Acta Math. Univ. Comenianae
Vol. LXXVI, 2(2007), pp. 179–188

SOME COMMENTS ON INJECTIVITY AND P-INJECTIVITY

R. YUE CHI MING

Abstract. A generalization of injective modules (noted GI-modules), distinct from p-injective modules, is introduced. Rings whose p-injective modules are GI are characterized. If M is a left GI-module, $E = \mathrm{End}({}_AM)$, then $E/J(E)$ is von Neumann regular, where $J(E)$ is the Jacobson radical of the ring E. A is semi-simple Artinian if, and only if, every left A-module is GI. If A is a left p. p., left GI-ring such that every non-zero complement left ideal of A contains a non-zero ideal of A, then A is strongly regular. Sufficient conditions are given for a ring to be either von Neumann regular or quasi-Frobenius. Quasi-Frobenius and von Neumann regular rings are characterized. Kasch rings are also considered.

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. A is called *semi-primitive* or *semi-simple* (resp. (a) left *non-singular*; (b) *right non-singular*) if $J = (0)$ (resp. (a) $Z = (0)$; (b) $Y = (0)$). An ideal of A will always mean a two-sided ideal of A. A is called *left* (resp. *right*) *quasi-duo* if every maximal left (resp. right) ideal of A is an ideal of A. It well-known that J, Z, Y are ideals of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent elements.

Following C. Faith, write "A is VNR" if A is a von Neumann regular ring [**8**]. A is called fully (resp. (1) fully left; (2) fully right) idempotent if every ideal (resp.(1) left ideal ; (2) right ideal) of A is idempotent.

It is well-known that A is VNR if and only if every left (right) A-module is flat (Harada ((1956); Auslander (1957)). Also, A is VNR if and only if every left (right) A-module is p-injective ([**2**], [**4**], [**12**], [**22**], [**23**]). Note that the Harada--Auslander's characterization may be weakened as follows: A is VNR if and only if every singular right A-module is flat (cf. [**38**, p. 147]).

Recall that a left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M ([**8**, p. 122], [**20**, p. 577], [**21**, p. 340], [**26**]). A is called a left p-injective ring if ${}_AA$ is p-injective. P-injectivity is similarly defined on the right side. A generalization of p-injectivity, noted YJ-injectivity, is introduced in [**29**](cf. also [**22**], [**39**]). YJ-injectivity is also called GP-injectivity by other authors (cf. [**4**], [**6**], [**15**]).

Received December 16, 2005.
2000 *Mathematics Subject Classification.* Primary 16D40, 16D50, 16E50.
Key words and phrases. Injective; GI-module; p-injective; YJ-injective; von Neumann regular; quasi-Frobenius ring.

${}_AM$ is called YJ-injective if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and every left A-homomorphism of Aa^n into M extends to one of A into M [**29**]. A is called a left YJ-injective ring if ${}_AA$ is YJ-injective. YJ-injectivity is similarly defined on the right side.

Note that A is left YJ-injective if and only if for every $0 \neq a \in A$, there exists a positive integer n such that a^nA is a non-zero right annihilator [**29**, Lemma 3].

Also, if A is right YJ-injective, then $Y = J$ [**28**, Proposition 1] (this is the origin of the notation). In recent years, p-injectivity and YJ-injectivity have drawn the attention of many authors ([**2**], [**4**], [**6**], [**8**, Theorem 6.4], [**11**], [**15**], [**16**], [**17**], [**20**], [**22**], [**23**], [**40**]).

We have consider the following generalization of injective modules.

Definition 1. A left A-module M is called GI (generalized injective) if, given any left submodule C of M which is isomorphic to a non-zero complement left submodule of M, any monomorphisms g, f of C into M, there exists a left A-homomorphism $h : M \to M$ such that $hf = g$. Write "A is a left GI-ring" if ${}_AA$ is GI.

Note that any simple left A-module is GI. Consequently, GI-modules generalize effectively injective modules.

GI-modules need not be p-injective (otherwise, any arbitrary ring would be fully left and right idempotent!).

The converse is not true either, as shown by the following result.

Theorem 1. *The following conditions are equivalent:*

(1) *A is a left Noetherian ring whose p-injective left modules are injective;*

(2) *Every p-injective left A-module is GI.*

Proof. (1) implies (2) evidently.

Assume (2). Let M be a p-injective left A-module, E the injective hull of ${}_AM$. Write $Q = {}_AM \oplus {}_AE$ and $S = \{(y, o); y \in M\}$. Then ${}_AS$ is a direct summand of ${}_AQ$ and ${}_AS \approx {}_AM$. If $i : M \to E$ is the inclusion map: $j : M \to Q$ and $k : E \to Q$ the canonical injections, since ${}_AQ$ is the direct sum of two p-injective left A-modules, then Q is p-injective and by hypothesis, ${}_AQ$ is GI. There exists a left A-homomorphism $h : Q \to Q$ such that $hki = j$. If $p : Q \to M$ is the canonical projection, then $v = phk : E \to M$ such that $vi = pj =$ identity map on M. Therefore ${}_AM$ is a direct summand of ${}_AE$ which yields $M = E$ is injective. We have shown that every p-injective left A-module is injective. Since any direct sum of p-injective left A-modules is p-injective, then every direct sum of injective left A-modules is injective which implies that A is left Noetherian [**7**, Theorem 20.1]. Thus (2) implies (1). □

As usual, A is called a left IF-ring if every injective left A-module is flat. The next theorem is motivated by [**38**, Proposition 6].

Theorem 2. *The following conditions are equivalent:*

(1) *A is quasi-Frobenius;*

(2) *A is a left IF-ring whose flat modules are GI;*

(3) *The direct sum of any injective and any projective left A-modules is GI.*

Proof. Assume (1). Since A is left perfect, any flat left A-module F is projective. Now F is injective by [**7**, Theorem 24.20], hence GI. Therefore (1) implies (2).

Assume (2). Let Q be a direct sum of an injective and a projective left A-modules. Then Q is the direct sum of two flat left A-modules which is therefore flat. By hypothesis, ${}_AQ$ is GI and therefore (2) implies (3).

Assume (3). Let P be a non-zero projective left A-module, E the injective hull of ${}_AP$. Write $Q ={}_A P \oplus_A E$ and $S = \{(y, 0); y \in P\}$. Then ${}_AS \approx_A P$ and ${}_AS$ is a direct summand of ${}_AQ$. By hypothesis, ${}_AQ$ is GI. The proof of Theorem 1 then shows that ${}_AP$ must be injective. By [**7**, Theorem 24.20], A is quasi-Frobenius and (3) implies (1). □

Corollary 2.1. *If flat left A-modules coincide with GI left A-modules, then A is quasi-Frobenius.*

Proof. By hypothesis, A is a left IF-ring. The corollary then follows from Theorem 2 (2). □

The proof of Theorem 1 shows that if the direct sum of any two GI left A-modules is GI, then every GI left A-module is injective. The next proposition then follows.

Proposition 3. *A is semi-simple Artinian if and only if every left A-module is GI.*

Given a left A-module M, $\mathrm{End}(M)$ denotes, as usual, the ring of endomorphisms of ${}_AM$. We now turn to an analogous result of a well-known theorem [**7**, Theorem 19.27].

Theorem 4. *Let M be a GI left A-module. If $E = \mathrm{End}(M)$, $\mathrm{J}(E)$ is Jacobson radical of E, then $E/\mathrm{J}(E)$ is VNR and $\mathrm{J}(E) = \{f \in E/ \ker f$ is essential in ${}_AM\}$.*

Proof. Write $E = \mathrm{End}(M)$, $\mathrm{J}(E)$ the Jacobson radical of E. Set $V = \{f \in E/ \ker f$ is essential in ${}_AM\}$. It is well-known that V is an ideal of E. We first show that $V \subseteq \mathrm{J}(E)$.

For any $f \in V$, $d \in E$, since $\ker f \cap \ker(1 - df) = 0$, then $\ker(1 - df) = 0$. With $u = 1 - df$, u is an isomorphism of M onto uM. Let $v : uM \to M$ be the inverse isomorphism of u. Since ${}_AM$ is GI, with $j : uM \to M$ the inclusion map, there exists an endomorphism h of ${}_AM$ such that $hj = v$.

Then

$$hu(m) = hj(u(m)) = v(u(m)) = m \qquad \text{for all } m \in M$$

which implies that hu is the identity map on M. Therefore $1-df$ is left invertible in E for every $d \in E$, proving that $f \in \mathrm{J}(E)$.

Now, let $\overline{0} \neq \overline{g} \in E/\mathrm{J}(E)$, $g \in E$. Then $g \notin V$ (because $V \subseteq \mathrm{J}(E)$). By Zorn's Lemma, there exists a non-zero complement submodule K of M such that $\ker g \oplus K$ is an essential submodule of ${}_AM$. If $r : K \to M$ is the restriction of g to K, then r is a monomorphism and consequently $r : K \to r(K)$ is an isomorphism. Let $s : r(K) \to K$ be the inverse isomorphism. Then $sr =$ identity map on K.

Since K is a non-zero complement submodule of M, if $i : K \to M$ is the canonical injection, then $is : r(K) \to M$ and is extends to an endomorphism t of ${}_AM$. For any $k \in K$,

$$t(g(k)) = t(r(k)) = isr(k) = k$$

which implies that $K + \ker g \subseteq \ker(gtg - g)$ and hence $gtg - g \in V \subseteq \mathrm{J}(E)$. Therefore $\overline{g}\overline{t}\overline{g} = \overline{g} \in E/\,\mathrm{J}(E)$ which proves that $E/\,\mathrm{J}(E)$ is VNR.

Now suppose there exists $w \in \mathrm{J}(E)$ such that $w \notin V$. Then the above proof shows that there exists $z \in E$ such that $y = w - wzw \in V$. But there exists $q \in E$ such that $(1 - zw)q = 1$. Therefore $y = w(1 - zw)$ yields $yq = w \cdot 1 = w$, whence $w \in V$ (since V is an ideal of E), which is a contradiction! Therefore $\mathrm{J}(E) \subseteq V$ and finally, $J(E) = V = \{f \in E/\ker f$ is essential in ${}_AM\}$. □

Proposition 5. *If A is a left GI-ring, then every non-zero-divisor of A is invertible in A. Consequently, A coincides with its classical left (and right) quotient ring.*

Proof. Let c be a non-zero divisor of A. Define $f : Ac \to A$ by $f(ac) = a$ for all $a \in A$. Then f is a well-defined left A-homomorphism which is a monomorphism. Now ${}_AAc \approx {}_AA$ and if $Ac \to A$ is the inclusion map, since ${}_AA$ is GI, there exists a left A-homomorphism $h : A \to A$ such that $hi = f$. If $h(1) = u \in A$, then

$$1 = f(c) = hi(c) = h(c) = ch(1) = cu.$$

Then $c = cuc$ which yields $c(1-uc) = 0$, whence $uc = 1$. Therefore c is invertible in A and consequently, A coincides with its classical left (and right) quotient ring. □

Call A a left TC-ring if every non-zero complement left ideal of A contains a non-zero ideal of A.

Corollary 5.1. *If A is a left TC, left p.p., left GI-ring, then A is strongly regular.*

Proof. Since A is left non-singular, left TC, then A is reduced by [**34**, Lemma 1]. Now A is a reduced left p.p. ring which implies that every element a of A is of the form $a = ce$, where c is a non-zero-divisor and e is a central idempotent in A [**30**, Theorem 2]. By Proposition 5, c is invertible in A. Then

$$a = ce = cec^{-1}c = cec^{-1}ce \qquad \text{(since } e \text{ is central)}$$

which yields $a = ac^{-1}a$. Therefore A is VNR and since A is reduced, then A is strongly regular. □

In [**17**, Example 2.4], the given ring A has the following property: for every $y \in J$, the Jacobson radical of A, $l(y) = r(y)$. This motivates the next result.

Proposition 6. *The following conditions are equivalent:*

(1) *A is strongly regular;*
(2) *A is a left quasi-duo ring whose simple left modules are either YJ-injective or flat and for every $u \in J$, $l(u) = r(u)$.*

Proof. (1) implies (2) evidently.

Assume (2). Suppose there exists $0 \neq v \in J$ such that $v^2 = 0$. If $I = AvA + l(v)$, suppose that $I \neq A$. Let M be a maximal left ideal of A containing I. If ${}_AA/M$ is YJ-injective, since $v^2 = 0$, every left A-homomorphism of Av into A/M extends to one of A into A/M.

Define

$$g : Av \to A/M \quad \text{by} \quad g(av) = a + M \qquad \text{for all } a \in A.$$

Then

$$1 + M = g(v) = vy + M \qquad \text{for some } y \in A.$$

Since $vy \in J \subseteq M$, then $1 \in M$, which contradicts $M \neq A$.

If ${}_AA/M$ is flat, then $v \in I \subseteq M$ implies that $v = vd$ for some $d \in M$ [**3**, p. 458]. Now $(1-d) \in r(v) = l(v) \subseteq M$ which yields $1 \in M$, again a contradiction! Therefore $I = A$. Then $1 = s+t$, $s \in AvA$, $t \in l(v)$ and $v = sv$. Since $s \in J$, $1-s$ is left invertible in A which yields $v = 0$, contradicting our original hypothesis. We have shown that J must be a reduced ideal of A.

Now suppose that $J \neq 0$. If $0 \neq w \in J$, since J is reduced, for any positive integer m,

$$l(w^m) = l(w) = r(w) = r(w^m).$$

Set $W = AwA + l(w)$. If $W \neq A$, let N be a maximal left ideal of A containing W. If ${}_AA/N$ is YJ-injective, there exists a positive integer n such that every left A-homomorphism of Aw^n into A/N extends to one of A into A/N. We may define a left A-homomorphism

$$h : Aw^n \to A/N \quad \text{by} \quad h(aw^n) = a + N \qquad \text{for all } a \in A.$$

Then

$$1 + N = h(w^n) = w^n z + N \qquad \text{for some } z \in A.$$

Now $w^n z \in J \subseteq N$ implies that $1 \in N$, contradicting $N \neq A$. If ${}_AA/N$ is flat, $w = wc$ for some $c \in N$.

Now $1 - c \in r(w) = l(w) \subseteq N$ implies that $1 \in N$, again a contradiction! Therefore $W = A$ and $1 = p + q$, $p \in AwA$, $q \in l(w)$, whence $w = pw$.

Now $1 - p$ is left invertible in A which yields $w = 0$, contradicting our first hypothesis. We have proved that $J = 0$. Since A is left quasi-duo, then A must be a reduced ring (cf. the proof of "(2) implies (3)" in [**27**, Theorem 2.1]). Now A is a left quasi-duo reduced ring whose simple left modules are either YJ-injective or flat which yields A strongly regular by a result of Chen and Ding [**5**, Corollary 7]. Thus (2) implies (1). □

In the above proposition, the expression "$l(u) = r(u)$" is not superfluous as shown by the following example.

Example. If A denotes the 2×2 upper triangular matrix ring over a field, then A is a left and right quasi-duo, Artinian, hereditary ring whose simple one-sided modules are either injective or projective but not semi-prime (indeed, the Jacobson radical J of A is non-zero and $J^2 = 0$).

Singular modules play an important role in the theory of modules and rings. It is well-known that A is a left non-singular ring if and only if A has a VNR maximal left quotient ring Q. In that case, ${}_AQ$ is the injective hull of ${}_AA$ and Q is a left self-injective ring. If A is left non-singular, then for any injective left A-module M, the singular submodule $Z(M)$ is injective [**25**, Theorem 4]. If A is left self-injective regular, then for any essentially finitely generated left A-module M, $Z(M)$ is a direct summand of ${}_AM$ [**39**, Corollary 10]. The right singular ideal will be crucial in the next result. Recall that M is a maximal right annihilator ideal of A if $M = r(S)$ for some non-zero subset S of A such that for any right annihilator R which strictly contains M, $R = A$. In that case, $M = r(s)$ for any $0 \neq s \in S$.

Proposition 7. *Let A be right YJ-injective such that each finitely generated right ideal is either a projective right annihilator or a maximal right annihilator. Then A is either VNR or quasi-Frobenius.*

Proof. First suppose that $Y \neq 0$. For any $0 \neq y \in Y$, since $r(y)$ is an essential right ideal of A, then yA cannot be a projective right annihilator. Therefore yA is a maximal right annihilator.

If $u \notin yA$, then $yA + uA = A$, whence $Y = yA$. We have just shown that Y is a minimal right ideal of A. If $a \in A$, $a \notin Y$, then $aA + yA = A$. This shows that Y must be a maximal right ideal of A. Since Y cannot contain a non-zero idempotent, then Y is an essential right ideal of A. For any non-zero proper right ideal I of A, $I \cap Y \neq 0$ which implies that $I \cap Y = Y$ by the minimality of Y. Therefore $Y \subseteq I$ which yields $Y = I$ by the maximality of Y. We have proved that Y is the unique non-zero proper right ideal of A. A is therefore right Artinian local with $J = Y$.

Let V denote a minimal left ideal of A. If $V = Av$, $v \in A$, either $V^2 = 0$ or V is a direct summand of ${}_AA$. If $v^2 = 0$, since A is right YJ-injective, Av is a left annihilator by [**29**, Lemma 3]. If V is a direct summand of ${}_AA$, then V is again a left annihilator. We have shown that every minimal left ideal of A must be a left annihilator. Since, by hypothesis, every finitely generated right ideal of A is a right annihilator, then A is quasi-Frobenius by [**18**, Proposition 1].

Now suppose that $Y = 0$. If $0 \neq b \in A$ such that bA is a maximal right annihilator, since $Y = 0$, bA cannot be an essential right ideal of A. Therefore $bA \cap cA = 0$ for some $0 \neq c \in A$. Now $bA \oplus cA = A$ (bA being a maximal right annihilator). Then every principal right ideal of A must be projective.

Now for any $0 \neq d \in A$, there exists a positive integer m such that Ad^m is a non-zero left annihilator [**29**, Lemma 3]. Since d^mA is a projective right A-module, then $r(d^m)$ is a direct summand of A_A. Therefore $Ad^m = l(r(Ad^m)) = l(r(d^m))$ is a direct summand of ${}_AA$. We have just proved that every left A-module must be YJ-injective. By [**40**, Theorem 9], A is VNR. □

Proposition 8. *The following conditions are equivalent for a ring A with centre C:*

(1) *A is VNR;*

(2) *A is a semi-prime ring whose essential left ideals are idempotent and for every maximal ideal M of C, A/AM is a VNR ring.*

Proof. (1) implies (2) evidently.

Assume (2). If $d \in C$ such that $d^2 = 0$, then $(Ad)^2 = Ad^2 = 0$ implies that $d = 0$. C is therefore a reduced ring. For any $c \in C$, let K be a complement left ideal of A such that $L = (Ac + l(c)) \oplus K$ is an essential left ideal of A. Then $Kc = cK \subseteq Ac \cap K = 0$ implies that $K \subseteq l(c)$, whence $K \subseteq K \cap (Ac + l(c)) = 0$. Therefore $L = Ac + l(c)$ and by hypothesis, $L = L^2$.

Now $c = \sum_{i=1}^{n}(a_ic + u_i)(b_ic + v_i)$, $a_i, bi \in A$, $ui, vi \in l(c)$, and

$$c - \sum_{i=1}^{n} a_icb_ic = \sum_{i=1}^{n}(a_icv_i + u_ib_ic + u_iv_i) = \sum_{i=1}^{n} u_iv_i,$$

since $a_icv_i = a_iv_ic = 0$, $u_ib_ic = u_icb_i = 0$. If $w \in Ac \cap l(c)$, $w = dc$, $d \in A$, $dc^2 = wc = 0$ and therefore $cAdc = 0$ which implies that $(Adc)^2 = 0$. Since A is semi-prime, $Adc = 0$ which yields $w = dc = 0$.

Now $c - \sum_{i=1}^{n} a_icb_ic = \sum_{i=1}^{n} u_iv_i \in Ac \cap l(c) = 0$ which yields $c = \sum_{i=1}^{n} a_icb_ic = czc$, where $z = \sum_{i=1}^{n} a_ib_i \in A$. Set $y = c^2z^3$. Then

$$cyc = (czc)zczc = (czc)zc = c \qquad \text{and} \qquad c^2z = zc^2 = czc = c.$$

For every $b \in A$,

$$zc^2b = cb = bc = bc^2z = c^2bz$$

and hence $z^3c^2b = c^2bz^3$ which shows that

$$yb = c^2z^3b = z^3c^2b = c^2bz^3 = bc^2z^3 = by,$$

whence $y \in C$. Therefore C is VNR. Then (2) implies (1) by [**1**, Theorem 3]. □

Proposition 9. *The following conditions are equivalent for a commutative ring A:*

(1) *A is VNR;*
(2) *For each non-zero principal ideal P of A, there exists a positive integer n such that P^n is generated by a non-zero idempotent;*
(3) *For each non-zero principal ideal P of A, there exists a positive integer n such that P^n is a non-zero flat complement ideal of A.*

Proof. It is clear that (1) implies (2) while (2) implies (3).

Assume (3). First suppose that A is not reduced. Then there exists $0 \neq b \in A$ such that $b^2 = 0$. By hypothesis, Ab is a non-zero flat complement ideal of A. Now $Ab \approx A/l(b)$ and since $b \in l(b)$, then $b = bd$ for some $d \in l(b)$ [**3**, p. 458]. Therefore $bd = db = 0$ implies that $b = 0$, a contradiction! We have shown that A must be reduced.

By [**33**, Proposition 1], every complement ideal of A is an annihilator. By hypothesis, for any $0 \neq a \in A$, there exists a positive integer n such that Aa^n

is a non-zero complement ideal of A and hence Aa^n is an annihilator. By [**29**, Lemma 3], A is YJ-injective. Then (3) implies (1) by [**29**, Lemma 5]. □

Question. Is A VNR if every finitely generated right ideal of A is a flat complement right ideal of A?

Recall that A is a right coherent ring if every finitely generated right ideal of A is finitely presented. For example, VNR rings are coherent.

Proposition 10. *If A is a commutative ring, then every factor ring of A is an IF-ring if and only if every factor ring of A is a coherent p-injective ring.*

Proof. Suppose that every factor ring B of A is a coherent p-injective ring. Then every factor ring B is a self FP-injective ring by [**35**, Proposition 3]. By [**13**, Corollary 2.5], every finitely generated ideal of B is an annihilator. Since B is coherent, then B is an IF-ring by [**9**, Theorem 2.1]. The converse is well-known. □

Proposition 11. *The following conditions are equivalent:*

(1) *Every factor ring of A is QF;*
(2) *A has the following properties: (a) A satisfies the maximum condition on left annihilators; (b) Every finitely generated left ideal of A is principal; (c) A left A-module M is p-injective if and only if M is flat.*

Proof. Assume (1). Then A is a principal left ideal ring which is QF [**7**, Proposition 25.4.6B]. Now every p-injective left A-module M is injective, which implies that M is flat [**7**, Theorem 24.12]. If ${}_AN$ is flat, since A is left perfect, then ${}_AN$ is projective [**21**, p. 392] which implies that ${}_AN$ is injective [**7**, Theorem 24.20]. Therefore ${}_AN$ is p-injective and (1) implies (2).

Assume (2). Then A is a left p-injective ring. Since A is a left IF-ring by (c), then A is right p-injective. □

Since A is left p-injective with maximum condition on left annihilators, then A is right Artinian [**22**, p. 34]. Then (2) implies (1) by [**18**, Proposition 2] and [**7**, Proposition 25.4.6B].

(Condition (a) is not superfluous since any VNR ring satisfies Conditions 2 (b), (c).)

Acknowledgment. I would like to thank the referee for helpful comments and suggestions leading to this revised version of my paper.

References

1. Armendariz E. P., Fisher J. W. and Steinberg S. A., *Central localizations of regular rings*, Proc. Amer. Math. Soc. **46** (1974), 315–321.
2. Baccella G., *Generalized V-ring and von Neumann regular rings*, Rend. Sem.Mat. Univ. Padova **72** (1984), 117–133.
3. Chase S. U., *Direct product of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.
4. Chen J. L. and Ding N. Q., *On regularity of rings*, Algebra Colloquium **8** (2001), 267–274.
5. ______, *On a generalization of V-rings and* SF*-rings*, Kobe J., Math. **11** (1994), 101–105.

6. Chen J. L., Zhou Y. Q. and Zhu Z. M., *GP-injective rings need not be P-injective*, Comm. Algebra **33** (2004), 2395–2402.
7. Faith C., *Algebra II: Ring Theory*, 191, Springer-Verlag 1976.
8. ______, *Rings and Things and a Fine Array of Twentieth Century Associative Algebra*, AMS Math. Surveys and Monographs 65, 1999.
9. Gomez-Pardo J. L. G. and Gonzales N. R., *On some properties of IF-rings,* Quaestiones Math. **5** (1983), 395–405.
10. Goodearl, K. R., *Von Neumann regular rings*, Pitman 1979.
11. Hirano Y., *On non-singular p-injective rings*, Publicacions Mat. **38** (1994), 455–461.
12. Hong C. Y., Kim N. K. and Lee Y., *On rings whose homomorphic images are p-injective*, Comm. Algebra **30** (2002), 261–271.
13. Jain S., *Flat and FP-injectivity*, Proc. Amer. Math. Soc. **41** (1973), 437–442.
14. Kasch F., *Modules and rings*, London Math. Soc. Monographs 17, 1982.
15. Nam S. B., Kim N. K. and Kim J. Y., *On simple GP-injective modules*, Comm. Algebra **23** (1995), 5437–5444.
16. Nicholson W. K. and Yousif M. F., *On completely principally injective rings*, Bull. Austral. Math. Soc. **187** (1994), 513–518.
17. Puninski G., Wisbauer R. and Yousif M., *On p-injective rings*, Glasgow Math. J. **37** (1995), 373–378.
18. Storrer H. H., *A note on quasi-Frobenius rings and ring epimorphisms*, Canad. Math. Bull. **12** (1969), 287–292.
19. Tuganbaev, A., *Rings close to regular*, Kluwer Acad. Publ. 545, 2002.
20. ______, *Maxrings and V-rings, Handbook of Algebra Vol. 3*, Elsevier North-Holland 2003.
21. Wisbauer, R., *Foundations of module and ring theory*, Gordon and Breach 1991.
22. Xue, W. M., *A note on YJ-injectivity*, Riv. Mat. Univ. Parma **6**(1) (1998), 31–37.
23. Yousif M. F., *SI-modules*, Math. J. Okayama Univ, **28** (1986), 133–146.
24. Yousif M., Zhou Y. Q. and Zeyada N., *On pseudo-Frobenius rings*, Canad. Math. Bull. **48**(2) (2005), 317–320.
25. Yue Chi Ming R., *A note on singular ideals*, Tôhoku Math. J. **21** (1969), 337–342.
26. ______, *On simple p-injective modules*, Math. Japonica **19** (1974), 173–176.
27. ______, *On von Neumann regular rings, VI*, Rend. Sem. Mat. Univ. Torino **39** (1981), 75–84.
28. ______, *On regular ring and self-injective rings, II*, Glasnik Matematički **18**(38) (1983), 221–229.
29. ______, *On regular rings and Artinian rings, II*, Riv. Mat. Univ. Parma **4**(11) (1985), 101–109.
30. ______, *On von Neumann regular rings, XI*, Bull. Math. Soc. Sci. Math. Roumanie **30**(76) (1986), 371–379.
31. ______, *A note on semi-prime rings*, Monatshefte für Math. **101** (1986), 173–182.
32. ______, *On Kasch rings and CC modules*, Tamkang J. Math. **17** (1986), 93–104.
33. ______, *Annihilators and strongly regular rings*, Rend. Sem. Fac. Sci. Univ. Cagliari **57** (1987), 51–59.
34. ______, *On strongly regular rings*, Rev. Roumanie Math. Pures Appl. **38** (1993), 281–287.
35. ______, *On injectivity and p-injectivity, III*, Riv. Mat. Univ. Parma **6**(4) (2001), 45–50.
36. ______, *On p-injectivity, YJ-injectivity and quasi-Frobeniusean Rings*, Comment. Math. Univ. Carolinae **43** (2002), 33–42.
37. ______, *On injectivity and p-injectivity, IV*, Bull. Korean Math. Soc. **40** (2003), 223–234.
38. ______, *On injektivity, p-injektivity and YJ-injektivity*, Acta Math. Univ. Comenianae **73** (2004), 141–149.
39. Zelmanowitz, J., *Injective hulls of torsionfree modules*, Canad. J. Math. **23**(1971), 1094–1101.
40. Zhang J. L. and Wu J., *Generalizations of principal injectivity*, Algebra Colloquium **6** (1999), 277–282.

R. Yue Chi Ming, Université Paris VII-Denis Diderot, UFR de Mathématiques, UMR 9994 CNRS,
2, Place Jussieu, 75251 Paris Cedex 05, France