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# LIMIT OF APPROXIMATE INVERSE SYSTEM OF TOTALLY REGULAR CONTINUA IS TOTALLY REGULAR

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Abstract. It is known that the limit of an inverse system of totally regular continua is a totally regular continuum. In this paper we shall prove that this is true for approximate limit of an approximate inverse system in the sense of S. Mardešić (Theorem 14).

## 1. Introduction

In this paper we shall use the notion of *inverse systems* $\mathbf{X} = \{X_a, p_{ab}, A\}$ and their limits in the usual sense [**1**, p. 135].

The cardinality of a set $X$ will be denoted by $\mathrm{card}(X)$. The cofinality of a cardinal number $m$ will be denoted by $\mathrm{cf}(m)$. $\mathrm{Cov}(X)$ is the set of all normal coverings of a topological space $X$. If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \leq \mathcal{U}$. For two mappings $f, g : Y \to X$ which are $\mathcal{U}$-near (for every $y \in Y$ there exists a $U \in \mathcal{U}$ with $f(y), g(y) \in U$), we write $(f, g) \leq \mathcal{U}$. A basis of (open) normal coverings of a space $X$ is a collection $\mathcal{C}$ of normal coverings such that every normal covering $\mathcal{U} \in \mathrm{Cov}(X)$ admits a refinement $\mathcal{V} \in \mathcal{C}$. We denote by $\mathrm{cw}(X)$ (*covering weight*) the minimal cardinal of a basis of normal coverings of $X$ [**9**, p. 181].

**Lemma 1.** [**9**, Example 2.2]. *If $X$ is a compact Hausdorff space, then* $\mathrm{cw}(X) = w(X)$.

The notion of *approximate inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ will be used in the sense of S. Mardešić [**11**].

**Definition 1.** An *approximate inverse system* is a collection $\mathbf{X} = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, $X_a$, $a \in A$, is a topological space and $p_{ab} : X_b \to X_a, a \leq b$, are mappings such that $p_{aa} = \mathrm{id}$ and the following condition (A2) is satisfied:

(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is an index $b \geq a$ such that $(p_{ac}p_{cd}, p_{ad}) \leq \mathcal{U}$ whenever $a \leq b \leq c \leq d$.

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An approximate map $\mathbf{p} = \{p_a : a \in A\} : X \to \mathbf{X}$ into an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a collection of maps $p_a : X \to X_a$, $a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $U \in \mathrm{Cov}(X_a)$ there is $b \geq a$ such that $(p_{ac}p_c, p_a) \leq U$ for each $c \geq b$. (See [**10**]).

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system and let $\mathbf{p} = \{p_a : a \in A\} : X \to \mathbf{X}$ be an approximate map. We say that $\mathbf{p}$ is a *limit* of $\mathbf{X}$ provided it has the following universal property:

(UL) For any approximate map $\mathbf{q} = \{q_a : a \in A\} : Y \to \mathbf{X}$ of a space $Y$ there exists a unique map $g : Y \to X$ such that $p_a g = q_a$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall U \in \mathrm{Cov}(X_a))(\exists b \geq a)(\forall c \geq b) p_{ac}(x_c) \in \mathrm{st}(x_a, U)$.

If $X_a$ is a $\mathrm{T}_{3.5}$ space, then the sets $\mathrm{st}(x_a, \mathcal{U})$, $\mathcal{U} \in \mathrm{Cov}(X_a)$, form a basis of the topology at the point $x_a$. Therefore, for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

(L*) $$(\forall a \in A) \lim\{p_{ac}(x_c) : c \geq a\} = x_a.$$

Some other properties of approximate systems and their subsystems are given in Appendix.

Let $\tau$ be an infinite cardinal. We say that a partially ordered set $A$ is $\tau$*-directed* if for each $B \subseteq A$ with $\mathrm{card}(B) \leq \tau$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. If $A$ is $\aleph_0$-directed, then we will say that $A$ is $\sigma$-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be $\tau$*-directed* if $A$ is $\tau$-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be $\sigma$*-directed* if $A$ is $\sigma$-directed.

The proof of the following theorem is similar to the proof of Theorem 1.1 of [**4**].

**Theorem 1.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a $\sigma$-directed approximate inverse system of compact spaces with surjective bonding mappings and limit $X$. Let $Y$ be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.*

**Theorem 2.** *Let $X$ be a compact spaces. There exists a $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces $X_a$ and surjective bonding mappings $p_{ab}$ such that $X$ is homeomorphic to $\lim \mathbf{X}$.*

**Theorem 3.** [**8**, p. 163, Theorem 2.]. *If $X$ is a locally connected compact space, then there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric locally connected compact space, each $p_{ab}$ is a monotone surjection and $X$ is homeomorphic to $\lim \mathbf{X}$. Conversely, the inverse limit of such system is always a locally connected compact space.*

**Remark 1.** *We may assume that $\mathbf{X} = \{X_a, p_{ab}, A\}$ in Theorem 3 is $\sigma$-directed* [**12**, Theorem 9.5].

**Theorem 4.** [**13**, Corollary 2.9]. *If $X$ is a hereditarily locally connected continuum, then there exists a $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metrizable hereditarily locally connected continuum, each $p_{ab}$ is a monotone surjection and $X$ is homeomorphic to* $\lim \mathbf{X}$.

**Theorem 5.** [**3**, Corollary 3]. *Let $\boldsymbol{X} = \{X_a,\ p_{ab},\ A\}$ be a $\sigma$-directed inverse system of hereditarily locally connected continua $X_a$. Then $X = lim\boldsymbol{X}$ is hereditarily locally connected.*

The following theorem is Theorem 1.7 from [**5**].

**Theorem 6.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a $\sigma$-directed inverse system of compact metrizable spaces and surjective bonding mappings. Then $X = \lim \mathbf{X}$ is metrizable if and only if there exists an $a \in A$ such that $p_b : X \to X_b$ is a homeomorphism for each $b \geq a$.*

## 2. Limit of approximate inverse system of totally regular continua

We shall say that a non-empty compact space is *perfect* if it has no isolated point.

A continuum is said to be *totally regular* [**12**, p. 47] if for each $x \neq y$ in $X$ there is a positive integer $n$ and perfect subsets $A_1, \ldots, A_n, \ldots$ of $X$ such that $x_i \in A_i$ for $i = 1, \ldots, n$ implies that $\{x_1, \ldots, x_n\}$ separates $x$ from $y$ in $X$.

**Lemma 2.** [**12**, Proposition 7.4]. *Each totally regular continuum is hereditarily locally connected and rim-finite.*

The following theorem is a part of [**12**, Theorem 7.15].

**Theorem 7.** *If $X$ is a continuum then the following conditions are equivalent:*

(1) *$X$ is totally regular,*
(2) *$X$ is homeomorphic to $\lim\{X_a, f_{ab}, \Gamma\}$ such that each $X_a$ is a totally regular continuum and each $f_{ab}$ is a monotone surjection.*

**Theorem 8.** [**12**, Theorem 7.7]. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of totally regular continua $X_a$ and monotone surjective mappings $p_{ab}$. Then $X = \lim \mathbf{X}$ is totally regular.*

**Theorem 9.** *Let $X$ be a non-metric totally regular continuum. There exists a $\sigma$-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each $X_a$ is totally regular, each $f_{ab}$ is a monotone surjection and $X$ is homeomorphic to* $\lim \mathbf{X}$.

*Proof.* Apply [**12**, Theorem 9.4], Theorem 8 and Lemma 3.5 of [**14**]. □

Now we consider approximate inverse systems of totally regular continua. We start with the following theorem.

**Theorem 10.** *Let $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of totally regular metric continua. If the bonding mappings are monotone and surjective, then $X = \lim \mathbf{X}$ is totally regular.*

*Proof.* There exists a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$, $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \dots p_{n_{j-1} n_j}$ for each $i, j \in \mathbb{N}$ and a homeomorphism $H : \lim \mathbf{X} \to \lim \mathbf{Y}$ [**2**, Proposition 8]. Each mapping $q_{ij}$ as a composition of the monotone mappings is monotone. This means that $\mathbf{Y}$ is a usual inverse sequence of totally regular continua with monotone bonding mappings $q_{ij}$. By virtue of Theorem 8 $\lim \mathbf{Y}$ is totally regular. We infer that $X = \lim \mathbf{X}$ is totally regular since there exists a homeomorphism $H : \lim \mathbf{X} \to \lim \mathbf{Y}$. □

**Theorem 11.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua such that* $\operatorname{card}(A) = \aleph_0$. *Then* $X = \lim \mathbf{X}$ *is totally regular.*

*Proof.* By virtue of Lemma 6 of Appendix there exists a countable well-ordered subset $B$ of $A$ such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. From Theorem 10 it follows that $\lim\{X_b, p_{bc}, B\}$ is totally regular. Hence $X = \lim \mathbf{X}$ is totally regular. □

**Theorem 12.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua and monotone bonding mappings. If* $w(X_a) \leq \tau < \operatorname{card}(A)$ *for each* $a \in A$*, then* $X = \lim \mathbf{X}$ *is totally regular continuum.*

*Proof.* By virtue of Theorem 15 (for $\lambda = \aleph_0$) of Appendix there exists a $\sigma$-directed inverse system $\{X_\alpha, q_{\alpha\beta}, T\}$, where each $X_\alpha$ is a limit of an approximate inverse subsystem $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \aleph_0$. From Theorem11 it follows that every $X_\alpha$ is totally regular. Theorem 8 completes the proof. □

**Theorem 13.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular metric continua and monotone bonding mappings. Then* $X = \lim \mathbf{X}$ *is totally regular continuum.*

*Proof.* If $\operatorname{card}(A) = \aleph_0$, then we apply Theorem 11. If $\operatorname{card}(A) \geq \aleph_1$, then from Theorem 12 it follows that $X$ is totally regular. □

A directed preordered set $(A, \leq)$ is said to be *cofinite* provided each $a \in A$ has only finitely many predecessors. If $a \in A$ has exactly $n$ predecessors, we shall write $p(a) = n + 1$. Hence, $a \in A$ is the first element of $(A, \leq)$ if and only if $p(a) = 1$.

**Lemma 3.** *If* $(A, \leq)$ *is cofinite, then it satisfies the following principle of induction:*

*Let* $B \subset A$ *be a set such that:*

(i) *$B$ contains all the first elements of $A$,*

(ii) *if $B$ contains all the predecessors of $a \in A$, then $a \in B$.*

*Then* $B = A$.

**Lemma 4.** [**15**, Lemma 1]. *Let* $q = (q_a) : Y \to \mathcal{Y} = \{Y_b, \mathcal{V}_b, q_{ab'}, B\}$ *be an approximate map (approximate resolution) of a space* $Y$. *Then there exists an approximate map (approximate resolution)* $q = (q_a) : Y \to \mathcal{Y} = \{Y_c^{'}, \mathcal{V}_c^{'}, q_{cc'}, C\}$ *of the space* $Y$ *and an increasing surjection* $t : C \to B$ *satisfying the following conditions:*

(i) *$C$ is directed, unbounded, antisymmetric and cofinite set,*
(ii) $(\forall c \in C)(\forall b \in B)(\exists c' > c) \quad t(c') > b$;
(iii) $(\forall c \in C) \quad Y'_c = Y_{t(c)},\ \mathcal{V}'_c = \mathcal{V}_{t(c)},\ q'_c = q_{t(c)}$ *and* $q'_{cc'} = q_{t(c)t(c')}$, *whenever* $c < c'$.

**Corollary 1.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact spaces. Then there exists an approximate inverse system $\mathbf{Y} = \{Y_c, p_{cc'}, C\}$ such that: a) each $Y_c$ is some $X_a$, b) each $p_{cc'}$ is some $p_{ab}$, c) $C$ is directed, unbounded, antisymmetric and cofinite set and $\lim \mathbf{X}$ is homeomorphic to $\lim \mathbf{Y}$.*

*Proof.* By virtue of Theorem 4.2 of [**10**] an approximate map $p : X \to \mathbf{X}$ is an approximate resolution if and only if it is a limit of $\mathbf{X} = \{X_a, p_{ab}, A\}$. Apply Lemma 4. □

Now we shall prove the main theorem of this paper.

**Theorem 14.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of totally regular continua with monotone surjective bonding mappings $p_{ab}$. Then $X = \lim X$ is totally regular.*

*Proof.* If every $X_a$ is a metric totally regular continuum, then we apply Theorem 13. Now, suppose that each $X_a$ is a non-metric totally regular continuum. The proof consists of several steps. In the Steps $0 - 11$ we shall define a usual inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit $X_D$ is homeomorphic to $X = \lim \mathbf{X}$. In *Step* 12 we shall use Theorem 8 which completes the proof.

*Step* 0.
From Corollary 1 it follows that we may assume that $A$ is cofinite.

*Step* 1.
By virtue of Theorem 9 for each $X_a$ there exists a $\sigma$ directed inverse system

(2.1) $$\mathbf{X}(\mathbf{a}) = \{X_{(a,\gamma)},\ f_{(a,\gamma)(a,\delta)},\ \Gamma_a\}$$

such that each $X_{(a,\gamma)}$ is a totally regular metric continuum, each $f_{(a,\gamma)(a,\delta)}$ is monotone and surjective and $X_a$ is homeomorphic to $\lim \mathbf{X}(\mathbf{a})$. Now we have the following diagram

(2.2) $$\begin{array}{ccccccc}
X_a & \xleftarrow{p_{ab}} & X_b & \xleftarrow{p_{bc}} & X_c & \xleftarrow{p_d} & X \\
\downarrow f_{(a,\gamma_a)} & & \downarrow f_{(b,\gamma_c)} & & \downarrow f_{(c,\gamma_c)} & & \\
X_{(a,\gamma_a)} & & X_{(b,\gamma_b)} & & X_{(c,\gamma_c)} & & \\
\downarrow f_{(a,\gamma_a)(a,\delta_a)} & & \downarrow f_{(b,\gamma_b)(b,\delta_b)} & & \downarrow f_{(c,\gamma_c)(c,\delta_c)} & & \\
X_{(a,\delta_a)} & & X_{(b,\delta_b)} & & X_{(c,\delta_c)} & & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}$$

*Step* 2.
Put B = $\{($a , $\gamma_a$ $)$ : a $\in$ A, $\gamma_a \in \Gamma_a\}$ and put $C$ to be the set of all subsets $c$ of B of the form

(2.3) $$c = \{(a, \gamma_a) : a \in A\},$$

where every $\gamma_a$ is the fixed element of $\Gamma_a$.

*Step* 3.

Let $D$ be a subset of $C$ containing all $c \in C$ for which there exist the mappings

$$g_{(a,\gamma_a)(b,\gamma_b)} : X_{(b,\gamma_b)} \to X_{(a,\gamma_a)}, b \geq a, \tag{2.4}$$

such that

$$\{X_{(a,\gamma_a)},\ g_{(a,\gamma_a)(b,\gamma_b)}, A\} \tag{2.5}$$

is an approximate inverse system and each diagram

$$\begin{array}{ccc} X_a & \overset{p_{ab}}{\longleftarrow} & X_b \\ \downarrow f_{(a,\gamma_a)} & & \downarrow f_{(b,\gamma_b)} \\ X_{(a,\gamma_a)} & \overset{g_{(a,\gamma_a)(b,\gamma_b)}}{\longleftarrow} & X_{(b,\gamma_b)} \end{array} \tag{2.6}$$

commutes, where $f_{(a,\gamma_a)} : X_a \to X_{(a,\gamma_a)}$ is the canonical projection.

*Step* 4.

The set D is non empty. Moreover, for each countable subset $S_a \subset \Gamma_a$, $a \in A$, there exists a $d \in D$ such that $d = \{(a, \gamma_a) : a \in A\}$, $\gamma_a \geq \gamma$ for every $\gamma \in S_a$. Let $a \in A$ be some first element of $A$ and let $\gamma_a \in \Gamma_a$ such that $\gamma_a \geq \gamma$ for every $\gamma \in S_a$. The space $X_{(a,\gamma_a)}$ is a metric compact space and there exist the mappings $f_{(a,\gamma_a)}p_{ab} : X_b \to X_{(a,\gamma_a)}, b \geq a$. By virtue of Theorem 1 for each $b \geq a$ there exist a $\gamma_b^1 \in \Gamma_b$ such that for each $\gamma_b \geq \gamma_b^1, \gamma$, where $\gamma \in S_b$, there exists a monotone surjective mapping $g_{(a,\gamma_a)(b,\gamma_b)} : X_{(b,\gamma_b)} \to X_{(a,\gamma_a)}$ with $f_{(a,\gamma_a)}p_{ab} = g_{(a,\gamma_a)(b,\gamma_b)}f_{(b,\gamma_b)}$, i.e., the diagram

$$\begin{array}{ccc} X_a & \overset{p_{ab}}{\longleftarrow} & X_b \\ \downarrow f_{(a,\gamma_a)} & & \downarrow f_{(b,\gamma_b)} \\ X_{(a,\gamma_a)} & \overset{g_{(a,\gamma_a)(b,\gamma_b)}}{\longleftarrow} & X_{(b,\gamma_b)} \end{array} \tag{2.7}$$

commutes. Suppose that $(a, \gamma_b^1), (a, \gamma_b^2), \ldots, (a, \gamma_b^{n-1})$ are defined for each $a \in A$ with $p(a) \leq n-1$ such that the each diagram (2.6) commutes. Let $a \in A$ be a member of $A$ with $p(a) = n$. This means that $(a, \gamma_b^1), (a, \gamma_b^2), \ldots, (a, \gamma_b^{n-1})$ are defined. From the cofinitness of $A$ it follows that the set of $\gamma_a^j$ which are defined in $\Gamma_a$ is finite. Hence there exists $\gamma_a^n \geq \gamma_a^{n-1}, \ldots, \gamma_a^1$. We define $\gamma_b^n \in \Gamma_b$ considering the space $X_{(a,\gamma_a^n)}$ and the mappings $f_{(a,\gamma_a^n)}p_{ab} : X_b \to X_{(a,\gamma_a^n)}$. Again, by Theorem 1 for each $b \geq a$ there exists an $\gamma_b^n \in \Gamma_b$ such that for each $\gamma_b \geq \gamma_b^n, \gamma_b^{n-1}, \ldots, \gamma_b^1$ and there is a mapping $g_{(a,\gamma_b)(b,\gamma_b)} : X_{(b,\gamma_b)} \to X_{(a,\gamma_a^n)}$ with $f_{(a,\gamma_a^n)}p_{ab} = g_{(a,\gamma_b)(b,\gamma_b)}f_{(b,\gamma_b)}$, i.e., the diagram

$$\begin{array}{ccc} X_a & \overset{p_{ab}}{\longleftarrow} & X_b \\ \downarrow f_{(a,\gamma_a^n)} & & \downarrow f_{(b,\gamma_b)} \\ X_{(a,\gamma_a^n)} & \overset{g_{(a,\gamma_a^n)(b,\gamma_b)}}{\longleftarrow} & X_{(b,\gamma_b)} \end{array} \tag{2.8}$$

commutes. By induction on $A$ (Lemma 3) the set $D$ is defined. It remains to prove that $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is an approximate inverse system. Let $\mathcal{U}$ be a normal cover of $X_{(a,\gamma_a)}$. Then $\mathcal{V} = f^{-1}_{(a,\gamma_a)}(\mathcal{U})$ is a normal cover of $X_a$. By virtue of (A2) there exists a $b \geq a$ such that for each $c \geq d \geq b$ we have $(p_{ad}, p_{ca}p_{cd}) \leq \mathcal{V}$. By virtue of the commutativity of the diagrams of the form (2.8) it follows that

$$(g_{(a,\gamma_a)(d,\gamma_d)},\ g_{(a,\gamma_a)(c,\gamma_c)}g_{(c,\gamma_c)(d,\gamma_d)}) \leq \mathcal{V}. \tag{2.9}$$

Thus, $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is an approximate inverse system.

*Step* 5.

We define a partial order on $D$ as follows. Let $d_1, d_2$ be a pair of members of $D$ such that $d_1 = \{(a, \gamma_a) : a \in A,\ \gamma_a \in \Gamma_a\}$ and $d_2 = \{(a, \delta_a) : a \in A,\ \delta_a \in \Gamma_a\}$. We write $d_2 \leq d_1$ if and only if $\delta_a \leq \gamma_a$ for each $a \in\ A$. From *Step* 4 it follows that $(D, \leq)$ is $\sigma$-directed.

*Step* 6.

*For each* $d \in D$ *a limit space* $X_d$ *of the inverse system (2.5) is a totally regular continuum (Theorem 13). Moreover, there exists a mapping* $F_d : X \to X_d$. The existence of $F_d$ follows from the commutativity of the diagram (2.6). The following diagram illustrates the construction of $d \in D$ and the space $X_d$.

$$\begin{array}{ccccccc}
X_a & \xleftarrow{p_{ab}} & X_b & \xleftarrow{p_{bc}} & X_c & \xleftarrow{p_d} & X \\
\downarrow f_{(a,\delta_a)} & & \downarrow f_{(b,\delta_c)} & & \downarrow f_{(c,\delta_c)} & & \\
X_{(a,\delta_a)} & & X_{(b,\delta_b)} & & X_{(c,\delta_c)} & & \\
\downarrow f_{(a,\gamma_a)(a,\delta_a)} & & \downarrow f_{(b,\gamma_b)(b,\delta_b)} & & \downarrow f_{(c,\gamma_c)(c,\delta_c)} & & \\
X_{(a,\gamma_a)} & \xleftarrow{g_{(a,\gamma_a)(b,\gamma_b)}} & X_{(b,\gamma_b)} & \xleftarrow{g_{(b,\gamma_b)(c,\gamma_c)}} & X_{(c,\gamma_c)} & \xleftarrow{g_{(c,\gamma_c)}} & X_d \\
\downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{2.10}$$

*Step* 7.

If $d_1, d_2$ is a pair of members of $D$ such that $d_1 = \{(a, \gamma_a) : a \in A,\ \gamma_a \in \Gamma_a\}$, $d_2 = \{(a, \delta_a) : a \in A,\ \delta_a \in \Gamma_a\}$ and $d_2 \geq d_1$, then for each $a \in A$ the following diagram commutes

$$\begin{array}{ccc}
X_{(a,\delta_a)} & \xleftarrow{g_{(a,\delta_a)(b,\delta_b)}} & X_{(b,\delta_b)} \\
\downarrow f_{(a,\gamma_a)(a,\delta_a)} & & \downarrow f_{(b,\gamma_b)(b,\delta_b)} \\
X_{(a,\gamma_a)} & \xleftarrow{g_{(a,\gamma_a)(b,\gamma_b)}} & X_{(b,\gamma_b)}
\end{array} \tag{2.11}$$

This follows from the commutativity of the diagrams of the form (2.6) for $d_1$ and $d_2$, i.e., from the commutativity of the diagrams

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
\downarrow f_{(a,\gamma_a)} & & \downarrow f_{(b,\gamma_b)} \\
X_{(a,\gamma_a)} & \xleftarrow{g_{(a,\gamma_a)(b,\gamma_b)}} & X_{(b,\gamma_b)}
\end{array} \tag{2.12}$$

and

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
\downarrow f_{(a,\delta_a)} & & \downarrow f_{(b,\delta_b)} \\
X_{(a,\delta_a)} & \xleftarrow{g_{(a,\delta_a)(b,\delta_b)}} & X_{(b,\delta_b)}
\end{array} \tag{2.13}$$

*Step* 8.

From *Step* 7 it follows that for $d_1, d_2 \in D$ with $d_2 \geq d_1$ there exists a mapping $F_{d_1 d_2} : X_{d_2} \to X_{d_1}$ (see [**1**, p. 138]) such that $F_{d_1} = F_{d_1 d_2} F_{d_2}$.
*Proof of Step 8.* Let $d_1, d_2, d_3 \in D$ and let $d_1 \leq d_2 \leq d_3$. Then $F_{d_1 d_3} = F_{d_1 d_2} F_{d_2 d_3}$. This follows from *Step* 7 and the commutativity condition in each inverse system $\mathbf{X}(\mathbf{a}) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ (see (2.1) of *Step* 1).

*Step* 9.

*The collection* $\{X_d, F_{de}, D\}$ *is a usual inverse system of totally regular metric continua.*

Apply *Steps* 1 – 8.

*Step* 10.

*There is a mapping* $F : X \to X_D$ *which is* $1 - 1$.

By *Step* 6 and *Step* 8 for each $d \in D$ there is a mapping $F_d : X \to X_d$ such that $F_{d_1} = F_{d_1 d_2} F_{d_2}$ for $d_2 \geq d_1$. This means that there exists a mapping $F : X \to X_D$ [**1**, p. 138]. Let us prove that F is $1 - 1$. Take a pair $x, y$ of distinct points of $X$. There exists an $a \in A$ such that $x_a = p_a(x)$ and $y_a = p_a(y)$ are distinct points of $X_a$. Now, there exists an $(a, \gamma_a)$ such that $f_{(a,\gamma_a)}(x_a)$ and $f_{(a,\gamma_a)}(y_a)$ are distinct points of $X_{(a,\gamma_a)}$. From *Step* 4 it follows that there is a $d \in D$ such that $F_d(x)$ and $F_d(y)$ are distinct points of $X_d$. Thus, $F$ is $1 - 1$.

*Step* 11.

*The mapping* $F$ *is a homeomorphism onto* $X_D$. Let $y$ be a point of $X_D$. Let us prove that there exists a point $x \in X$ such that $F(x) = y$. For each $d \in D$ we have a point $y_d = F_d(y)$. Now, we have the points $g_{(a,\gamma_a)} F_d(y)$ in $X_{(a,\gamma_a)}$ and the subsets $Y_a = f^{-1}_{(a,\gamma_a)}(g_{(a,\gamma_a)} F_d(y))$ of $X_a$. Let $U$ be an open neighborhood $Y_a$. There exists an open neighborhood $V$ of $g_{(a,\gamma_a)} F_d(y)$ such that $f^{-1}_{(a,\gamma_a)}(V) \subseteq U$. We infer that $\mathrm{Ls}\{g_{(b,\gamma_b)}(Y_b) : b \geq a\} \subseteq Y_a$ since $g_{(a,\gamma_a)} F_d(y) = \lim\{g_{(a,\gamma_a)(b,\gamma_b)} g_{(b,\gamma_b)} F_d(y) : b \geq a\}$ and the diagrams (2.6) commute. By virtue of [**6**, Lemma 2.1] it follows that there exists a non-empty closed subset $C_d$ of $\lim \mathbf{X}$ such that $p_b(C_d) \subseteq Y_b$. The family $\{C_d : d \in D\}$ has the finite intersection property. This means that $X' = \bigcap\{C_d : d \in D\}$ is non-empty. For each $x \in X'$ we have $F_d(x) = F_d(y), d \in D$. Thus, $F(y) = x$. The proof of this *Step* is completed.

*Step* 12.

By virtue of Theorem 8 it follows that $X_D = \lim\{X_d, F_{de}, D\}$ is totally regular. We infer that $X$ is totally regular since the mapping $F$ is a homeomorphism of $X$ onto $X_D$ (*Step* 11). □

## 3. Appendix

In this Appendix we investigate the approximate subsystem of an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$. We start with the following definition.

**Definition 2.** Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system and let $B$ be a directed subset of $A$ such that $\{X_b, p_{bc}, B\}$ is an approximate inverse system. We say that $\{X_b, p_{bc}, B\}$ is an approximate subsystem of $\mathbf{X} = \{X_a, p_{ab}, A\}$ if there exists a mapping $q : \lim \mathbf{X} \to \lim\{X_b, p_{bc}, B\}$ such that

$$p_b q = P_b, \quad b \in B,$$

where $p_b : \lim\{X_b, p_{bc}, B\} \to X_b$ and $P_b : \lim \mathbf{X} \to X_b, b \in B$, are natural projections.

We say that an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *irreducible* if for each $B \subset A$ with $\text{card}(B) < \text{card}(A)$ it follows that $B$ is not cofinal in $A$.

**Lemma 5.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system. There exists a cofinal subset $B$ of $A$ such that $\mathbf{X} = \{X_a, p_{ab}, B\}$ is irreducible.*

*Proof.* Consider the family $\mathcal{B}$ of all cofinal subset of $B$ of $A$. The set $\{\text{card}(B) : B \in \mathcal{B}\}$ has a minimal element $b$ since each $\text{card}(B)$ is some initial ordinal number. Let $B \in \mathcal{B}$ be such that $\text{card}(B) = b$. It is clear that $\{X_a, p_{ab}, B\}$ is irreducible. □

In the sequel we will assume that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is irreducible.

**Lemma 6.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact spaces such that $\text{card}(A) = \aleph_0$. Then there exists a countable well-ordered subset $B$ of $A$ such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$.*

*Proof.* Let $\nu$ be any finite subset of $A$. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. Since $A$ is infinite, there exists a sequence $\{\nu_n : n \in \mathbb{N}\}$ such that $\nu_1 \subseteq \ldots \nu_n \subseteq \ldots$ and $A = \bigcup\{\nu_n : n \in \mathbb{N}\}$. Recursively, we define the sets $A_1, \ldots, A_n, \ldots$ by

$$A_1 = \nu_1 \bigcup \{\delta(\nu_1)\},$$

and

$$A_{n+1} = A_n \bigcup \nu_{n+1} \bigcup \{\delta(A_n \bigcup \nu_{n+1})\}.$$

It follows that there exists a sequence

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \ldots$$

of finite sets $A_n$ such that $A = \bigcup\{A_n : n \in \mathbb{N}\}$. Using a $\delta(A_n)$ for each $A_n$, we obtain a sequence $B = \{b_n : n \in \mathbb{N}\}$ such that $B$ is cofinal in $A$. Let us prove that $\{X_b, p_{bc}, B\}$ is an approximate inverse system, i.e., that (A2) is satisfied for $\{X_b, p_{bc}, B\}$. For each $X_b$ and each normal cover of $X_b$ there exists an $a' \in A$ such that (A2) is satisfied for $b \leq a' \leq c \leq d$ since (A2) is satisfied for $\mathbf{X} = \{X_a, p_{ab}, A\}$. There exists a $b'$ such that $b' \in B, b' \geq a'$, since $B$ is cofinal in $A$. It is obvious that (A2) is satisfied for each $c, d \in B$ such that $b \leq b' \leq c \leq d$. By virtue of [**10**, Theorem 1.19] it follows that $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. □

Now we consider irreducible approximate inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ with $\text{card}(A) \geq \aleph_1$.

**Lemma 7.** *Let $A$ be a directed set. For each subset $B$ of $A$ there exists a directed set $F_\infty(B)$ such that $\text{card}(F_\infty(B)) = \text{card}(B)$.*

*Proof.* For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup \{\delta(\nu) : \nu \in B\}$, where $\nu$ is a finite subset of $B$ and $\delta(\nu)$ is defined as in the proof of Lemma 6. Put

$$F_{n+1} = F_1(F_n(B),$$

and

$$F_\infty(B) = \bigcup \{F_n(B) : n \in \mathbb{N}\}.$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \ldots \subseteq F_n(B) \subseteq \ldots$$

The set $F_\infty(B)$ is directed since each finite subset $\nu$ of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$.

If $B$ is finite, then $\mathrm{card}(F_\infty(B)) = \aleph_0$. If $\mathrm{card}(B) \geq \aleph_0$, then we have

$$\mathrm{card}(\{\delta(\nu) : \nu \in B\}) \leq \mathrm{card}(B)\aleph_0.$$

We infer that $\mathrm{card}(F_1(B)) \leq \mathrm{card}(B)\aleph_0$. Similarly, $\mathrm{card}(F_n(B)) \leq \mathrm{card}(B)\aleph_0$. This means that $\mathrm{card}(F_\infty(B)) \leq \mathrm{card}(B)\aleph_0$. Thus

$$\mathrm{card}(F_\infty(B)) \leq \mathrm{card}(B)\aleph_0, \qquad \text{if} \qquad \mathrm{card}(B) < \mathrm{card}(A).$$

The proof is completed. □

**Lemma 8.** *Let* $\{X_a, p_{ab}, A\}$ *be an approximate inverse system such that* $\mathrm{cw}(X_a) < \mathrm{card}(A)$, $a \in A$. *For each subset* $B$ *of* $A$ *with* $\mathrm{card}(B) < \mathrm{card}(A)$, *there exists a directed set* $G_\infty(B) \supseteq B$ *such that the collection* $\{X_a, p_{ab}, G_\infty(B)\}$ *is an approximate system and* $\mathrm{card}(G_\infty(B)) = \mathrm{card}(B)$.

*Proof.* Let $\mathcal{B}_a$ be a base of normal coverings of $X_a$. Let $\mathcal{U}_a$ be a normal covering of $\mathcal{B}_a$. By virtue of (A2) there exists an $a(\mathcal{U}_a) \in A$ such that $(p_{ad}, p_{ac}p_{cd}) \leq \mathcal{U}_a$, $a \leq a(\mathcal{U}_a) \leq c \leq d$. For each subset $B$ of $A$ we define $G_\infty(B)$ by induction as follows:

a) Let $G_1(B) = F_\infty(B)$. From Lemma 7 it follows that $\mathrm{card}(G_1(B)) = \mathrm{card}(F_\infty(B)) = \mathrm{card}(B)$.

b) For each $n > 1$ we define $G_n(B)$ as follows:
   1) If $n$ is odd then $G_n(B) = F_\infty(G_{n-1}(B))$,
   2) If $n$ is even, then $G_n(B) = G_{n-1}(B) \cup \{a(U_a) : U_a \in B_a, a \in G_{n-1}(B)\}$.
   Since $\mathrm{card}(\mathcal{B}_a) < \mathrm{card}(A)$ the set $G_n(B)$ has the cardinality $< \mathrm{card}(A)$.
   Now we define $G_\infty(B) = \cup\{G_n(B) : n \in \mathbb{N}\}$. It is obvious that $\mathrm{card}(G_\infty(B)) < \mathrm{card}(A)$.

*The set* $G_\infty(B)$ *is directed.* Let $a$, $b$ be a pair of the elements of $G_\infty(B)$. There exists a $n \in \mathbb{N}$ such that $a, b \in G_n(B)$. We may assume that $n$ is odd. Then $a, b \in F_\infty(G_{n-1}(B))$. Thus there exists a $c \in F_\infty(G_{n-1}(B))$ such that $c \geq a, b$. It is clear that $c \in G_\infty(B)$. The proof of directedness of $G_\infty(B)$ is completed.

*The collection* $\{X_a, p_{ab}, G_\infty(B)\}$ *is an approximate system.* It suffices to prove that the condition (A2) is satisfied. Let $a$ be any member of $G_\infty(B)$. There exists a $n \in N$ such that $a \in G_n(B)$. We have two cases.

1) If $n$ is odd then $G_n(B) = F_\infty(G_{n-1}(B))$. This means that $a \in F_\infty(G_{n-1}(B))$. By definition of $F_\infty(G_{n-1}(B))$ we infer that $a(\mathcal{U}_a) \in F_\infty(G_{n-1}(B))$. Thus (A2) is satisfied.

2) If $n$ is even, then $G_n(B) = G_{n-1}(B) \cup \{a(\mathcal{U}_a) : \mathcal{U}_a \in \text{Cov}(X_a),\ a \in G_{n-1}(B)\}$. In this case $a \in G_{n+1}(B) \subseteq G_\infty(B)$. Arguing as in the case 1, we infer that (A2) is satisfied.

□

**Theorem 15.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces. If* $\lambda \leq w(X_a) \leq \tau <$card$(A)$ *for each* $a \in A$*, then* $\lim \mathbf{X}$ *is homeomorphic to a limit of a* $\lambda$*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$*, where each* $X_\alpha$ *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$*,* card$(\Phi) = \lambda$*.*

*Proof.* The proof consists of several steps.
*Step 1.*
Let $\mathcal{B} = \{B_\mu : \mu \in M\}$ be a family of all subsets $B_a$ of $A$ with card$(B_\alpha) = \lambda$. Put $A_\mu = G_\infty(B_\mu)$ (Lemma 8) and let $\Delta = \{A_\mu : \mu \in M\}$ be ordered by inclusion $\subseteq$.

*Step 2.*
*If* $\Phi$ *and* $\Psi$ *are in* $\Delta$ *such that* $\Phi \subset \Psi$*, then there exists a mapping* $q_{\Phi\Psi}$ : $\lim\{X_\alpha, p_{\alpha\beta}, \Psi\} \to \lim\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$.
Namely, if $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$, then by definition of the threads of $\{X_\alpha, p_{\alpha\beta}, \Psi\}$ the condition (L) is satisfied. If (L) is satisfied for $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$, then it is satisfied for $(x_\gamma, \gamma \in \Phi)$ since the required $a'$ in (L) lies – by definition of the set $\Phi$ – in the set $\Phi$. This means that $(x_\gamma, \gamma \in \Phi) \in \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$. Now we define $q_{\Phi\Psi}(x) = (x_\gamma, \gamma \in \Phi)$.

*Step 3.*
*The collection* $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ *is a usual inverse system.* It suffices to prove the transitivity, i.e., if $\Phi \subseteq \Psi \subseteq \Omega$ , then $q_{\Phi\Psi} q_{\Psi\Omega} = q_{\Phi\Omega}$. This easily follows from the definition of $q_{\Phi\Psi}$.

*Step 4.*
*The space* lim$X$ *is homeomorphic to* $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$*, where* $X_\Phi = \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$. We shall define a homeomorphism $H : \lim X \to \lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Let $x = (x_a : a \in A)$ be any point of $\lim \mathbf{X}$. Each collection $\{x_a : a \in \Phi \in \Delta\}$ is a point $x_\Phi$ of $X_\Phi$ since $X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$. Moreover, from the definition of $q_{\Phi\Psi}$ (*Step* 2) it follows that $q_{\Phi\Psi}(x_\Psi) = x_\Phi$, $\Psi \supseteq \Phi$. Thus, the collection $\{x_\Phi : \Phi \in \Delta\}$ is a point of $\lim\{X_\Phi, q_{\Phi\Psi}, \Delta\}$. Let $H(x) = \{x_\Phi, \Phi \in\in \Delta\}$. Thus, $H$ is a continuous mapping of $\lim \mathbf{X}$ to $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. In order to complete the proof it suffices to prove that $H$ is $1-1$ and onto. Let us prove that $H$ is $1-1$. Let $x = (x_a : a \in A)$ and $y = (y_a : a \in A)$ be a pair of points of $\lim \mathbf{X}$. This means that there exists an $a \in A$ such that $y_a \neq x_a$. There exists a $\Phi \in \Delta$ such that $a \in \Phi$. Thus, the collections $\{x_a : a \in \Phi\}$ and $\{x_a : a \in \Phi\}$ are different. From this we conclude that $x_\Phi \neq y_\Phi, x_\Phi, y_\Phi \in X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$. Hence $H$ is $1-1$. Let us prove that $H$ is onto. Let $y = (y_\Phi : \Phi \in \Delta)$ be any point of $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Each $y_\Phi$ is a collection $\{x_a : a \in \Phi\}$ and if $\Psi \supseteq \Phi$, then the collection $\{x_a : a \in \Phi\}$ is the restriction of the collection $\{x_a : a \in \Psi\}$ on $\Phi$. Let $x$ be the collection which is the union of all collections $\{x_a : a \in \Phi\}$, $\Phi \in \Delta$. Hence $x$ is a collection $(x_a : a \in A)$ which is a point of $\lim \mathbf{X}$ and $H(x) = y$.

*Step 5.*

*Inverse system* $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ *is a* $\lambda$*-directed inverse system.* Let $\{\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\} : \kappa \leq \lambda\}$ be a collection of approximate subsystems $\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\}$. The set $\Phi = \cup\{\Phi_\kappa : \kappa \leq \lambda\}$ has the cardinality $\leq \lambda$ since $\mathrm{card}(\Phi_\kappa) \leq \lambda$. By virtue of *Steps* $1-4$ there exists an approximate subsystem $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \lambda$. This means that $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ is a $\lambda$-directed inverse system. □

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an approximate inverse system of compact metric spaces, then $\mathrm{w}(X_a) = \aleph_0$, for each $a \in A$. It follows that $\lambda = \aleph_0$ if $\mathrm{card}(A) \geq \aleph_1$. Hence we have the following theorem.

**Corollary 2.** *Let* $X = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact metric spaces such that* $\mathrm{card}(A) \geq \aleph_1$. *Then* $\lim \mathbf{X}$ *is homeomorphic to the limit of a* $\sigma$*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \Delta\}$*, where each* $X_\alpha$ *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$*,* $\mathrm{card}(\Phi) = \aleph_0$.

**Lemma 9.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate system such that* $X_a, a \in A$*, are compact locally connected spaces and* $p_{ab}$ *are monotone surjections. If* $\mathbf{Y} = \{X_b, p_{cd}, B\}$ *is an approximate subsystem of* $\mathbf{X}$*, then the mapping* $q_{AB} : \lim \mathbf{X} \to \lim \mathbf{Y}$ *(defined in Step 2 of the proof of Theorem 15) is a monotone surjection.*

*Proof.* Let $P_a : \lim \mathbf{X} \to X_a, a \in A$, be the natural projection. Similarly, let $p_a : \lim \mathbf{Y} \to X_a, a \in B$, be the natural projection. From the definition of $q_{AB}$ (Step 2 of the proof of Theorem 15) it follows that $p_a q_{AB} = P_a$ for each $a \in B$. By virtue of [**10**, Corollary 4.5] and [**7**, Corollary 5.6] it follows that $P_a$ and $p_a$ are monotone surjections. Let us prove that $q_{AB}$ is a surjection. Let $y = (y_a : a \in B) \in \lim \mathbf{Y}$. The sets $P_a^{-1}(y_a), a \in B$, are non-empty since $P_a$ is surjective for each $a \in A$. From the compactness of $\lim \mathbf{X}$ it follows that a limit superior $Z = \mathrm{Ls}\{P_a^{-1}(y_a), a \in B\}$ is a non-empty subset of $\lim \mathbf{X}$. We shall prove that for each $z = (z_a : a \in A) \in Z$ we have $P_a(z) = y_a$. Suppose that $P_a(z) \neq y_a$. There exists a pair $U, V$ of open disjoint subsets of $X_a$ such that $y_a \in U$ and $P_a(z) \in V$. For a sufficiently large $b \in B$ the set $P_a(P_b^{-1}(_b))$ is in $U$ because (AS). This means that $P_a^{-1}(V) \bigcap P_b^{-1}(y_b) = \emptyset$ for a sufficiently large $b \in B$. This contradicts the assumption $z \in \mathrm{Ls}\{P_a^{-1}(y_a), a \in B\}$. Hence $q_{AB}$ is a surjection. In order to complete the proof it suffices to prove that $q_{AB}$ is monotone. Take a point $y \in \lim \mathbf{Y}$ and suppose that $q_{AB}^{-1}(y)$ is disconnected. There exists a pair $U, V$ of disjoint open sets in $\lim \mathbf{X}$ such that $q_{AB}^{-1}(y) \subseteq U \bigcup V$. From the compactness of $\lim \mathbf{X}$ it follows that $q_{AB}$ is closed. This means that there exists an open neighborhood $W$ of $y$ such that $q_{AB}^{-1}(y) \subseteq q_{AB}^{-1}(W) \subseteq U \bigcup V$. From the definition of the basis in $\lim \mathbf{Y}$ it follows that there exists an open set $W_a$ in some $X_a, a \in B$, such that $y \in p_a^{-1}(W_a) \subseteq W$. Moreover, we may assume that $W_a$ is connected since $X_a$ is locally connected. Then $P_a^{-1}(W_a)$ is connected since $P_a$ is monotone [**7**, Corollary 5.6]. Moreover, $q_{AB}^{-1}(y) \subseteq P_a^{-1}(W_a)$ and $P_a^{-1}(W_a) \subseteq U \bigcup V$ since $P_a = p_a q_{AB}$. This is impossible since $U$ and $V$ are disjoint open sets and $P_a^{-1}(W_a)$ is connected. □

**Theorem 16.** *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces such that* $\lambda \leq w(X_a) < \mathrm{card}(A)$ *for each* $a \in A$*. If* $cf(\mathrm{card}(A)) \neq \lambda$*, then* $X = \lim \mathbf{X}$ *is homeomorphic to a limit of a* $\lambda$*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$*, where each* $X_\alpha$ *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$*,* $\mathrm{card}(\Phi) = \lambda$*. Moreover, if* $\mathrm{card}(A)$ *is a regular cardinal, then* $X = \lim \mathbf{X}$ *is homeomorphic to a limit of a* $\lambda$*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$*, where each* $X_\alpha$ *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$*,* $\mathrm{card}(\Phi) = \lambda$*.*

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