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# ERGODIC DYNAMICAL SYSTEMS CONJUGATE TO THEIR COMPOSITION SQUARES

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Abstract. We investigate the question of when an ergodic automorphism $T$ is conjugate to its composition square $T^2$, i.e., when does there exist an automorphism $S$ with the property that $ST = T^2S$. This is a non–generic property of automorphisms which seems to be quite exceptional. The situation for ergodic automorphisms having discrete spectrum and automorphisms having the weak closure property is investigated.

## 0. Introduction

Let $T$ be an invertible measure–preserving transformation (*automorphism*) defined on a standard Borel probability space $(X, \mathcal{F}, \mu)$. We investigate the question of when $T$ is isomorphic to its composition square $T^2$. If the conjugating automorphism is $S$, i.e., $ST = T^2S$, properties of $S$ are also investigated. There are some well known cases of automorphisms conjugate to their squares. For example, the map arising from the horocycle flow and the Bernoulli shift of infinite entropy. Maps having finite non–zero entropy cannot have this property because of the identity $h(T^2) = 2h(T)$. Consequently, we are mainly interested in maps having zero entropy. It was shown by del Junco (1981) that the property of being conjugate to its square, is a non–generic property of automorphisms.

In Section 2 we give some basic results concerning automorphisms conjugate to their squares. In particular we show that if $T$ is conjugate to $T^2$ and $C(T)$ is abelian then $C(T) = C(T^2)$. In Section 3 we consider the situation for transformations having the weak closure property. This includes the rank one transformations and, in particular, ergodic transformations having discrete spectrum. In this case, if $T$ is conjugate to its square, every member of the centralizer of $T$, $C(T)$, has a unique square root in $C(T)$. It is an open problem whether or not there exists a weakly mixing transformation having rank one and conjugate to its square.

In Section 4 we consider the situation for ergodic automorphisms having discrete spectrum. The Discrete Spectrum Theorem tells us that $T$ is conjugate to $T^2$ if they have the same eigenvalue group and they are both ergodic. New necessary and sufficient conditions for ergodic $T$ with discrete spectrum to be conjugate to its square are given, and properties of the conjugating automorphism $S$ are studied.

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It is shown that $S$ can be mixing. Much of our exposition generalizes to the case where there are automorphisms $S$ and $T$ satisfying $ST = T^pS$ for some $p > 1$. I wish to thank the referee for some helpful suggestions.

## 1. Preliminaries

By a *dynamical system* we mean a 4–tuple $(X, \mathcal{F}, \mu, T)$ consisting of an automorphism $T : (X, \mathcal{F}, \mu) \to (X, \mathcal{F}, \mu)$ defined on a non-atomic standard Borel probability space. The identity automorphism will be denoted by $I$. The group of all automorphisms $\mathrm{Aut}(X)$ of $(X, \mathcal{F}, \mu)$, becomes a completely metrizable topological group when endowed with weak convergence of transformations ($T_n \to T$ if for all $A \in \mathcal{F}$, $\mu(T_n^{-1}(A) \triangle T^{-1}(A)) + \mu(T_n(A) \triangle T(A)) \to 0$ as $n \to \infty$). Denote by $C(T)$ the *centralizer* of $T$, i.e., the set of those members of $\mathrm{Aut}(X)$ which commute with $T$ (more generally it is usual to define $C(T)$ to be those measure–preserving transformations which commute with $T$, but it will be convenient to assume that $C(T)$ is a group).

If $S$ is measure–preserving and $ST = T^2S$ then $ST^n = T^{2n}S$ for $n \in \mathbb{Z}$ and $S^nT = T^{2^n}S^n$ for all $n \in \mathbb{Z}^+$.

## 2. Basic Results

It was shown by del Junco (1981) that the property of being conjugate to the composition square is non–generic in $\mathrm{Aut}(X)$, and in fact it is difficult to find examples with this property. If $T$ is ergodic, then $T^2$ has to be ergodic, so $-1$ cannot be an eigenvalue of $T$.

If $S^n = I$ for some $n \in \mathbb{Z}^+$, then the equation $S^nT = T^{2^n}S^n$ implies that $T^{2^n-1} = I$. This is impossible for $T$ *aperiodic* ($T$ is aperiodic if the set $\{x \in X : T^nx = x\}$ has $\mu$–measure zero for each $n \in \mathbb{Z}^+$). We improve on this result slightly in **Proposition 1** below, where we show that for $T$ aperiodic, $S$ is necessarily aperiodic.

**Proposition 1.** *If $ST = T^2S$ for automorphisms $S$ and $T$ with $T$ aperiodic, then $S$ is aperiodic.*

*Proof.* Set $A = \{x \in X : Sx = x\}$, then if $x \in A, n \in \mathbb{Z}$, $ST^nx = T^{2n}Sx = T^{2n}x$, so $T^nx \notin A$ for otherwise $T^nx = T^{2n}x$ or $T^nx = x$, which is impossible (mod 0) for $T$ aperiodic. Consequently $A \cap T^nA = \emptyset$ for all $n \in \mathbb{Z}$, and we deduce that $T^nA \cap T^mA = \emptyset$ for all $m \neq n$ in $\mathbb{Z}$.

We have shown that the sets $T^nA$, $n \in \mathbb{Z}$ are pairwise disjoint, so if $\mu(A) > 0$ we get a contradiction, since $T$ is measure preserving and $\mu(X) = 1$.

Similarly, if we put $A_n = \{x \in X : S^nx = x\}$, using $S^nT^m = T^{2^nm}S^n$, we again deduce that $\mu(A_n) = 0$ for all $n \in \mathbb{Z}^+$, so that $S$ is aperiodic. □

Although $S$ has to be aperiodic, it need not be ergodic even when $T$ is ergodic, as we shall see for discrete spectrum $T$. On the other hand, if $S$ is ergodic we obtain:

**Proposition 2.** *Suppose that $S$ and $T$ are automorphisms with $ST = T^2S$. If $S$ is ergodic, then either $T$ is aperiodic or $T^n = I$ for some $n > 0$ odd.*

*Proof.* Suppose that $T$ is not aperiodic, then there exists $n > 0$ for which

$$A_n = \{x \in X : T^n x = x\} \quad \text{has} \quad \mu(A_n) > 0.$$

If $x \in A_n$, then $T^n S^{-1}x = S^{-1}T^{2n}x = S^{-1}x$, so that $S^{-1}x \in A_n$, i.e., $A_n$ is $S^{-1}$ invariant, and $S$ ergodic implies $T^n = I$. If the order of $T$ is even, say $n = 2m$, then $ST^m = T^{2m}S = S$ which gives $T^m = I$, a contradiction. □

In a similar way to above, it can be shown that if $ST = T^2S$ where $S$ is ergodic, but $T$ is not ergodic, then $S$ has a proper factor.

**Proposition 3.** *Suppose that $T$ is an automorphism for which $KT = T^2K$ for some automorphism $K$. Define a map*

$$\Phi : C(T) \to C(T^2) \quad \text{by} \quad \Phi(S) = KSK^{-1},$$

*then*

(a) *$\Phi$ is a continuous group isomorphism.*

(b) *$T$ has a square root in $C(T)$, conjugate to $T$.*

(c) *If $C(T)$ is abelian, then $\Phi$ is independent of the conjugating automorphism $K$ and $C(T) = C(T^2)$. Furthermore, $T$ has a unique square root in $C(T)$ conjugate to $T$.*

*Proof.* (a) Let $S \in C(T)$, then

$$\Phi(S)T^2 = KSK^{-1}T^2 = KSTK^{-1} = KTSK^{-1} = T^2KSK^{-1} = T^2\Phi(S),$$

so that $\Phi(S) \in C(T^2)$.

Clearly $\Phi$ is one–to–one, and it is a homomorphism since

$$\Phi(RS) = KRSK^{-1} = KRK^{-1}KSK^{-1} = \Phi(R)\Phi(S).$$

$\Phi$ is onto, for if $R \in C(T^2)$, set $S = K^{-1}RK$, then $\Phi(S) = R$ and $S \in C(T)$.

The continuity of $\Phi$ is a consequence of the continuity of multiplication in a topological group.

(b) Since $\Phi : C(T) \to C(T^2)$ is a group isomorphism with $C(T) \subseteq C(T^2)$, there is an $S \in C(T)$ with $\Phi(S) = T$, i.e., $KSK^{-1} = T$, or $S = K^{-1}TK$, so that

$$S^2 = K^{-1}TKK^{-1}TK = K^{-1}KT = T,$$

i.e., $S$ is a square root of $T$ which is conjugate to $T$.

(c) Suppose that $K_iT = T^2K_i$ for $i = 1, 2$, then $K_2^{-1}K_1 \in C(T)$, and since $C(T)$ is abelian

$$(K_2^{-1}K_1)S = S(K_2^{-1}K_1) \;\; \forall S \in C(T),$$

or $K_1SK_1^{-1} = K_2SK_2^{-1}$, which says that $\Phi$ is independent of the conjugating map.

Since $\Phi$ is independent of the conjugating map $K$, the square root $S$ of $T$ from (b) above is the unique square root which is conjugate to $T$. Write $S = T^{\frac{1}{2}}$, then for $R \in C(T)$

$$\Phi(R)T = KRK^{-1}T = KRT^{\frac{1}{2}}K^{-1} = KT^{\frac{1}{2}}RK^{-1} = TKRK^{-1} = T\Phi(R),$$

(since $C(T)$ is abelian), so that $\Phi(R) \in C(T)$ and it follows that $C(T) = C(T^2)$. $\square$

**Examples** 1. An irrational rotation $T_\alpha : [0,1) \to [0,1)$, $T_\alpha x = x + \alpha (\text{mod } 1)$, cannot be conjugate to its square as $T_\alpha$ and $T_\alpha^2$ have distinct eigenvalue groups. Necessary and sufficient conditions for an ergodic automorphism having discrete spectrum to be conjugate to its square are given in Section 4.

2. The square root of $T$ in the above proposition need not be unique. For example, suppose that $KT = T^2K$ and $S^2 = T$, then $T \times T$ is also conjugate to its square and $S \times S$ is a square root. However, $R(x,y) = (y, Tx)$ is also a square root of $T \times T$ which is not isomorphic to $S \times S$. In fact $T$ may have more than one square root in $C(T)$ (even when $C(T)$ is abelian), only one of which is conjugate to $T$.

3. Maps having a certain type of "trivial" centralizer cannot be conjugate to their squares. Substitutions and many Morse automorphisms have a centralizer which is an abelian group of the form $\{T^n\sigma^k : n \in \mathbb{Z},\ 0 \leq k \leq m-1\}$, where $\sigma^m = I$. If $C(T)$ is a finitely generated abelian group, then $T$ cannot be conjugate to its square:

**Proposition 4.** *Suppose that $T$ is an automorphism whose centralizer is a finitely generated abelian group, then $T$ cannot be conjugate to its square unless $T^m = I$ for some $m \in \mathbb{Z}^+$.*

*Proof.* By Proposition 3 there are unique transformations $\{T_i\}_{i=0}^\infty$ in $C(T)$ such that $T_0 = T$ and for each $i > 0$

$$T_i^2 = T_{i-1}.$$

But since $C(T)$ is a finitely generated abelian group, these $T_i$ cannot all be distinct, so either $T = I$, or for some $i \neq j$, $T_i = T_j$, which would imply that for some $m > 1$, $T^m = I$. $\square$

4. We give a new construction of an automorphism conjugate to its square. Start with a weakly mixing transformation $T : X \to X$ on a Lebesgue probability space as usual. We assume that $T$ has an infinite square root chain (i.e. $T^{\frac{1}{2}}, T^{\frac{1}{4}}$ etc., all exist). Define $\tilde{T} : \Omega \to \Omega$ where $\Omega = \Pi_{-\infty}^{\infty} X$ by

$$\tilde{T}(\ldots, x_{-1}, \overset{*}{x_0}, x_1, x_2, \ldots) = (\ldots, T^{\frac{1}{2}}x_{-1}, \overset{*}{T}x_0, T^2x_1, T^4x_2, \ldots).$$

If $S : \Omega \to \Omega$ is the left shift map

$$S(\ldots, x_{-1}, \overset{*}{x_0}, x_1, x_2, \ldots) = (\ldots, x_0, \overset{*}{x_1}, x_2, \ldots),$$

then we can check that $S\tilde{T} = \tilde{T}^2S$. The map $\tilde{T}$ may be realized as a homeomorphism of the Hilbert cube $[0,1]^{\mathbb{Z}}$.

5. We now show that finite rank mixing maps cannot be conjugate to their squares. This will follow from a slightly more general result concerning locally rank one maps which have no partial rigidity. It is known that if $T$ has local rank one and is mixing, then $T$ has finite joining rank (King 1988). Using the results of King and Thouvenot (1991) concerning the centralizer of such maps, it can be seen that they cannot be conjugate to their squares. More generally:

**Theorem 1.** *If $T$ has local rank one with no partial rigidity, then $T$ cannot be conjugate to its square.*

*Proof.* We use a result from Goodson and Ryzhikov (1997): If $T$ has local rank one with constant $\beta > 0$, and no partial rigidity and if $S \in C(T)$, then there exists $p \in \mathbb{Z}^+$, $p < 1/\beta$ with $S^p = T^m$ for some $m \in \mathbb{Z}$.

Now if $KT = T^2K$, then $S_1 = K^{-1}TK \in C(T)$, and inductively defining $S_{n+1} = K^{-1}S_nK \in C(T)$, we see that $S_n^{2^n} = T$ and that there is no smaller $p < 2^n$ satisfying $S_n^p = T^m$ for any $m \in \mathbb{Z}$. Choosing $n$ so that $2^n > 1/\beta$, we obtain a contradiction. □

## 3. Maps with the Weak Closure Property

Suppose that $T$ is rank one and there is an automorphism $S$ with $ST = T^2S$. From the dichotomy of J. King (1986), the centralizer $C(T)$ of $T$ is either trivial ($C(T) = \{T^n : n \in \mathbb{Z}\}$), or $T$ is rigid (there is a sub–sequence $k_n$ of integers such that $T^{k_n} \to I$ as $n \to \infty$), so the centralizer is uncountable. Now if $C(T)$ is trivial, with $T^2$ conjugate to $T$, $T^2$ must also have rank one, so must also have a trivial centralizer, which is impossible as $T \in C(T^2)$. Consequently, $T$ must be rigid.

Let $\mathrm{WC}(T)$ denote the weak closure of the powers of the ergodic transformation $T$, i.e., the closure in $\mathrm{Aut}(X)$ of the set $\{T^n : n \in \mathbb{Z}\}$. We say that $T$ has the weak closure property if $\mathrm{WC}(T) = C(T)$. It is known (King (1986)) that any rank one map has the weak closure property, so any ergodic transformation having discrete spectrum has the weak closure property. It is clear that if $T$ has this property, then $C(T)$ is an abelian group. King (1986) also showed that if $T^2$ is rank one then $C(T) = C(T^2)$.

**Proposition 5.** *Suppose that $T$ and $K$ are automorphisms with $KT = T^2K$, and $\Phi : C(T) \to C(T^2)$ is the group isomorphism defined by $\Phi(S) = KSK^{-1}$, then:*

(a) $\Phi(WC(T)) = WC(T^2)$.

(b) $\Phi(S) = S^2$ *for all* $S \in WC(T)$.

*Proof.* (a) If $S \in \mathrm{WC}(T)$, then $S = \lim_{i\to\infty} T^{n_i}$, for some sequence $n_i$, so $\Phi(S) = \lim_{i\to\infty} \Phi(T^{n_i}) = \lim_{i\to\infty} T^{2n_i}$, therefore $\Phi(S) \in \mathrm{WC}(T^2)$.

On the other hand, if $R \in \mathrm{WC}(T^2)$, then $R = \lim_{i\to\infty} T^{2n_i}$, so that

$$K^{-1}RK = \lim_{i\to\infty} K^{-1}T^{2n_i}K = \lim_{i\to\infty} T^{n_i} \in \mathrm{WC}(T)$$

and $\Phi(K^{-1}RK) = R$, so the result follows.

(b) If $S \in \mathrm{WC}(T)$, then there is a sequence $n_i$ with $T^{n_i} \to S$ as $i \to \infty$, so also $T^{2n_i} \to S^2$. It follows that both $\Phi(T^{n_i}) = T^{2n_i} \to S^2$ and $\Phi(T^{n_i}) \to \Phi(S)$, and hence $\Phi(S) = S^2$. $\square$

Next we show that if $T$ has the weak closure property with $T$ conjugate to $T^2$, then $C(T) = C(T^2)$ and the map $\Phi$ is independent of the conjugating map $K$ and may be written as $\Phi : C(T) \to C(T)$, $\Phi(S) = S^2$, a continuous isomorphism of the group $C(T)$. It follows that every member of the group $C(T)$ has a unique square root in $C(T)$. In particular, $T$ has a unique infinite square root chain: $T \to T^{1/2} \to T^{1/4} \to \ldots$. A group with the property that every element has a square root is said to be *2–divisible.*

**Theorem 2.** *Let $T$ be conjugate to $T^2$ and $T^n \neq I$ for any $n \in \mathbb{Z} - \{0\}$. Suppose that $T$ has the weak closure property, then the map $\Phi : C(T) \to C(T)$, $\Phi(S) = S^2$ is a group automorphism. Consequently, every member of the uncountable abelian group $C(T)$ has a unique square root in $C(T)$ which is conjugate to its square.*

*Proof.* If $T$ has the weak closure property, then

$$\mathrm{WC}(T^2) \subseteq \mathrm{WC}(T) = C(T) \subseteq C(T^2) = \Phi(C(T)) = \Phi(\mathrm{WC}(T)) = \mathrm{WC}(T^2),$$

so we must have equality throughout. It follows that $T^2$ has the weak closure property and $C(T^2) = C(T)$. The previous theorem implies that $\Phi(S) = KSK^{-1} = S^2$ defines an isomorphism from $C(T)$ to $C(T^2)$ which may be regarded as an automorphism of the group $C(T)$. Consequently, given $S \in C(T)$, there exists a unique $R \in C(T)$ such that $R^2 = S$, i.e., every member of $C(T)$ has a unique square root in C(T) which is conjugate to its square. $\square$

**Remark.** Although every member of $C(T)$ has a unique square root in $C(T)$, a given $S \in C(T)$ may have other square roots. For example, suppose that $T$ is ergodic with discrete spectrum and also conjugate to $T^2$. Then $T$ has the weak closure property and is conjugate to its inverse, i.e., there exists an automorphism $S$ with $ST = T^{-1}S$. Now $I \in C(T)$ with unique square root $I$ in $C(T)$. However, $I$ has other square roots since for example $S^2 = I$ and clearly $S \notin C(T)$ (see Goodson (1999)). However, if $T$ has simple spectrum (for example if $T$ is of rank one), then every root of $T$ is in $C(T)$ because they are "functions of $T$", so that $T$ itself will have a unique square root in this case.

The proof of the next result, which tells us that if $T$ is conjugate to its square and has the weak closure property, then all conjugations are isomorphic, does not generalize to the situation where $T$ is conjugate to $T^3$.

**Proposition 6.** *If $T$ has the weak closure property, then all conjugations between $T$ and $T^2$ are isomorphic.*

*Proof.* Suppose that $S_1$ and $S_2$ are two conjugating maps between $T$ and $T^2$, then $S_2^{-1}S_1 \in C(T)$, and since $T$ has the weak closure property

$$S_2^{-1}S_1 = \lim_{i\to\infty} T^{n_i}$$

for some subsequence $n_i$. Now since $S_1 T^{n_i} = T^{2n_i} S_1$, we deduce, on taking the weak limit, that

$$S_1(S_2^{-1}S_1) = (S_2^{-1}S_1)^2 S_1,$$

or $S_2(S_1S_2^{-1}) = (S_1S_2^{-1})S_1$, so that $S_1$ and $S_2$ are isomorphic via the conjugating map $S_1S_2^{-1}$. □

**Examples.** 1. We apply the above results to show that a $\mathbb{Z}_2$–extension (or any $\mathbb{Z}_n$–extension, $n$ even) which has the weak closure property cannot be conjugate to its square.

**Proposition 7.** *Suppose that $T_\phi$ is a $\mathbb{Z}_n$–extension ($n$ even), of an automorphism $T$. If $T_\phi$ has the weak closure property, then it cannot be conjugate to its square.*

*Proof.* Let $\sigma : X \times \mathbb{Z}_n \to X \times \mathbb{Z}_n$ be the flip map: $\sigma(x, j) = (x, j + 1)$, then $\sigma \in C(T_\phi)$ and $\sigma^n = I$, $n = 2m$ with $\sigma^m \neq I$.

Now from the previous theorem, if $T_\phi$ is conjugate to its square it has a unique square root $S$ in $C(T_\phi)$. But $\sigma^{n/2}S$ is also a square root of $T_\phi$ in $C(T_\phi)$, contradicting the uniqueness. □

In the above Proposition, if it is only assumed that $C(T_\phi)$ is abelian, the above argument will not work as $T_\phi$ may have other square roots in $C(T_\phi)$ (but not necessarily conjugate to $T_\phi$).

2. M. Lemańczyk (1985) has shown a dichotomy for Morse automorphisms which are two point extensions of maps having rational discrete spectrum. They are either of rank one, so have the weak closure property, and cannot be conjugate to their squares by the above proposition, or they have centralizers which are finitely generated abelian groups, so again cannot be conjugate to their squares (Proposition 4). More general types of Morse automorphisms and also substitutions of constant length, typically have centralizers which are finitely generated abelian groups, so cannot be conjugate to their squares.

The situation for a 3–point extension having the weak closure property is not clear since the requirement that the map $\sigma(x, j) = (x, j + 1)$ with $\sigma^3 = I$ be conjugate to its square (and have a square root) does not lead to a contradiction (since for example if $\tau = \sigma^2$, then $\tau^2 = \sigma$).

## 4. Ergodic Automorphisms Having Discrete Spectrum and Conjugate to their Squares

Throughout this section we assume that $T$ is an ergodic automorphism having discrete spectrum, i.e., there is a complete orthonormal basis of eigenfunctions $f_n(x)$, $n \in \mathbb{Z}$ for $L^2(X, \mu)$. Recall the result of Halmos, that $T$ has a square root $S$ (i.e., $S^2 = T$) if and only if $-1$ is not an eigenvalue of $T$. The issue for the existence of a conjugating map between $T$ and $T^2$ is related, but a little more complex:

The discrete spectrum theorem of Halmos and von Neumann (see Halmos (1956)), tells us that we can represent $T$ as a rotation on a compact abelian group, $T : G \to G$ say, defined by $T(g) = a + g$ for some $a \in G$. If we set $S(g) = 2g$ then we see that $ST = T^2S$ since

$$ST(g) = S(a+g) = 2a + 2g, \quad \text{and} \quad T^2S(g) = T^2(2g) = 2a + 2g.$$

It does not follow that $T$ and $T^2$ are conjugate, as $S$ need not be onto or one–to––one (modulo sets of measure zero). If $S$ is onto, then $T^2$ is a factor of $T$, and if $S$ is both one–to–one and onto, then $T$ and $T^2$ are conjugate.

For example, let $S^1$ be the unit circle in the complex plane and define $T : S^1 \to S^1$ by $T(z) = az$ where $a \in S^1$ is not a root of unity. Then $T$ is ergodic, and $S(z) = z^2$ is onto but not one–to–one. Now $ST = T^2S$, so that $T^2$ is a factor of $T$, but they are not conjugate as they have a different eigenvalue group. In this example $S$ is also ergodic (in fact mixing).

In order to study transformations having discrete spectrum, we shall need the following consequences of the discrete spectrum theorem.

**Proposition 8.** *Let $T$ be ergodic with discrete spectrum and suppose that $S$ is an automorphism satisfying $ST = T^2S$. Then $T$ can be represented as a rotation $T : G \to G$, $T(g) = a + g$ on a compact abelian group and $S$ can be represented as $S : G \to G$, $S(g) = b + 2g$ for some $a, b \in G$.*

The above proposition is implicit in the work of Halmos and von Neumann. We use this to prove the following:

**Theorem 3.** *Suppose $ST = T^2S$ where $T$ is ergodic with discrete spectrum. $S$ is ergodic if and only if $T$ has no eigenvalues of finite order.*

*Proof.* Represent $S$ and $T$ on a compact abelian group $G$ by $T(g) = a + g$ and $S(g) = b + 2g$.

Suppose that for some $n > 1$, $\omega$, a primitive $n$th root of unity is an eigenvalue for $T$ and there exists $\chi \in \widehat{G}$ (the character group of $G$) such that $\chi(Tg) = \omega\chi(g)$ for $g \in G$. $n$ has to be odd, for otherwise $-1$ is an eigenvalue for $T$, contradicting the ergodicity of $T^2$.

Then $\chi^n(Tg) = \chi^n(g)$ and $T$ ergodic implies $\chi^n = 1$. Set

$$f(g) = \sum_{k=1}^{n-1} \chi^{k-1}(b)\chi^k(g) = \chi(g) + \chi(b)\chi^2(g) + \ldots + \chi^{n-2}(b)\chi^{n-1}(g),$$

then since $\chi(Sg) = \chi(b)\chi^2(g)$, we have

$$\begin{aligned} f(Sg) = \sum_{k=1}^{n-1} \chi^{k-1}(b)\chi^k(Sg) &= \sum_{k=1}^{n-1} \chi^{k-1}(b)\chi^k(b)\chi^{2k}(g) \\ &= \sum_{k=1}^{n-1} \chi^{2k-1}(b)\chi^{2k}(g) = f(g) \end{aligned}$$

since $n$ has to be odd. But $f \neq$ constant because $\chi, \chi^2, \ldots, \chi^{n-1}$ are orthogonal eigenfunctions, so that $S$ is not ergodic.

Conversely suppose that $T$ has no eigenvalues of finite order. If $f(Tg) = \lambda f(g)$, then

$$f \circ S^n(Tg) = f(T^{2^n} S^n g) = \lambda^{2^n} f(S^n g),$$

so if $f$ is an eigenfunction of $T$, then $f \circ S^n$ is also, but corresponding to a distinct eigenvalue, so they must be orthogonal. Thus if $\chi \in \widehat{G}$, $\chi(Tg) = \chi(a)\chi(g)$, then $\langle \widehat{S}^n(\chi), \chi \rangle = 0$ for all $n \in \mathbb{Z}$, $n \neq 0$. It follows that $S$ has countable Lebesgue spectrum and hence is mixing, and the result follows. $\square$

An immediate consequence of the above proof is the following:

**Corollary 1.** *If $ST = T^2 S$ where $T$ is ergodic having discrete spectrum, but with no eigenvalues of finite order, then $S$ has countable Lebesgue spectrum, and so is mixing.*

Below we give an example of such a mixing conjugation $S$. Recall that since an ergodic discrete spectrum transformation has the weak closure property, all conjugations between $T$ and $T^2$ are isomorphic (in fact del Junco (1976) showed that any ergodic discrete spectrum transformation has rank one). First we give necessary and sufficient conditions for an ergodic discrete spectrum transformation to be conjugate to its square.

**Theorem 4.** *Suppose that $T$ is ergodic with discrete spectrum and eigenvalue group $\Lambda$, then $T$ is conjugate to $T^2$ if and only if the map $\phi : \Lambda \to \Lambda$, $\phi(\lambda) = \lambda^2$ is a group automorphism.*

*Proof.* Suppose $T$ is ergodic with discrete spectrum and conjugate to $T^2$. Then $T^2$ is also ergodic with discrete spectrum. Moreover, the eigenvalue groups $\Lambda(T)$ and $\Lambda(T^2)$ satisfy

$$\Lambda(T^2) = [\Lambda(T)]^2.$$

(that is $\Lambda(T^2) = \{\lambda^2 : \lambda \in \Lambda(T)\}$). Thus the map $\sigma : \Lambda(T) \to \Lambda(T^2)$ defined by $\sigma(\lambda) = \lambda^2$ is an onto homomorphism. If it were not one-to-one, then for some $\lambda \in \Lambda(T) - \{1\}$, $\lambda^2 = 1$, or $\lambda = -1$. But if $-1$ were an eigenvalue of $T$, then $T^2$ would not be ergodic.

Conversely, we may assume that $T$ is a rotation on a compact abelian group $G$ with eigenvalue group $\Lambda$ for which the map $\phi : \Lambda \to \Lambda$, $\phi(\lambda) = \lambda^2$ is a group automorphism. If $\widehat{\phi}$ is the induced map on the character group $\widehat{\Lambda}$, then $\widehat{\phi}$ may be identified with $S : G \to G$, $S(g) = g^2$, again an automorphism of $G$ (written multiplicatively). We then see that $ST = T^2 S$, and the result follows. $\square$

**Examples.** 1. Fix $0 < \theta < 2\pi$ for which $e^{in\theta} \neq 1$ for all $n \in \mathbb{Z} - \{0\}$. Set

$$\Lambda = \{e^{in\theta/2^m} : n \in \mathbb{Z}, m = 1, 2, \ldots\}.$$

$\Lambda$ is a countable subgroup of the circle, so by the discrete spectrum theorem there is an ergodic automorphism $T : G \to G$ ($G$ a compact abelian group) having discrete spectrum, and which has $\Lambda$ as its eigenvalue group.

If we define $\phi : \Lambda \to \Lambda$, by $\phi(\lambda) = \lambda^2$, then $\phi$ is a group automorphism. It follows from the previous theorem that $T$ is conjugate to its square and that any conjugating automorphism is mixing.

2. Let $G =$ the group of 3–adic integers, and $T$ the adding machine

$$T(g) = g + \mathbf{1},$$

where $\mathbf{1} = (1, 0, 0, \ldots)$. Then the map $S(g) = 2g$ is a group automorphism of $G$ which conjugates $T$ to $T^2$ ($S$ is not ergodic in this case, but is aperiodic).

$T$ can be realized as a rank one (rational discrete spectrum) transformation whose eigenvalues are the $3^n$th roots of unity.

3. An example of $ST = T^2S$ with $S$ ergodic and $T$ aperiodic, but not ergodic can be given by taking $T_0$ to be ergodic with discrete spectrum, but no eigenvalues of finite order, and $S_0$ to be mixing with $S_0T_0 = T_0^2S_0$, then $T = T_0 \times T_0$ is not ergodic, but is aperiodic, and $S = S_0 \times S_0$ is mixing, so $S$ and $T$ have the required properties.

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