ALMOST NEARLY CONTINUOUS MULTIFUNCTIONS

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ABSTRACT. Erdal Ekici has introduced and studied nearly continuous multifunctions [Acta Math. Univ. Comenianae, Vol. LXXII, 2 (2003), pp. 1–7]. The purpose of the present paper is to introduce and study upper and lower almost nearly continuous multifunctions as a weaker form of upper and lower nearly continuous multifunctions. Basic characterizations, several properties of upper and lower almost nearly continuous multifunctions are investigated.

1. Introduction and preliminaries

In a recent paper author [5] presented the notion of nearly continuous multifunctions. It is well known that continuity and multifunctions are two of the basic topics in general topology and in set valued analysis. Many authors have researched several forms of continuous multifunctions.

The aim of this paper is to study a new weaker form of continuous multifunctions. In this paper, almost nearly continuity is introduced and studied. Basic properties and theorems of almost nearly continuous multifunctions are investigated.

In this paper, spaces (X, τ) and (Y, υ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ, respectively.

A subset A of a space X is said to be regular open (respectively regular closed) if $A = \mathrm{int}(\mathrm{cl}(A))$ (respectively $A = \mathrm{cl}(\mathrm{int}(A))$) [17].

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By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1], [13] we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup\limits_{x \in A} F(x)$, and F is said to be a surjection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover, $F : X \to Y$ is called upper semi continuous (resp. lower semi continuous) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [13].

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $(\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$ [16].

A topological space (X, τ) is called nearly compact [15] if every cover of X by regular open sets has a finite subcover.

The class of nearly compact spaces is properly placed between the classes of quasi-H-closed (i. e., almost compact) spaces and the spaces satisfying the finite chain condition, i. e., every space satisfying the finite chain condition is nearly compact and every nearly compact space is almost compact. Recall that a space is called quasi-H-closed [14] if every open cover of X has a finite proximate subcover, i. e., finite subfamily the closures of whose members cover X.

Let (X, τ) be a topological space and let A be a subset of X. If every cover of A by regular open subsets of (X, τ) has a finite subfamily whose union covers A, then A is called N-closed (relative to X) [10]. Sometimes, such sets are called N-sets or α-nearly compact. It is known the fact that every compact set is N-closed.

Many authors studied N-closed sets in the literature (for example, see [2], [4], [8], [6]). The class of N-closed sets is important in the study of functions with strongly closed graphs [11].

A subset U of X is called a N-neighbourhood of a point $x \in X$ if there exists an open set V having N-closed complement such that $x \in V \subset U$.

The graph multifunction G_F of a multifunction $F : X \to Y$ is said to be strongly closed if for each $(x, y) \notin G_F(x)$, there exists open sets U and V containing x and containing y, respectively, such that $(U \times \mathrm{cl}(V)) \cap G_F(x) = \emptyset$ [7].

Definition 1. A multifunction $F : X \to Y$ is said to be:

1. Lower nearly continuous [5] at $x \in X$ if for each open set V having N-closed complement such that $x \in F^-(V)$, there exists an open set U containing x such that $U \subset F^-(V)$,
2. Upper nearly continuous [5] at $x \in X$ if for each open set V having N-closed complement such that $x \in F^+(V)$, there exists an open set U containing x such that $U \subset F^+(V)$.
3. Lower (upper) nearly continuous if F has this property at each point of X.

2. Almost nearly continuous multifunctions

In this section, the notion of almost nearly continuous multifunctions is introduced and characterizations and some relationships of almost nearly continuous multifunctions and basic properties of almost nearly continuous multifunctions are investigated and obtained.

Definition 2. A multifunction $F : X \to Y$ is said to be:

1. Lower almost nearly continuous at a point $x \in X$ if for each open set V of Y having N-closed complement such that $x \in F^-(V)$, there exists an open set U containing x such that $U \subset F^-(\mathrm{int}(\mathrm{cl}(V)))$,
2. Upper almost nearly continuous at a point $x \in X$ if for each open set V of Y having N-closed complement such that $x \in F^+(V)$, there exists an open set U containing x such that $U \subset F^+(\mathrm{int}(\mathrm{cl}(V)))$.
3. Lower (upper) almost nearly continuous if F has this property at each point of X.

The following theorem gives some characterizations of upper almost nearly continuous multifunction.

Theorem 3. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(1) *F is upper almost nearly continuous multifunction;*
(2) *For each $x \in X$ and for each open set V having N-closed complement such that $F(x) \subset V$, there exists an open set U containing x such that if $y \in U$, then $F(y) \subset \mathrm{int}(\mathrm{cl}(V))$;*
(3) *For each $x \in X$ and for each regular open set G having N-closed complement such that $F(x) \subset G$, there exists an open set U containing x such that $F(U) \subset G$;*
(4) *For each $x \in X$ and for each closed N-closed set K such that $x \in F^+(Y \backslash K)$, there exists a closed set H such that $x \in X \backslash H$ and $F^-(\mathrm{cl}(\mathrm{int}(K))) \subset H$;*
(5) *$F^+(\mathrm{int}(\mathrm{cl}(V)))$ is an open set for any open set $V \subset Y$ having N-closed complement;*
(6) *$F^-(\mathrm{cl}(\mathrm{int}(K)))$ is a closed set for any closed N-closed set $K \subset Y$;*
(7) *$F^+(G)$ is an open set for any regular open set G having N-closed complement;*
(8) *$F^-(K)$ is a closed set for any regular closed N-closed set K of Y;*
(9) *For each point x of X and each N-neighbourhood V of $F(x)$, $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is a neighbourhood of x;*
(10) *For each point x of X and each N-neighbourhood V of $F(x)$, there exists a neighbourhood U of x such that $F(U) \subset \mathrm{int}(\mathrm{cl}(V))$.*

Proof. (1)$\Leftrightarrow$(2). Obvious.

(2)$\Rightarrow$(3). Let $x \in X$ and G be a regular open set having N-closed complement such that $F(x) \subset G$. By (2), there exists an open set U containing x such that if $y \in U$, then $F(y) \subset \mathrm{int}(\mathrm{cl}(G)) = G$. We obtain $F(U) \subset G$.

(3)$\Rightarrow$(2). Let $x \in X$ and V be an open set having N-closed complement such that $F(x) \subset V$. Then, $\mathrm{int}(\mathrm{cl}(V))$ is a regular open set having N-closed complement. By (3), there exists an open set U containing x such that $F(U) \subset \mathrm{int}(\mathrm{cl}(V))$.

(2)⇒(4). Let $x \in X$ and K be a closed N-closed set of Y such that $x \in F^+(Y \backslash K)$. By (2), there exists an open set U containing x such that $F(U) \subset \mathrm{int}(\mathrm{cl}(Y \backslash K))$. We have

$$\mathrm{int}(\mathrm{cl}(Y \backslash K)) = Y \backslash \mathrm{cl}(\mathrm{int}(K))$$

and

$$U \subset F^+(Y \backslash \mathrm{cl}(\mathrm{int}(K))) = X \backslash F^-(\mathrm{cl}(\mathrm{int}(K))).$$

We obtain $F^-(\mathrm{cl}(\mathrm{int}(K))) \subset X \backslash U$. Take $H = X \backslash U$. Then, $x \in X \backslash H$ and H is a closed set.

(4)⇒(2). It can be obtained similarly as (2)⇒(4).

(1)⇒(5). Let V be any open set having N-closed complement and $x \in F^+(\mathrm{int}(\mathrm{cl}(V)))$. We know that $\mathrm{int}(\mathrm{cl}(V))$ has N-closed complement. By (1), there exists an open set U containing x such that $U \subset F^+(\mathrm{int}(\mathrm{cl}(V)))$. Hence, $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is an open set.

(5)⇒(1). Let V be any open set having N-closed complement and $x \in F^+(V)$. By (5), $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is an open set. Take $U = F^+(\mathrm{int}(\mathrm{cl}(V)))$. Then, $F(U) \subset \mathrm{int}(\mathrm{cl}(V))$. Hence, F is upper almost nearly continuous.

(5)⇒(6). Let K be any closed N-closed set of Y. Then, $Y \backslash K$ is an open set having N-closed complement. By (5), $F^+(\mathrm{int}(\mathrm{cl}(Y \backslash K)))$ is an open set. Since $\mathrm{int}(\mathrm{cl}(Y \backslash K)) = Y \backslash \mathrm{cl}(\mathrm{int}(K))$, it follows that

$$F^+(\mathrm{int}(\mathrm{cl}(Y \backslash K))) = F^+(Y \backslash \mathrm{cl}(\mathrm{int}(K))) = X \backslash F^-(\mathrm{cl}(\mathrm{int}(K))).$$

We obtain that $F^-(\mathrm{cl}(\mathrm{int}(K)))$ is closed in X.

(6)⇒(5). It can be obtained similarly as (5)⇒(6).

(5)⇒(7). Let G be any regular open set having N-closed complement. By (5), $F^+(\mathrm{int}(\mathrm{cl}(G))) = F^+(G)$ is an open set.

(7)⇒(5). Let V be any open set having N-closed complement. Then, $\mathrm{int}(\mathrm{cl}(V))$ is a regular open set having N-closed complement. By (7), $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is an open set.

(6)⇒(8). It can be obtained similarly as (5)⇒(7).

(8)⇒(6). It can be obtained similarly as (7)⇒(5).

(5)⇒(9). Let $x \in X$ and V be a N-neighbourhood of $F(x)$. Then there exists an open set G having N-closed complement such that $F(x) \subset G \subset V$. therefore, we obtain $x \in F^+(G) \subset F^+(V)$. Since $F^+(\mathrm{int}(\mathrm{cl}(G)))$ is an open set, $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is a neighbourhood of x.

(9)⇒(10). Let $x \in X$ and V be a N-neighbourhood of $F(x)$. By (9), $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is a neighbourhood of x. Take $U = F^+(\mathrm{int}(\mathrm{cl}(V)))$. Then $F(U) \subset \mathrm{int}(\mathrm{cl}(V))$.

(10)⇒(1). Let $x \in X$ and V be any open set having N-closed complement such that $F(x) \subset V$. Then V is a N-neighbourhood of $F(x)$. By (10), there exists a neighbourhood U of x such that $F(U) \subset \mathrm{int}(\mathrm{cl}(V))$. Therefore, there exists an open set G such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset \mathrm{int}(\mathrm{cl}(V))$. We obtain that F is upper almost nearly continuous. □

Remark 4. For a multifunction $F : X \to Y$ from a topological space (X, τ) to a topological space (Y, υ), the following implications hold:

$$\begin{array}{c}\text{upper semi-continuity}\\ \Downarrow \\ \text{upper nearly continuity}\\ \Downarrow \\ \text{upper almost nearly continuity}\end{array}$$

Note that none of these implications is reversible. We give an example for the last implication. Another example is given in [5].

Example 5. Let $X = Y = \{a, b, c, d\}$. Let τ and σ be respectively topologies on X and on Y given by

$$\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\} \qquad \text{and} \qquad \sigma = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, b, c\}\}.$$

Define the multifunction

$$F : X \to Y \quad \text{by} \quad F(a) = \{c\},\ F(b) = \{a, b\},\ F(c) = \{d\},\ F(d) = \{a, b\}.$$

Then F is upper almost nearly continuous but not upper nearly continuous, since $\{a, b\} \in \sigma$ having N-closed complement and $F^+(\{a, b\}) = \{b, d\}$ is not open in (X, τ).

The following theorem gives some characterizations of lower almost nearly continuous multifunction.

Theorem 6. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent:*

(1) *F is lower almost nearly continuous multifunction;*
(2) *For each $x \in X$ and for each open set V having N-closed complement such that $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that if $y \in U$, then $F(y) \cap \mathrm{int}(\mathrm{cl}(V)) \neq \emptyset$;*
(3) *For each $x \in X$ and for each regular open set G having N-closed complement such that $F(x) \cap G \neq \emptyset$, there exists an open set U containing x such that if $y \in U$, $F(y) \cap G \neq \emptyset$;*
(4) *For each $x \in X$ and for each closed N-closed set K such that $x \in F^-(Y \backslash K)$, there exists a closed set H such that $x \in X \backslash H$ and $F^+(\mathrm{cl}(\mathrm{int}(K))) \subset H$;*
(5) *$F^-(\mathrm{int}(\mathrm{cl}(V)))$ is an open set for any open set $V \subset Y$ having N-closed complement;*
(6) *$F^+(\mathrm{cl}(\mathrm{int}(K)))$ is a closed set for any closed N-closed set $K \subset Y$;*
(7) *$F^-(G)$ an open set for any regular open set G having N-closed complement;*
(8) *$F^+(K)$ is a closed set for any regular closed N-closed set K of Y.*

Proof. It can be obtained similarly as the previous theorem. □

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Theorem 7. *Let $F : X \to Y$ be a multifunction. F is upper (lower) almost nearly continuous multifunction if and only if for each $x \in X$ and for each net (x_α) which converges to x in X and for each open set $V \subset Y$ having N-closed complement such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), the net (x_α) is eventually in $F^+(\mathrm{int}(\mathrm{cl}(V)))$ (resp. $F^-(\mathrm{int}(\mathrm{cl}(V)))$).*

Proof. We prove only the case for F upper almost nearly continuous, the proof for F lower almost nearly continuous being analogous.

($\Rightarrow$) Let (x_α) be a net which converges to x in X and let $V \subset Y$ be any open set having N-closed complement such that $x \in F^+(V)$. When F is upper almost nearly continuous multifuction, it follows that there exists an open set $U \subset X$ containing x such that $U \subset F^+(\text{int}(\text{cl}(V)))$. Since (x_α) converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in U \subset F^+(\text{int}(\text{cl}(V)))$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^+(\text{int}(\text{cl}(V)))$.

($\Leftarrow$) Suppose that F is not upper almost nearly continuous. Then, there exists a point x and an open set V having N-closed complement with $x \in F^+(V)$ such that $U \not\subseteq F^+(\text{int}(\text{cl}(V)))$ for each open set $U \subset X$ containing x. Let $x_U \in U$ and $x_U \notin F^+(\text{int}(\text{cl}(V)))$ for each open set $U \subset X$ containing x. Hence, $x_U \to x$ but (x_U) is not eventually in $F^+(\text{int}(\text{cl}(V)))$ where (x_U) is a neighbourhood net. This is a contradiction. Thus, F is upper almost nearly continuous multifunction. $\square$

Definition 8. Let (X, τ) be a topological space. X is said to be a N-normal space if for every disjoint closed subsets K and F of X, there exists two open sets U and V having N-closed complements such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$ [5].

Recall that a multifunction $F : X \to Y$ is said to be point closed if, for each $x \in X$, $F(x)$ is closed.

Theorem 9. *If Y is N-normal space and $F_i : X_i \to Y$ is upper almost nearly continuous multifunction such that F_i is point closed for $i = 1, 2$, then a set $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is closed set in $X_1 \times X_2$.*

Proof. Let

$$A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\} \quad \text{and} \quad (x_1, x_2) \in (X_1 \times X_2)\backslash A.$$

Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is N-normal and F_i is point closed for $i = 1, 2$, there exist disjoint open sets V_1, V_2 having N-closed complements such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. Since F_i is upper almost nearly continuous, $F_i^+(\mathrm{int}(\mathrm{cl}(V_i)))$ is open for $i = 1, 2$. Put

$$U = F_1^+(\mathrm{int}(\mathrm{cl}(V_1))) \times F_2^+(\mathrm{int}(\mathrm{cl}(V_2))),$$

then U is open and $(x_1, x_2) \in U \subset (X_1 \times X_2)\backslash A$.

This shows that $(X_1 \times X_2) \setminus A$ is open and hence A is closed in $X_1 \times X_2$. □

Theorem 10. *Let F and G be upper almost nearly continuous and point closed multifunctions from a topological space X to a N-normal topological space Y. Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.*

Proof. Let $x \in X\backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a N-normal space, then there exist disjoint open sets U and V having N-closed complements containing $F(x)$ and $G(x)$, respectively. Since F and G are upper almost nearly continuous, then the sets $F^+(\mathrm{int}(\mathrm{cl}(U)))$ and $G^+(\mathrm{int}(\mathrm{cl}(V)))$ are open and contain x. Let

$$H = F^+(\mathrm{int}(\mathrm{cl}(U))) \cup G^+(\mathrm{int}(\mathrm{cl}(V))).$$

Then H is an open set containing x and $H \cap K = \emptyset$. Hence, K is closed in X. □

Theorem 11. *Let $F : X \to Y$ be a multifunction and let U be a subset in X. If F is a lower (upper) almost nearly continuous, then the restriction multifunction $F|_U : U \to Y$ is a lower (resp. upper) almost nearly continuous.*

Proof. We prove only the case for $F|_U$ lower almost nearly continuous, the proof for $F|_U$ upper almost nearly continuous being analogous.

Suppose that V is an open set having N-closed complement. Let $x \in U$ and let $x \in (F|_U)^-(V)$. Since F is lower almost nearly continuous multifunction, then there exists an open set G such that $x \in G \subset F^-(\mathrm{int}(\mathrm{cl}(V)))$. Then $x \in G \cap U$ and $G \cap U$ is open in U. Moreover,

$$G \cap U \subset (F|_U)^-(\mathrm{int}(\mathrm{cl}(V))).$$

Hence, we show that the restriction multifunction $F|_U$ is a lower almost nearly continuous. □

Theorem 12. *Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of a space X. Then a multifunction $F : X \to Y$ is upper almost nearly continuous (resp. lower almost nearly continuous) if and only if the restriction $F\,|_{U_\alpha} : U_\alpha \to Y$ is upper almost nearly continuous (resp. lower almost nearly continuous) for each $\alpha \in \Lambda$.*

Proof. We prove only the case for F upper almost nearly continuous, the proof for F lower almost nearly continuous being analogous.

($\Rightarrow$) Let $\alpha \in \Lambda$ and V be any open set having N-closed complement. Since F is upper almost nearly continuous, $F^+(\mathrm{int}(\mathrm{cl}(V)))$ is open in X. We have

$$(F\,|_{U_\alpha})^+(\mathrm{int}(\mathrm{cl}(V))) = F^+(\mathrm{int}(\mathrm{cl}(V))) \cap U_\alpha$$

is open in U_α and hence $F\,|_{U_\alpha}$ is upper almost nearly continuous.

($\Leftarrow$) Let V be any open set having N-closed complement. Since F_α is upper almost nearly continuous for each α, from Theorem 3,

$$F_\alpha^+(\mathrm{int}(\mathrm{cl}(V))) \subset \mathrm{int}_{H_\alpha}(F_\alpha^+(\mathrm{int}(\mathrm{cl}(V))))$$

and since H_α is open, we have

$$F^+(\mathrm{int}(\mathrm{cl}(V))) \cap H_\alpha \subset \mathrm{int}_{H_\alpha}(F^+(\mathrm{int}(\mathrm{cl}(V))) \cap H_\alpha)$$

and

$$F^+(\mathrm{int}(\mathrm{cl}(V))) \cap H_\alpha \subset \mathrm{int}(F^+(\mathrm{int}(\mathrm{cl}(V)))) \cap H_\alpha.$$

Since $\{H_\alpha : \alpha \in \Lambda\}$ is an open cover of X, it follows that

$$F^+(\mathrm{int}(\mathrm{cl}(V))) \subset \mathrm{int}(F^+(\mathrm{int}(\mathrm{cl}(V)))).$$

Hence, from Theorem 3, we obtain that F is an upper almost nearly continuous multifunction. □

Recall that a multifunction $F : X \to Y$ is said to be point connected if, for each $x \in X$, $F(x)$ is connected.

Definition 13. A space X is called N-connected provided that X is not the union of two disjoint nonempty open sets having N-closed complements.

Theorem 14. *Let F be a multifunction from a connected topological space X onto a topological space Y such that F is point connected. If F is upper almost nearly continuous multifunction, then Y is a N-connected space.*

Proof. Let $F : X \to Y$ be a upper almost nearly continuous multifunction from a connected topological space X onto a topological space Y. Suppose that Y is not N-connected and let $Y = H \cup K$ be a partition of Y having N-closed complements. Then both H and K are open and closed subsets of Y. Since F is upper almost nearly continuous multifunction, $F^+(H)$ and $F^+(K)$ are open subsets of X. In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is point connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the connectedness of X. Hence, it is obtained that Y is a N-connected space. $\square$

Theorem 15. *Let $F : X \to Y$ be an upper almost nearly continuous multifunction and point closed from a topological space X to a N-normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a N-normal space, then there exists disjoints open sets U and V having N-closed complements containing $F(x)$ and $F(y)$, respectively. Thus, $F^+(\mathrm{int}(\mathrm{cl}(U)))$ and $F^+(\mathrm{int}(\mathrm{cl}(V)))$ are disjoint open sets containing x and y, respectively. Thus, X is Hausdorff. $\square$

Theorem 16. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Suppose that Y has a base of neigbourhoods such that complement of each set of the base of neighbourhoods is a regular closed N-closed set. If F is lower almost nearly continuous multifunction, then F is lower nearly continuous.*

Proof. Let $x \in X$ and let U be any open set having N-closed complement such that $x \in F^-(U)$. Due to assumption, we can write $U = \bigcup\limits_{i \in I} V_i$ where V_i is a regular open set having N-closed complement for $i \in I$. We have $F^-(\bigcup_{i\in I} V_i) = \bigcup_{i\in I} F^-(V_i)$. Since F is lower almost nearly continuous multifunction, then $F^-(V_i)$ is an open set for $i \in I$. So, $F^-(U)$ is an open set. Hence, F is lower nearly continuous multifunction. $\square$

Let X, Y and Z be topological spaces. If $F : X \to Y$ and $G : Y \to Z$ are multifunctions, then the composite multifunction $G \circ F : X \to Z$ is defined by $(G \circ F)(x) = G(F(x))$ for each $x \in X$.

Theorem 17. *Let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. The following statements hold:*

(1) *If F is upper (lower) semi-continuous and G is upper (lower) almost nearly continuous multifunction, then $G \circ F : X \to Z$ is an upper (lower) almost nearly continuous multifunction.*
(2) *If F is upper (lower) semi-continuous and G is upper (lower) nearly continuous multifunction, then $G \circ F : X \to Z$ is an upper (lower) almost nearly continuous multifunction.*

Proof. (1) Let $V \subset Z$ be any regular open set having N-closed complement. Since G is upper (lower) almost nearly continuous multifunction, it follows that $G^+(V)$ (resp. $G^-(V)$) is an open set. Since F is upper (lower) semi-continuous multifunction, it follows that $F^+(G^+(V)) = (G \circ F)^+(V)$ (resp. $F^-(G^-(V)) = (G \circ F)^-(V)$) is an open set. It shows that $G \circ F$ is a upper (resp. lower) almost nearly continuous multifunction.

The other proof can be obtained similarly. $\square$

3. Graphs and product spaces

In this section, the relationships between almost nearly continuity and graphs in product spaces are obtained.

Lemma 18. *For a multifunction $F : X \to Y$, the following hold:*

(1) $G_F^+(A \times B) = A \cap F^+(B)$,
(2) $G_F^-(A \times B) = A \cap F^-(B)$

for any subsets $A \subset X$ and $B \subset Y$ [12].

Theorem 19. *Let $A_i \subset X_i$ be nonempty for each $i \in I$. Set $A = \prod_{i\in I} A_i$ and $X = \prod_{i\in I} X_i$. Then A is N-closed relative to X if and only if each A_i is N-closed relative to X_i* [9].

Theorem 20. *Let X and X_α be topological spaces and X_α be nearly compact for each $\alpha \in J$. Let $F : X \to \prod_{\alpha\in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha\in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ be the projection for each $\alpha \in J$. If F is upper (lower) almost nearly continuous multifunction, then $P_\alpha \circ F$ is upper (resp. lower) almost nearly continuous multifunction for each $\alpha \in J$.*

Proof. We prove only the case for F upper almost nearly continuous, the proof for F lower almost nearly continuous being analogous.

Take any $\alpha_0 \in J$. Let V_{α_0} be an open set having N-closed complement in X_{α_0}. We have

$$(P_{\alpha_0} \circ F)^+(\mathrm{int}(\mathrm{cl}(V_{\alpha_0})) = F^+(P^+_{\alpha_0}(\mathrm{int}(\mathrm{cl}(V_{\alpha_0}))) = F^+(\mathrm{int}(\mathrm{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha).$$

Since F is upper almost nearly continuous and $\mathrm{int}(\mathrm{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha$ is a regular open set having N-closed complement to $\prod_{\alpha\in J} X_\alpha$, then $F^+(\mathrm{int}(\mathrm{cl}(V_{\alpha_0})) \times \prod_{\alpha\neq\alpha_0} X_\alpha)$ is an open set in (X, τ). Thus, $P_{\alpha_0} \circ F$ is upper almost nearly continuous multifunction. □

Theorem 21. *Let X_α and Y_α be topological spaces for each $\alpha \in J$. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha\in J} X_\alpha$ to the product space $\prod_{\alpha\in J} Y_\alpha$. If F is upper (lower) almost nearly continuous multifunction and Y_α is nearly compact for each $\alpha \in J$, then each F_α is upper (resp. lower) almost nearly continuous multifunction for each $\alpha \in J$.*

Proof. Let $V_\alpha \subset Y_\alpha$ be an open set having N-closed complement. Then

$$\operatorname{int}(\operatorname{cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta$$

is a regular open set having N-closed complement relative to $\prod_{\alpha\in J} Y_\alpha$. Since F is upper almost nearly continuous multifunction, it follows that

$$F^+(\operatorname{int}(\operatorname{cl}(V_\alpha)) \times \prod_{\alpha\neq\beta} Y_\beta) = F_\alpha^+(\operatorname{int}(\operatorname{cl}(V_\alpha))) \times \prod_{\alpha\neq\beta} X_\beta$$

is an open set. We obtain that $F_\alpha^+(\operatorname{int}(\operatorname{cl}(V_\alpha)))$ is an open set. Hence, F_α is upper almost nearly continuous multifunction.

The proof for lower almost continuity is similar. □

Remark 22. In view of Theorem 19, the nearly compactness in Theorem 20 and Theorem 21 can not be omitted.

Theorem 23. *Let $F : X \to Y$ be an upper almost nearly continuous multifunction. If every two distinct points of Y are contained in disjoint open sets such that one of them may be chosen to have N-closed complement, then the graph multifunction G_F is strongly closed.*

Proof. Suppose that $(x, y) \notin G_F$. Then $y \notin F(x)$. Due to the assumption, there exist disjoint open sets H and K such that $F(x) \subset H$, $y \in K$ and H has N-closed complement. Since F is upper almost nearly continuous multifunction, then $U = F^+(\operatorname{int}(\operatorname{cl}(H)))$ is an open set containing x and $F(U) \subset \operatorname{int}(\operatorname{cl}(H)) \subset Y\backslash K$. We have $x \in U$ and $y \in K$. Hence, $U \times \operatorname{cl}(K)$ does not contain any points of G_F and thus, G_F is strongly closed. □

Theorem 24. *Let X be a nearly compact space and let $F : X \to Y$ be a multifunction. If the graph multifunction of F is upper almost nearly continuous, then F is upper almost nearly continuous.*

Proof. Suppose that $G_F : X \to X \times Y$ is upper almost nearly continuous. Let $x \in X$ and V be any open set having N-closed complement and containing $F(x)$. Since $X \times V$ is an open set having N-closed complement relative to $X \times Y$ and $G_F(x) \subset X \times V$, there exists an open set U containing x such that

$$G_F(U) \subset \operatorname{int}(\operatorname{cl}(X \times V)) = X \times \operatorname{int}(\operatorname{cl}(V)).$$

By Lemma 18, we have

$$U \subset G_F^+(X \times \operatorname{int}(\operatorname{cl}(V))) = F^+(\operatorname{int}(\operatorname{cl}(V))) \quad \text{and} \quad F(U) \subset \operatorname{int}(\operatorname{cl}(V)).$$

Thus, F is upper almost nearly continuous. □

Theorem 25. *Let X be a nearly compact space. A multifunction $F : X \to Y$ is lower almost nearly continuous if $G_F : X \to X \times Y$ is lower almost nearly continuous.*

Proof. Let $x \in X$ and V be any open set having N-closed complement such that $x \in F^-(V)$. Then $X \times V$ is an open set having N-closed complement relative to $X \times Y$ and

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset.$$

Since G_F is lower almost nearly continuous, there exists an open set U containing x such that $U \subset G_F^-(\operatorname{int}(\operatorname{cl}(X \times V)))$. Since

$$G_F^-(\operatorname{int}(\operatorname{cl}(X \times V))) = G_F^-(X \times \operatorname{int}(\operatorname{cl}(V))),$$

by Lemma 18, we have $U \subset F^-(\operatorname{int}(\operatorname{cl}(V)))$.

This shows that F is lower almost nearly continuous. □

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