum_i\sum_\lambda u_{j'i}(x_{i\lambda}\omega)v_{\lambda\mu'} = \begin{cases} u_i^{-1}(g\omega)v_\lambda & \text{if } \; j'=i\xi, i=k, \lambda=\nu, \mu'=\lambda\eta \\ 0 & \text{otherwise}\end{cases} \\
&= \begin{cases} u_{j'\xi^{-1}}^{-1}(g\omega)v_{\mu'\eta^{-1}} & \text{if } \; j'=k\xi,\ \mu'=\nu\eta \\ 0 & \text{otherwise}\end{cases} \\
&= \begin{cases} u_k^{-1}(g\omega)v_\nu & \text{if } \; j'=k\xi,\ \mu'=\nu\eta \\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

which proves that

$$U^{-1}([k,g,\nu]\omega)V=[k\xi,u_k^{-1}(g\omega)v_\nu,\nu\eta].$$

Hence $R_0\chi\subseteq R_0'$.

By Theorem 4.3, we have $\chi^{-1}=\chi(U^{-1}\omega^{-1},\omega^{-1},V^{-1}\omega^{-1})$. Here both $U^{-1}\omega^{-1}$ and $V^{-1}\omega^{-1}$ are permutational. So we may apply the above analysis to χ^{-1} thereby obtaining the inclusion $R_0'\chi^{-1}\subseteq R_0$ whence $R_0'\subseteq R_0\chi$. Therefore $R_0\chi= R_0'$. □

It is of particular interest to identify those Rees matrix rings which are isomorphic to symmetric ones. This is the content of the next result.

Proposition 4.10. *Let $R=\mathcal{M}(I,\Delta,\Lambda;P)$. Then $R\cong\Gamma_K$ for some division ring Γ and set K if and only if $|I|=|K|=|\Lambda|$, $\Delta\cong\Gamma$ and P is the product of an invertible row finite $\Lambda\times K$ - matrix over Δ and an invertible column finite $I\times K$ - matrix over Δ.*

Proof. Necessity. By [**11**, Corollary II.7.7], we have $|I|=|K|=|\Lambda|$. Further, by Theorem 4.3, there exist:

an isomorphism $\omega:\Delta\to\Gamma$,

an invertible row finite $\Lambda\times K$ - matrix U over Γ,

an invertible column finite $I\times K$ - matrix V over Γ

such that $(P\omega)U=V\iota_K$ whence $P=(V\omega^{-1})(U^{-1}\omega^{-1})$. The matrices $V\omega^{-1}$ and $U^{-1}\omega^{-1}$ satisfy the above specifications.

Sufficiency. Let $P=VU$ be as in the statement of the proposition and set $V'=V\omega, U'=U^{-1}\omega$. Then

$$(P\omega)U'=(V\omega)(U\omega)(U^{-1}\omega)=V'=V'\iota_K$$

with the conditions needed to define $\chi=\chi(U',\omega,V')$ which is an isomorphism of R onto Γ_K according to Theorem 4.3. □

There are two special cases in which the conditions of Proposition 4.10. are fulfilled.

Corollary 4.11. *Let $R=\mathcal{M}(I,\Delta,\Lambda;P)$ where P is permutational. Then $R\cong\Delta_I$.*

Proof. The hypothesis on P immediately yields that $|I|=|\Lambda|$ and $P=P\iota_I$, the needed factorization of P in Proposition 4.10. □

The next case can be handled directly.

Proposition 4.12. *Let $R = \mathcal{M}(I, \Delta, \Lambda; P)$ where both I and Λ are denumerable. Then $R \cong \Delta_I$.*

Proof. By Theorems 3.4 and 3.5, we have $R \cong \mathcal{F}_U(V)$ for some dual pair (U, V) of vector spaces. By [**11**, Proposition II.2.18], vector spaces U and V have biorthogonal bases which by Theorem 3.5 implies that R admits a symmetric Rees matrix representation. The assertion now follows by [**11**, Corollary 11.7.7]. $\square$

We have assumed in this section that the ring R, and thus also the ring R', is not a division ring. This was needed in order to use the result that in such a case the rings R and R' have unique addition. To include this exception, we may apply the same construction with the extra proviso that the multiplicative isomorphism ω be also additive. However, in the case when R is a division ring, the nonzero elements of R form a group under multiplication so that in case $\mathfrak{M}R \cong \mathfrak{M}R'$, we get that also that R' is a division ring. The construction of a Rees matrix ring in this case trivializes to $|I| = |\Lambda| = 1$ so that, by possibly changing the coordinates, the sole entry of P may be assumed to the identity of Δ, we are faced with isomorphisms of division rings. But, in general, a division ring need not have a unique addition. This seems to have been noticed first by Sushkevich [**13**]. An example is not far to look for: $\mathbb{Z}_5 = \mathbb{Z}/(5)$ already gives such an instance, Indeed, $\varphi = (\bar{2}\ \bar{3})$ is an automorphism of $\mathfrak{M}\mathbb{Z}_5$ which is not additive. The new addition on the set $\mathbb{Z}_5$ is given by $a \oplus b = (a\varphi + b\varphi)\varphi^{-1}$, see [**7**].

5. The translational hull of a Rees matrix ring

We construct here a suitable isomorphic copy of the translational hull of a Rees matrix ring in terms of matrices and explore the relationship of several related rings. Starting with the necessary concepts, we prove a sequence of lemmas which lead to the main result of this section.

Let R be a ring. A transformation ρ of R written on the right, additive and satisfying $(xy)\rho = x(y\rho)$ for all $x, y \in R$ is a **right translation** of R. A **left translation** of R is a transformation of R written on the left, additive and satisfying $\lambda(xy) = (\lambda x)y$ for all $x, y \in R$.

Right translations of R added pointwise and composed on the right form a ring $P(R)$. Similarly left translations of R added pointwise and composed on the left form a ring $\Lambda(R)$. For $\lambda \in \Lambda(R)$ and $\rho \in P(R)$, we say that they are **linked**, and that (λ, ρ) is a **bitranslation** of R if $x(\lambda y) = (x\rho)y$ for all $x, y \in R$. The set $\Omega(R)$ of all bitranslations of R under componentwise operations is a ring, the **translational** hull of R.

For every $a \in R$, the mappings ρ_a and λ_a, defined by

$$x\rho_a = xa, \qquad \lambda_a x = ax \qquad (x \in R)$$

are the **inner right** and the **inner left translations** of R, respectively, and $\pi_a = (\lambda_a, \rho_a)$ the **inner bitranslation** of R induced by a. The mapping

$$\pi : a \to \pi_a \qquad (a \in R)$$

is the **canonical homomorphism** of R into $\Omega(R)$.

We can now proceed with the specific material.

Notation 5.1. Throughout this section, we fix

$$R = \mathcal{M}(I, \Delta, \Lambda; P).$$

Let $RF(\Delta, \Lambda)$ denote the set of all row finite $\Lambda \times \Lambda$ - matrices over Δ with componentwise addition and the row by column multiplication of matrices. Similarly let $CF(I, \Delta)$ be the set of all column finite $I \times I$ - matrices over Δ with the same operations.

According to ([6], IX.4), $RF(\Delta, \Lambda)$ is a ring; analogously for $CF(I, \Delta)$.

Lemma 5.2.

(i) *For $b \in \mathcal{L}(V)$, define a matrix $b\psi = (b_{\lambda\mu})$, where $z_\lambda b = \sum\limits_{\mu} b_{\lambda\mu} z_\mu$ for every $\lambda \in \Lambda$. Then ψ is an isomorphism of $\mathcal{L}(V)$ onto $RF(\Delta, \Lambda)$.*

(ii) *For $a \in \mathcal{L}'(U)$, define a matrix $a\varphi = (a_{ji})$, where $aw_i = \sum\limits_{j} w_j a_{ji}$ for every $i \in I$. Then φ is an isomorphism of $\mathcal{L}'(U)$ onto $CF(I, \Delta)$.*

Proof. For part (i), see([6], IX.4); part (ii) is dual. □

Lemma 5.3.

(i) *For $B \in RF(\Delta, \Lambda)$, define a function ρ^B by*

$$X\rho^B = XB \qquad (X \in R).$$

Then the mapping

$$\rho : B \to \rho^B \qquad (B \in RF(\Delta, \Lambda))$$

is an isomorphism of $RF(\Delta, \Lambda)$ onto $P(R)$.

(ii) *For $A \in CF(I, \Delta)$, define a function λ^A by*

$$\lambda^A X = AX \qquad (X \in R).$$

Then the mapping

$$\lambda : A \to \lambda^A \qquad (A \in CF(I, \Delta))$$

is an isomorphism of $CF(I, \Delta)$ onto $\Lambda(R)$.

Proof. (i) Let $X, Y \in R$ and $B \in RF(\Delta, \Lambda)$. Then $XB = \left(\sum\limits_{\lambda} x_{i\lambda} b_{\lambda\mu}\right)$. There is a finite number of nonzero elements $x_{i\lambda}$ and for each λ, there exists only a finite number of nonzero elements $b_{\lambda\mu}$. Hence there is only a finite number of nonzero elements $x_{i\lambda} b_{\lambda\mu}$. It follows that the matrix XB has only a finite number of nonzero entries so that $XB \in R$. Therefore ρ_B maps R into itself. Also

$$(X + Y)\rho^B = (X + Y)B = XB + YB = X\rho^B + Y\rho^B,$$

$$(X * Y)\rho^B = (XPY)B = XP(YB) = X * (Y\rho^B),$$

and ρ^B is a right translation of R. Hence ρ maps $RF(\Delta, \Lambda)$ into $P(R)$.

Now let $X \in R$ and $B, B' \in RF(\Delta, \Lambda)$. Then

$$X(\rho^B + \rho^{B'}) = X\rho^B + X\rho^{B'} = XB + XB' = X(B + B') = X\rho^{B+B'},$$
$$X(\rho^B \rho^{B'}) = (X\rho^B)\rho^{B'} = (XB)B' = (XBB') = X\rho^{BB'}$$

so that $\rho^B + \rho^{B'} = \rho^{B+B'}$ and $\rho^B\rho^{B'} = \rho^{BB'}$ and ρ is a homomorphism.

Let $B \in RF(\Delta, \Lambda), B = (b_{\lambda\mu})$. If $b_{\nu\theta} \neq 0$, then $[i, 1, \nu]B = (x_{j\tau})$ with $x_{i\theta} = b_{\nu\theta} \neq 0$. Thus if $B \neq 0$, then $XB \neq 0$ for some $X \in R$. By contrapositive, we get that $\rho^B = 0$ implies that $B = 0$. Therefore the kernel of the homomorphism ρ is equal to zero and thus ρ is injective.

In order to prove that ρ is surjective, we let $\eta \in P(R)$. Let $[i, 1, \mu] \in R$. Since P is right column independent, there exists $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$. Hence

$$\begin{aligned}[i, 1, \mu]\eta &= ([i, p_{\lambda i}^{-1}, \lambda] * [i, 1, \mu])\eta = [i, p_{\lambda i}^{-1}, \lambda] * ([i, 1, \mu]\eta) \\ &= [i, p_{\lambda i}^{-1}, \lambda]P([i, 1, \mu]\eta) = (m_{k\theta})\end{aligned}$$

has all its nonzero entries in the i-th row. Similarly $[j, 1, \mu]\eta = (n_{k\theta})$ has all its nonzero entries in the j-th row. With $p_{\nu j} \neq 0$, we get

$$[i, 1, \mu] = [i, p_{\lambda j}^{-1}, \nu] * [j, 1, \mu]$$

which implies that

$$[i, 1, \mu]\eta = [i, p_{\nu j}^{-1}, \nu] * ([j, 1, \mu]\eta)$$

and thus

$$(m_{k\theta}) = [i, p_{\nu j}^{-1}, \nu]P(n_{k\theta}),$$

whence

$$m_{i\theta} = p_{\nu j}^{-1} p_{\nu j} n_{j\theta} = n_{j\theta}.$$

This proves that $[i, 1, \mu]\eta$ has all nonzero entries in the i-th row and these entries are independent of i.

Further,

$$\begin{aligned}[i, g, \mu]\eta &= ([i, gp_{\lambda i}^{-1}, \lambda] * ([i, 1, \mu])\eta = [i, gp_{\lambda i}^{-1}, \lambda] * ([i, 1, \mu]\eta) \\ &= [i, gp_{\lambda i}^{-1}, \lambda]P(m_{k\theta}) = (gm_{k\theta}).\end{aligned}$$

In addition, every nonzero element of R is uniquely a sum of matrices of the form $[i, g, \mu]$ and thus η is uniquely determined by its values on elements of the form $[i, 1, \mu]$.

Now let B be the $\Lambda \times \Lambda$ - matrix over Δ whose μ-th row is the i-th row of the matrix $[i, g, \mu]\eta$. Hence B is row finite so that $B \in RF(\Delta, \Lambda)$. For any $i \in I$ and $\mu \in \Lambda$, we get

$$[i, 1, \mu]\rho^B = [i, 1, \mu]B = [i, 1, \mu]\eta$$

so that $\rho^B = \eta$ which proves that ρ is surjective.

(ii) This is the dual of part (i). □

Lemma 5.4. *The mappings*

$$\delta : X \to PX, \qquad \gamma : X \to XP \qquad (X \in R)$$

are embeddings of R into $RF(\Delta, \Lambda)$ and $CF(I, \Delta)$, respectively.

Proof. Let $X \in R$. Then PX is defined since X is column finite. Also PX has only a finite number of nonzero columns since this is true for X. Hence PX is row finite and thus δ maps R into $RF(\Delta, \Lambda)$.

Now let $X, Y \in R$. Then

$$(X+Y)\delta = (X+Y)P = XP + YP = X\delta + Y\delta,$$
$$(X * Y)\delta = (XPY)P = XP(YP) = (X\delta)(Y\delta),$$

so that δ is a homomorphism. Further,

$$\ker \delta = \{X \in R \mid PX = 0\},$$

which is an ideal of R. Since R is simple, either $\ker \delta = 0$ or $\ker \delta = R$. For any $[i, 1, \lambda] \in R$, by column independence of P, there exist $\mu \in \Lambda$ such that $p_{\mu i} \neq 0$. Hence $P[i, 1, \lambda] = (x_{\sigma\tau})$ where $x_{\mu\lambda} = p_{\mu i} \neq 0$ and thus $\ker \delta = 0$. So δ is an embedding.

The case of γ is dual. □

Lemma 5.5.

(i) *Letting i be the inclusion map, for $i : \mathcal{F}_U(V) \to \mathcal{L}(V)$, we have $i\psi = \eta\delta$ and for $i : \mathcal{F}'_U(V) \to \mathcal{L}'(V)$, we have $i\varphi = \eta\gamma$.*

(ii) *The mappings*

$$r : X \to \rho_X, \qquad l : X \to \lambda_X \qquad (X \in R)$$

are embeddings of R into $P(R)$ and $\Lambda(R)$, respectively and $\delta\rho = r, \gamma\lambda = l$.

Proof. (i) With the notation (1) and (2) in Section 3, we obtain

$$bi\psi = b\psi = (b_{\lambda\mu}), \qquad z_\lambda b = \sum_\mu b_{\lambda\mu} z_\mu$$

and

$$\begin{aligned} z_\lambda b &= z_\lambda \sum_{k=1}^n (i_k, \gamma_k, \lambda_k) = \sum_{k=1}^n (z_\lambda, u_{i_k})\gamma_k v_{\lambda_k} \\ &= \sum_{k=1}^n \left(z_\lambda, \sum_j w_j \tau_{j i_k} \right) \gamma_k v_{\lambda_k} = \sum_{k=1}^n \sum_j (z_\lambda, w_j)\tau_{j i_k}\gamma_k v_{\lambda_k} \\ &= \sum_{k=1}^n \sum_j p_{\lambda j}\tau_{j i_k}\gamma_k \sum_\mu \sigma_{\lambda_k \mu} z_\mu = \sum_\mu \left(\sum_{k=1}^n \sum_j p_{\lambda j}\tau_{j i_k}\gamma_k \sigma_{\lambda_k \mu} \right) z_\mu \end{aligned}$$

so that $b_{\lambda\mu} = \sum_{k=1}^n \sum_j p_{\lambda j}\tau_{j i_k}\gamma_k\sigma_{\lambda_k\mu}$. Also

$$b\eta\delta = \left(\sum_{k=1}^n \tau_{j i_k}\gamma_k\sigma_{\lambda_k\mu} \right)\delta = P\left(\sum_{k=1}^n \tau_{j i_k}\gamma_k\sigma_{\lambda_k\mu} \right) = \left(\sum_j \sum_{k=1}^n p_{\lambda j}\tau_{j i_k}\gamma_k\sigma_{\lambda_k\mu} \right)$$

whence the desired equality.

The second assertion is dual.

(ii) For any $X \in R$, we have

$$X\delta\rho = (PX)\rho = \rho^{PX} = \rho_X$$

since for any $Y \in R$,

$$Y\rho^{PX} = Y(PX) = Y * X = Y\rho_X.$$

By Lemmas 5.2(i) and 5.3(i), we have that r is an embedding.

The remaining assertions are dual. □

Lemma 5.6. *Let $a \in \mathcal{L}'(U)$ and $b \in \mathcal{L}(V)$. Then the following statements are equivalent.*

(i) *a is the adjoint of b.*
(ii) *$P(a\varphi) = (b\psi)P$.*
(iii) *$a\varphi\lambda$ and $b\psi\rho$ are linked translations.*

Proof. Let $A = a\varphi = (a_{ij})$ and $B = b\psi = (b_{\lambda\mu})$ and assume that $XPAY = XBPY$ for all $X, Y \in R$. For $X = Y = [i, 1, \lambda] = (x_{m\theta})$, we get

$$\sum_{\theta,l,n} x_{m\theta}p_{\theta l}a_{ln}x_{n\sigma} = \sum_{\eta,\tau,s} x_{m\eta}b_{\eta\tau}p_{\tau s}x_{s\sigma}$$

whence

$$\begin{cases} \sum_l p_{\lambda l}a_{li} & \text{if} \quad m = i,\ \sigma = \lambda \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \sum_\tau b_{\lambda\tau}p_{\tau i} & \text{if} \quad m = i,\ \sigma = \lambda \\ 0 & \text{otherwise} \end{cases}$$

which evidently implies that $PA = BP$. Next

$$\begin{aligned}
&(v, au) = (vb, u) \quad \text{for all } v \in V, u \in U \\
&\Leftrightarrow (v_\lambda, au_i) = (v_\lambda b, u_i) \quad \text{for all } \lambda \in \Lambda, i \in I \\
&\Leftrightarrow \left(v_\lambda, \sum_j w_j a_{ji}\right) = \left(\sum_\mu b_{\lambda\mu}z_\mu, u_i\right) \quad \text{for all } \lambda \in \Lambda, i \in I \\
&\Leftrightarrow \sum_j (v_\lambda, w_j)a_{ji} = \sum_\mu b_{\lambda\mu}(z_\mu, u_i) \quad \text{for all } \lambda \in \Lambda, i \in I \\
&\Leftrightarrow \sum_j p_{\lambda j}a_{ji} = \sum_\mu b_{\lambda\mu}p_{\mu i} \quad \text{for all } \lambda \in \Lambda, i \in I \\
&\Leftrightarrow PA = BP \\
&\Leftrightarrow XPAY = XBPY \quad \text{for all } X, Y \in R \qquad \text{by the above} \\
&\Leftrightarrow X * (AY) = (XB) * Y \quad \text{for all } X, Y \in R \\
&\Leftrightarrow X * (\lambda^A Y) = (X\rho^B) * Y \quad \text{for all } X, Y \in R \\
&\Leftrightarrow \lambda^A \text{ and } \rho^B \text{are linked}
\end{aligned}$$

and the assertion follows. □

The following notation will be used frequently.

Notation 5.7. Let

$$M(R) = \{(A, B) \in CF(I, \Delta) \times RF(\Delta, \Lambda) \mid PA = BP\}$$

with componentwise operations.

Lemma 5.8. *$M(R)$ is closed under its operations and thus forms a ring. The mapping*

$$\tau : X \to (XP, PX) \qquad (X \in R)$$

is an embedding of R into $M(R)$.

Proof. Straightforward verification, see Lemma 5.4. □

We are finally ready for the main result of this section.

Theorem 5.9.

(i) *The mapping*

$$\alpha : b \to (a\varphi, b\psi) \qquad (b \in \mathcal{L}(U),\ a \text{ is an adjoint of } b)$$

is an isomorphism of $\mathcal{L}_U(V)$ onto $M(R)$.

(ii) *The mapping*

$$\mu : (A, B) \to (A\lambda, B\rho) \qquad ((A, B) \in M(R))$$

is an isomorphism of $M(R)$ onto $\Omega(R)$.

(iii) *The following diagram commutes*

$$\begin{tikzcd}
\mathcal{L}_U(V) \arrow[rr, "\alpha"] \arrow[dd, "i"'] & & M(R) \arrow[r, "\mu"] & \Omega(R) \arrow[dd, "(\lambda, \rho) \to \rho"] \\
& \mathcal{F}_U(V) \arrow[ul, "i"] \arrow[dl, "i"'] \arrow[r, "\eta"] & R \arrow[u, "\tau"] \arrow[ur, "\pi"'] \arrow[d, "\delta"'] \arrow[dr, "r"] & \\
\mathcal{L}(V) \arrow[rr, "\psi"] & & RF(\Delta, \Lambda) \arrow[r, "\rho"] & P(R)
\end{tikzcd}$$

where i denotes the inclusion mappings.

Proof. (i) This follows directly from Lemmas 5.2 and 5.6.

(ii) This follows directly from Lemmas 5.3 and 5.6.

(iii) Commutativity of the extreme left and the extreme right triangles is obvious; that of the lower quadrangle follows from Lemma 5.5(i) and that of the lower triangle by Lemma 5.5(ii).

In order to prove commutativity of the upper quadrangle, we let b be as in (1) with (2) and a be its adjoint in U. Letting

$$X = b\eta = \left(\sum_{k=1}^{n}(i_k, \gamma_k, \lambda_k)\right)\eta = \left(\sum_{k=1}^{n} \tau_{ji_k}\gamma_k\sigma_{\lambda_k\mu}\right) = (x_{j\mu})$$

we have $b_{\eta\tau} = (XP, PX)$ where

$$XP = \left(\sum_{\mu} x_{j\mu}p_{\mu i}\right) = \left(\sum_{\mu}\sum_{k=1}^{n} \tau_{ji_k}\gamma_k\sigma_{\lambda_k\mu}p_{\mu i}\right), \tag{5}$$

$$PX = \left(\sum_{j} p_{\lambda j}x_{j\mu}\right) = \left(\sum_{j}\sum_{k=1}^{n} p_{\lambda j}\tau_{ji_k}\gamma_k\sigma_{\lambda_k\mu}\right). \tag{6}$$

For any $\lambda \in \Lambda$, we get

$$\begin{aligned} z_\lambda b &= z_\lambda \left(\sum_{k=1}^{n}(i_k, \gamma_k, \lambda_k)\right) = \sum_{k=1}^{n}(z_\lambda, u_{i_k})\gamma_k v_{\lambda_k} \\ &= \sum_{k=1}^{n}\left(z_\lambda, \sum_j w_j\tau_{ji_k}\right)\gamma_k\left(\sum_\mu \sigma_{\lambda_k\mu}z_\mu\right) \\ &= \sum_\mu\left(\sum_j\sum_{k=1}^{n}(z_\lambda, w_j)\tau_{ji_k}\gamma_k\sigma_{\lambda_k\mu}\right)z_\mu \\ &= \sum_\mu\left(\sum_j\sum_{k=1}^{n}p_{\lambda j}\tau_{ji_k}\gamma_k\sigma_{\lambda_k\mu}\right)z_\mu \end{aligned}$$

and similarly for every $i \in I$,

$$aw_i = \sum_j w_j\left(\sum_\mu\sum_{k=1}^{n}\tau_{ji_k}\gamma_k\sigma_{\lambda_k\mu}p_{\mu i}\right).$$

Comparing the last two expressions with (5) and (6), we conclude that $b\eta\tau = (a\varphi, b\psi) = b\alpha$. Therefore $\eta\tau = \alpha$, as required.

For commutativity of the upper triangle, we let $X \in R$ and apply Lemma 5.5(ii) obtaining

$$X\tau\mu = (X\gamma, X\delta)\mu = (X\gamma\lambda, X\delta\rho) = (Xl, Xr) = X\pi$$

so that $\tau\mu = \pi$, as required. □

The situation is particularly transparent for the case when $R = \Delta_I$. Indeed, then

$$M(R) = \{(A, B) \in CF(\Delta, I) \times RF(\Delta, I) \mid A = B\}$$

and we may identify $M(R)$ with

$$M'(R) = \{A \mid A \text{ is a row and column finite } I \times I\text{-matrix over } \Delta\}$$

with the usual addition and multiplication of matrices, cf. [**6**, IX.10, Exercise 2]. This ring contains Δ_I, as a subring. For this case, we have the following result.

Proposition 5.10. *Let $R = \Delta_I$. Then $M(R)$ is directly finite if and only if I is finite.*

Proof. Suppose first that I is infinite. We may assume that $N = \{1, 2, 3, \dots\}$ is a subset of I and that the matrices in $M'(R)$ are of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $A \in M'(\Delta_N)$. We shall use the notation $\iota_\emptyset = \emptyset$.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ & . & . & . & & \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ & . & . & . & \end{bmatrix},$$

cf. [**6**, IX.10, Exercise 3]. Then

$$AB = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ & . & . & . & \end{bmatrix}, \qquad BA = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ & . & . & . & \end{bmatrix},$$

so that $AB = \iota_N \neq BA$. Now letting

$$A' = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \iota_{I \setminus N} \end{array}\right], \qquad B' = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & \iota_{I \setminus N} \end{array}\right],$$

we get $A'B' = \iota_I \neq B'A'$, and $M(R')$ is not directly finite.

Since $M(R') \cong M(R)$, by contrapositive, we deduce that if $M(R)$ is directly finite, then I must be finite. The converse follows from [**5**, Proposition 5.2]. □

6. Isomorphisms of the translational hulls of Rees matrix rings

We shall need the following simple result.

Lemma 6.1. *Let R and R' be rings and χ be an isomorphism of R onto R'. For $(\lambda, \rho) \in \Omega(R)$, define the functions $\overline{\lambda}$ and $\overline{\rho}$ by*

$$\overline{\lambda}x = (\lambda(x\chi^{-1}))\chi, \qquad x\overline{\rho} = ((x\chi^{-1})\rho)\chi \qquad (x \in R').$$

Then the mapping

$$\overline{\chi} : (\lambda, \rho) \to (\overline{\lambda}, \overline{\rho}) \qquad ((\lambda, \rho) \in \Omega(R))$$

is an isomorphism of $\Omega(R)$ onto $\Omega(R')$ such that for all $r \in R$, $\pi_r\overline{\chi} = \pi_{r\chi}$.

Proof. See [**11**, Lemma II.7.2]. □

We are now ready for isomorphisms of rings $M(R)$ and $M(R')$ for Rees matrix rings R and R'. For R we write μ, τ and π and for R' we write μ', τ' and π' respectively, for the mappings introduced in the preceding section.

Theorem 6.2. *Let*

$$\chi = \chi(U, \omega, V) : R = \mathcal{M}(I, \Delta, \Lambda; P) \to R' = \mathcal{M}(I', \Delta', \Lambda'; P')$$

be an isomorphism where R is not a division ring.

(i) *The mapping $\widehat{\chi}$ defined by*

$$\widehat{\chi} : (A, B) \to (U, V)^{-1}(A\omega, B\omega)(U, V) \qquad ((A, B) \in M(R))$$

is an isomorphism of $M(R)$ onto $M(R')$.

(ii) $\widehat{\chi}^{-1} = \widehat{\chi^{-1}}$.

(iii) *The following diagram commutes*

$$\begin{tikzcd}
M(R) \arrow[rrr, "\hat{\chi}"] \arrow[dd, "\mu"'] & & & M(R') \arrow[dd, "\mu'"] \\
& R \arrow[ul, "\tau"'] \arrow[dl, "\pi"] \arrow[r, "\chi"] & R' \arrow[ur, "\tau'"] \arrow[dr, "\pi'"'] & \\
\Omega(R) \arrow[rrr, "\bar{\chi}"] & & & \Omega(R')
\end{tikzcd}$$

Proof. (i) The multiplicative part of the argument is the same as in the proof of [**12**, Theorem 4.5(i)] and can be omitted here. By [**11**, Theorem II.1.25, Corollary II.1.27, Corollary II.7.5] and Theorem 5.9(i), we conclude that the rings $M(R)$ and $M(R')$ have unique addition. Hence the (multiplicative) isomorphism of $\mathfrak{M}R$ onto $\mathfrak{M}R'$ is additive and thus an isomorphism of R onto R'.

(ii) The argument here is the same as in [**12**, Theorem 4.5(ii)].

(iii) This follows from Theorem 5.9(iii), [**12**, Theorem 4.5(iii)] and Lemma 6.1. □

For a Rees matrix ring R, we write

$$M_0(R) = R\tau,$$

see Lemma 5.8 for the mapping τ. In view of [**11**, Corollary II.1.27], Theorem 6.2 implies that $M_0(R)$ is the socle of $M(R)$ and $M(R)$ is a maximal essential extension of $M_0(R)$. As a kind of converse of Theorem 6.2, we have the following result.

Theorem 6.3. *Let R and R' be Rees matrix rings where R is not a division ring.*

(i) Every isomorphism of $M(R)$ onto $M(R')$ maps $M_0(R)$ onto $M_0(R')$ and is of the form as in Theorem 6.2(i).

(ii) Every isomorphism of $\mathfrak{M}M_0(R)$ onto $\mathfrak{M}M_0(R')$ extends uniquely to a homomorphism of $M(R)$ into $M(R')$ and this extension is actually an isomorphism of $M(R)$ onto $M(R')$.

Proof. The argument in the proof of [**12**, Theorem 4.6] **mutatis mutandis** carries over to this case. The precise argument is omitted. □

In the next three items we set $R = \mathcal{M}(I, \Delta, \Lambda; P)$. The proofs of these statements are essentially identical to those in [**12**] for Rees matrix semigroups. Recall Definition 4.1.

Lemma 6.4. *Let $\chi = \chi(U, \varepsilon_c, V)$ be an automorphism of R. Then $\widehat{\chi} = \varepsilon_{(cU,cV)}$.*

Proof. See [**12**, Lemma 4.7]. □

Proposition 6.5. *Let $\chi = \chi(U, \omega, V)$ be an automorphism of R. Then $\omega \in \mathcal{I}(\Delta)$ if and only if $\widehat{\chi} \in \mathcal{I}(M(R))$.*

Proof. See [**12**, Proposition 4.8]. □

Proposition 6.6. *Denote by $Z(A)$ the center of any algebra A. Then*

$$Z(M(R)) = \{(c\iota_I, c\iota_\Lambda) \mid c \in Z(\Delta)\} \cong Z(\Delta).$$

Proof. See [**12**, Proposition 4.9]. □

7. r-maximal Rees matrix rings

Recall from Section 3 that a ring containing a minimal one-sided ideal is termed atomic. We have defined in [**11**] a simple atomic ring R to be **r-maximal** if R can not be embedded as a proper right ideal in a simple atomic ring. From [**12**] we have an analogue of this concept for Rees matrix semigroups. Indeed, let $S = \mathcal{M}^o(I, \Delta, \Lambda; P)$. Then P is **column tight** if P has no identical columns. If so, S is an **r-maximal** Rees matrix semigroup if the matrix P can not be augmented to a column tight $\Lambda \times I'$ - matrix P' over G^o with $I' \supset I$ and $P'|_{\Lambda \times I} = P$.

Recall that a subspace U of $V^\star$ is a **t-subspace** if for any $v \in V$, $v \neq 0$, there exists $f \in U$ such that $vf \neq 0$. The next result involves all these concepts.

Theorem 7.1. *Let V be a left vector space over a division ring Δ and U be a t-subspace of $V^\star$. Let Z be a basis of V and W be a basis of U. Also set*

$$R = \mathcal{M}(W, \Delta, Z; P), \; P = (p_{zw}), \; p_{zw} = zw,$$
$$S = \mathcal{M}(U\backslash\{0\}, \Delta, Z; Q), \; Q = (q_{zu}), \; q_{zu} = zu.$$

Then the following statements are equivalent.

(i) $U = V^{\star}$.
(ii) *R is an r-maximal Rees matrix ring.*
(iii) *S is an r-maximal Rees matrix semigroup.*

Proof. The equivalence of parts (i) and (ii) follows easily from [**11**, Theorems II.2.8 and II.3.5].

(i) implies (iii). We shall apply [**12**, Theorem 6.4]. Hence let $\varphi : Z \to \Delta$ be a function such that $z\varphi \neq 0$ for some $z \in Z$. Then φ may be extended uniquely to a linear function f from V to Δ, the latter considered as a vector space over itself. Then f is a linear form on V and thus $f \in V^{\star}$. By hypothesis, we get that $f \in U \backslash \{0\}$ and for all $z \in Z$, we have $p_{zf} = zf = z\varphi$. Now Q being clearly column tight, the cited result yields that S is an r-maximal Rees matrix semigroup.

(iii) implies (i). Let $f \in V^{\star} \backslash \{0\}$. Then $\varphi = f|_Z$ is a nonzero mapping from Z to Δ. The cited result, in view of the hypothesis, yields the existence of $u \in U \backslash \{0\}$ such that $q_{zu} = z\varphi$ for all $z \in Z$. But then $zu = z\varphi$ for all $z \in Z$ which by linearity implies that $u = f$. Hence $f \in U$ which implies that $U = V^{\star}$. □

The next result treats general Rees matrix rings.

Proposition 7.2. *Let $R = \mathcal{M}(I, \Delta, \Lambda; P)$. Then R is r-maximal if and only if for every mapping $\varphi : \Lambda \to \Delta$, there exist elements $i_k \in I$ and $g_k \in \Delta$, $k = 1, 2, \ldots, n$, such that $\lambda\varphi = p_{\lambda i_1} g_1 + p_{\lambda i_2} g_2 + \cdots + p_{\lambda i_n} g_n$ for all $\lambda \in \Lambda$.*

Proof. According to [**11**, Theorem II.2.8], we may represent R as $\mathcal{F}_U(V)$ and in view of [**11**, Theorem II.3.5], R is r-maximal if and only if $U = V^{\star}$. The last condition is equivalent to:

for every $f \in V^{\star}$, there exists $u \in U$ such that $vf = (v, u)$ for all $v \in V$. Now using the basis Z of V, we can write equivalently

for every $\psi : Z \to \Delta$, there exists $u \in U$ such that $z_\lambda \psi = (z_\lambda, u)$ for all $z_\lambda \in Z$ since such a ψ extends uniquely to a linear form on V. Since the basis elements in Z are indexed by Λ, we may rewrite this condition in the form:

for every $\varphi : \Lambda \to \Delta$, there exists $u \in U$ such that $\lambda\varphi = (z_\lambda, u)$ for all $\lambda \in \Lambda$. We can now use the basis W of U to write such a u as a linear combination of basis vectors, say

for every $\varphi : \Lambda \to \Delta$, there exist $w_{i_k} \in W$ and $g_k \in \Delta$, $k = 1, 2, \ldots, n$, such that

$$\lambda\varphi = \left(z_\lambda, \sum_{k=1}^{n} w_{i_k} g_k \right) = \sum_{k=1}^{n} (z_\lambda, w_{i_k}) g_k = \sum_{k=1}^{n} p_{\lambda i_k} g_k$$

as asserted. □

For isomorphisms of r-maximal Rees matrix rings, we have the following simple result.

Proposition 7.3. *Let*

$$R = \mathcal{M}(I, \Delta, \Lambda; P) \quad \text{and} \quad R' = \mathcal{M}(I', \Delta', \Lambda'; P')$$

be r-maximal Rees matrix rings. Then $R \cong R'$ if and only if $\Delta \cong \Delta'$ and $|\Lambda| = |\Lambda'|$.

Proof. We have observed in Theorem 3.4 that there exists a pair of dual vector spaces (U, V) over the given division ring Δ such that there exists an isomorphism η of $\mathcal{F}_U(V)$ onto R. In view of [**11**, Theorem II.3.5] and its proof, η is an isomorphism of $\mathcal{F}(V)$ onto R if and only if R is r-maximal. The hypothesis then implies that

$$\eta : \mathcal{F}(V) \to R, \qquad \eta' : \mathcal{F}(V') \to R'$$

in the obvious notation. Hence $\mathcal{F}(V) \cong \mathcal{F}(V')$ if and only if $R \cong R'$. It follows from [**11**, Corollary I.5.15] that

$$\mathcal{F}(V) \cong \mathcal{F}(V') \Leftrightarrow \Delta \cong \Delta', \ \dim V = \dim V'.$$

But $\dim V = |\Lambda|$ and $\dim V' = |\Lambda'|$. Now combining these statements, we get the assertion. □

Proposition 7.3 indicates that an r-maximal Rees matrix ring $\mathcal{M}(I, \Delta, \Lambda; P)$ depends only on Δ and Λ so that these determine the remaining two parameters I and P. Indeed, $|I| = \dim V^\star$ according to the proof of Proposition 7.3 and P is then defined by the bilinear form associated to the dual pair $(V^\star, V)$.

The next result concerns general simple atomic rings. Recall that if a ring R is a subring of the rings R' and R'', then an isomorphism of R' onto R'' which leaves R elementwise fixed is an **R-isomorphism** and R' and R'' are said to be **R-isomorphic**.

Theorem 7.4. *Let R be a simple atomic ring.*

(i) R is a right ideal of an r-maximal simple atomic ring.

(ii) If R is a right ideal of r-maximal simple atomic rings R' and R'', then R' and R'' are R-isomorphic.

Proof. (i) By [**11**, Theorem II.2.8], R is isomorphic to $\mathcal{F}_U(V)$ for some pair of dual vector spaces (U, V) over a division ring Δ. By [**11**, Theorem I.3.4], $\mathcal{F}_U(V)$ is a right ideal of $\mathcal{F}(V)$. Finally, by [**11**, Theorem II.3.5], $\mathcal{F}(V)$ is an r-maximal simple atomic ring. Hence R is a isomorphic to a right ideal of the r-maximal simple atomic ring $\mathcal{F}(V)$. The assertion follows.

(ii) By [**11**, Theorem II.3.5], there exist left vector spaces V' over Δ' and V'' over Δ'' and isomorphisms $\varphi' : R' \to \mathcal{F}(V')$ and $\varphi'' : R'' \to \mathcal{F}(V'')$. Letting $\psi' = \varphi'|_R$ and $\psi'' = \varphi''|_R$, we get that $R\psi'$ is a right ideal of $\mathcal{F}(V')$ and $R\psi''$ is a right ideal of $\mathcal{F}(V'')$. By [**11**, Theorem I.3.4], there exist t-subspaces U' of $V'^\star$ and U'' of $V''^\star$ such that $R\psi' = \mathcal{F}_{U'}(V')$ and $R\psi'' = \mathcal{F}_{U''}(V'')$. The mapping $\chi = {\psi'}^{-1}\psi''$ is an isomorphism of $\mathcal{F}_{U'}(V')$ onto $\mathcal{F}_{U''}(V'')$. In view of [**11**, Theorem I.5.12], χ is induced by a semilinear isomorphism (ω, a) of V' onto V'' with an adjoint $b : U'' \to U'$. But then $\zeta_{(\omega,a)} : c \to a^{-1}ca$ is a semilinear isomorphism of $\mathcal{F}(V')$ onto $\mathcal{F}(V'')$. Hence the mapping $\varphi'\zeta_{(\omega,a)}{\varphi''}^{-1}$ is an isomorphism of R' onto R'' whose restriction to R equals

$$\psi'(\zeta_{(\omega,a)}\ |_{\mathcal{F}_{U'}(V')}){\psi''}^{-1} = \psi'\chi{\psi''}^{-1} = \psi'{\psi'}^{-1}\psi''{\psi''}^{-1}$$

the identity map on R since $\zeta_{(\omega,a)}$ extends χ. Therefore $\varphi'\zeta_{(\omega,a)}{\varphi''}^{-1}$ is an R-isomorphism of R' onto R''. □

We conclude by considering direct finiteness of $M(R)$ where R is an r-maximal simple atomic ring.

Proposition 7.5. *Let $R = \mathcal{M}(I, \Delta, \Lambda; P)$ be r-maximal. Then $M(R)$ is directly finite if and only if I is finite.*

Proof. According to [**11**, Theorem II.3.5], we have $R \cong \mathcal{F}(V)$ for a left vector space V over Δ with $|\Lambda| = \dim V$, so by Theorem 5.9, we obtain that $M(R) \cong \mathcal{L}(U)$. Suppose that I is infinite. Then so is Λ and hence V is infinite dimensional. The same idea as in the proof of Proposition 5.10 produces here two linear transformations φ and ψ with the property that $\varphi\psi = \iota_V \neq \psi\varphi$. Hence $\mathcal{L}(V)$ is not directly finite.

Necessity of the condition now follows by contrapositive. Sufficiency follows from [**5**, Proposition 5.2]. □

Proposition 7.5 is a faithful analogue of Proposition 5.10. The general case, namely when is $M(\mathcal{M}(I, \Delta, \Lambda; P))$ directly finite remains open.

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