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NORMAL GENERATION OF UNITARY GROUPS OF CUNTZ ALGEBRAS BY INVOLUTIONS

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Abstract. In purely infinite factors, P. de la Harpe proved that a normal subgroup of the unitary group which contains a non-trivial self-adjoint unitary contains all self-adjoint unitaries of the factor. Also he proved the same result in finite continuous factors. In a previous work the author proved a similar result in some types of unital AF-algebras. In this paper we extend the result of de la Harpe, concerning the purely infinite factors to a main example of purely infinite C^*-algebras called the Cuntz algebras $\mathcal{O}_n (2 \leq n \leq \infty)$ and we prove that $\mathcal{U}(\mathcal{O}_n)$ is normally generated by some non-trivial involution. In particular, in the Cuntz algebra $\mathcal{O}_\infty$ we prove that $\mathcal{U}(\mathcal{O}_\infty)$ is normally generated by self-adjoint unitary of odd type.

1. Introduction

Let $\mathcal{A}$ be any unital C^*-algebra. The group of unitaries and the set of projections of $\mathcal{A}$ are denoted by $\mathcal{U}(\mathcal{A})$, $\mathcal{P}(\mathcal{A})$ respectively. The involutions of $\mathcal{A}$ are the set of self-adjoint unitaries ($*$-symmetries). In several types of C^*-algebras, we have that the involutions generate all the unitaries. In the case of von Neumann factors, M. Broise in [**3**]; proved the following main theorem.

Theorem 1.1. [**3**, Theorem 1] *If $\mathcal{B}$ is a factor of type II_1 or III, then the set of involutions generates $\mathcal{U}(\mathcal{B})$.*

Also, in the case of simple, purely infinite C^*-algebras, M. Leen proved the following result.

Theorem 1.2. [**10**, Theorem 3.8] *If A is a simple, unital purely infinite C^*-algebra, then the set of $*$-symmetries of A forms a set of generators for $\mathcal{U}_0(A)$, (where $\mathcal{U}_0(A)$ denotes the identity component of the unitary group of A).*

The Cuntz algebras are interesting examples of simple, unital purely infinite C^*-algebras, which was introduced by Cuntz in [**5**] this C^*-algebra is generated by isometries that have orthogonal ranges (for more information see [**6**, V. 4]). As shown in [**5**] the unitary group of the Cuntz algebras are connected. Now let us recall these definitions.

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Definition 1.3. The Cuntz algebra $\mathcal{O}_n$, where $2 \leq n$, is the universal C^*-algebra which is generated by isometries $s_1, s_2, \ldots, s_n$, such that

$$\sum_{i=1}^{n} s_i s_i^* = 1 \tag{1}$$

with $s_i^* s_j = 0$, when $i \neq j$. The Cuntz algebra $\mathcal{O}_\infty$ is generated by infinite number of such isometries.

Remark 1.4. [**6**, V. 4] Recall that a universal C^*-algebra $\mathcal{O}_n$ means, whenever $t_1, t_2 \ldots t_n$ form another set of isometries satisfying (1), then there is a unique $*$-homomorphism ρ of $\mathcal{O}_n$ onto $C^*(\{t_1, t_2, \ldots t_n\})$ such that $\rho(s_i) = t_i$, for all $1 \leq i \leq n$.

In this paper, the projection $s_i s_i^*$ is denoted by p_i, and these projections are called the standard projections of the Cuntz algebras. The corresponding involution $1 - 2p_i$ is denoted by u_i.

Let us recall the following main results concerning the Cuntz algebras.

Theorem 1.5. [**5**] *The Cuntz algebras $\mathcal{O}_n(2 \leq n \leq \infty)$ are simple unital purely infinite C^*-algebras.*

Using the fact that $K_1(\mathcal{O}_n) \cong 0$ (see [**4**, 3.8]) and $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}(A)_0$ (see [**4**, p. 188], M. Leen's result (Theorem 1.2) shows that the set of $*$-symmetries of $\mathcal{O}_n(2 \leq n \leq \infty)$ generates the unitary group $\mathcal{U}(\mathcal{O}_n)$.

Definition 1.6. A group G is normally generated by an element x if the only normal subgroup of G containing x is G itself.

If $u = 1 - 2p$ is an involution in a factor $\mathcal{B}$, then P. de la Harpe defined the notion of the type of u to be the pair (x, y), where $x = D(1-p)$ and $y = D(p)$, as D denotes a normalized dimension function on $\mathcal{B}$, see [**7**]. He proved that any normal subgroup $\mathcal{N}$ of $\mathcal{U}(\mathcal{B})$, which is not contained in the circle $\mathbb{S}^1$, contains a non-trivial involution, and then contains all the involutions of $\mathcal{B}$ (see [**8**, Proposition 2]). Afterwards, P. de la Harpe used Broise's result (Theorem 1.1), and he proved the following theorem.

Theorem 1.7. [**8**] *If $\mathcal{B}$ is a factor of type II_1 or III and $\mathcal{N}$ is any normal subgroup of $\mathcal{U}(\mathcal{B})$, which is not contained in the circle $\mathbb{S}^1$, then $\mathcal{N} = \mathcal{U}(\mathcal{B})$.*

If v is an involution of $\mathcal{O}_n(2 \leq n \leq \infty)$, then as introduced in [**1**], we define the type of v to be the element $[p]$ in $K_0(\mathcal{O}_n)$, where $v = 1 - 2p$. Since the $K_0(\mathcal{O}_n)$ is a cyclic group, the type of v is an integer. In Section 2, we show that a normal subgroup $\mathcal{N}$ of $\mathcal{U}(\mathcal{O}_n)$, $n < \infty$ contains all the involutions if

1. $\mathcal{N}$ contains an involution of the type 1 (i.e. [1]), or
2. $\mathcal{N}$ contains a non-trivial involution and $n - 1$ is a prime number, or
3. $\mathcal{N}$ contains a non-trivial involution such that its type and $n-1$ are relatively prime. Then using M. Leen's result in Theorem 1.2, we prove that $\mathcal{U}(\mathcal{O}_n)$ is normally generated by a non-trivial involution.

In Section 3, we show that if $\mathcal{N}$ contains an involution of odd type, then $\mathcal{N}$ contains all the involutions of $\mathcal{O}_\infty$. Consequently, we use M. Leen's result in order to prove that $\mathcal{U}(\mathcal{O}_n)$ is normally generated by an involution of odd type.

Now, let us recall main results concerning purely infinite C^*-algebras, that might be used throughout this paper.

Proposition 1.8. [**4**, 1.5] *In any C^*-algebra A, the following hold:*

(i) *If p, q are infinite projections and $pq = 0$, then $p+q$ is an infinite projection.*
(ii) *If p is an infinite projection, and $p' \sim p$, then p' is an infinite projection.*
(iii) *If p and q are infinite projections, then there exists an infinite projection p' such that $p \sim p'$ and $p' < q$, moreover $q - p'$ is an infinite projection.*

Theorem 1.9. [**2**, 6.11.9] *Two infinite projections in a simple unital C^*-algebra are equivalent if and only if they have the same K_0-class. Two non-trivial projections with the same K_0-class in a purely infinite C^*-algebra are unitarily equivalent.*

2. The $\mathcal{O}_n (2 \leq n < \infty)$ Case

We prove the following result which is valid for the Cuntz algebras $\mathcal{O}_n (2 \leq n \leq \infty)$. The proof is similar to [**1**, Lemma 2.2], in the case of the UHF-algebras. For completeness we have.

Lemma 2.1. *Let u and v be two involutions of $\mathcal{O}_n (2 \leq n \leq \infty)$. Then u is conjugate to v if and only if they have the same type.*

Proof. If u and v are conjugate involutions of $\mathcal{O}_n (2 \leq n \leq \infty)$, then as in [**1**, Lemma 2.2], there exists a unitary w in $\mathcal{U}(\mathcal{O}_n)$ such that $u = wvw^*$. But $u = 1-2e$ and $v = 1 - 2f$ for some projections e, f in A, so $u = w(1 - 2f)w^* = 1 - 2wfw^*$, therefore $e = wfw^*$ and by Theorem 1.9, $[e] = [f]$.

Conversely, assume that the involutions u and v have the same type. Then $u = 1 - 2p$ and $v = 1 - 2q$, for some $p, q \in \mathcal{P}(\mathcal{O}_n)$ with $[p] = [q]$ in $K_0(\mathcal{O}_n)$ group. Then by Theorem 1.9, the projections p and q are unitarily equivalent, and therefore $u = wvw^*$ for some $w \in \mathcal{U}(\mathcal{O}_n)$. □

Proposition 2.2. *In $\mathcal{O}_n (2 \leq n \leq \infty)$, the involution $u_i (i = 1, \dots n)$ has type 1.*

Proof. As $u_i = 1 - 2p_i$, the type of u_i is $[p_i]$. By definition $p_i = s_i s_i^*$ and $s_i^* s_i = 1$; therefore by Theorem 1.9, we have $[p_i] = [1]$. □

The following result is based on [**4**, 3.7, 3.8]; that is $K_0(\mathcal{O}_n) \simeq \mathbb{Z}_{n-1}$.

Proposition 2.3. *If $0 \leq k \leq n - 2$; $n < \infty$, then there exists an involution in $\mathcal{O}_n$ of type k (in fact, of type $k[1]$).*

Proof. Let $p_1, p_2, \dots p_n$ be the standard projections of $\mathcal{O}_n$, and $v_k = 1 - 2(p_1 + p_2 + \cdots p_k)$; for $0 \leq k \leq n - 2$. Then v_k is an involution in $\mathcal{O}_n$ of type equal to k. □

Lemma 2.4. *If $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(\mathcal{O}_n)(n < \infty)$, which contains an involution of the type 1([**1**]), then $\mathcal{N}$ contains an involution of any given type.*

Proof. As $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(\mathcal{O}_n)$, and it contains an involution of the type 1, then by Lemma 2.1, $\mathcal{N}$ contains $u_i(i = 1, \ldots n)$. Then $u_1u_2 = (1-2p_1)(1-2p_2) = 1-2(p_1+p_2)$, which is an involution of type 2, contained in $\mathcal{N}$. Also $u_1u_2u_3$ is an involution in $\mathcal{N}$ of type 3. Keep going we have $u_1u_2 \ldots u_k = 1-2(p_1+p_2+\cdots+p_k)$ is an involution in $\mathcal{N}$ of type $k(1 \leq k \leq n-2)$, hence $\mathcal{N}$ contains an involution of any given type, which proves the required. □

Lemma 2.5. *Let $\mathcal{N}$ be a normal subgroup of $\mathcal{U}(\mathcal{O}_n)$, and suppose that $n-1$ is a prime number. If $\mathcal{N}$ contains a non-trivial involution of $\mathcal{O}_n$, then $\mathcal{N}$ contains the involution u_1.*

Proof. Suppose that $v \in \mathcal{N}$ such that $v = 1-2p$ and v is of type k, i.e. $[p] = k$. If $k = n-1$, then $[p] = 0 \in K_0(\mathcal{O}_n)$, by Proposition 1.8(ii) we must have $p = 0$ and then $v = 1$ which gives a contradiction as v is non-trivial. Therefore we consider $1 \leq k \leq n-2$. We may assume that $p < 1$, since if $p = 1$, then $v = -1$ which is an involution of type one, and this ends the proof. As $n-1$ is a prime, there exist integers s and t such that $sk + t(n-1) = 1$, then $sk = 1$ in $\mathbb{Z}_{n-1}$. By Proposition 1.8(iii), we can find mutually orthogonal projections $q_1, q_2, \ldots q_s$, with $[q_i] = [p]$, $i = 1, \ldots s$. Let $v_i = 1-2q_i$, $i = 1, \ldots s$. Then for every i, v_i there is an involution of the type k, which belongs to $\mathcal{N}$ as it is conjugate to v. Therefore

$$v_1v_2 \ldots v_s = 1-2(q_1+q_2+\cdots+q_s)$$

is an involution in $\mathcal{N}$, and the type of $v_1v_2 \ldots v_s$ is $sk = 1 \in \mathbb{Z}_{n-1}$. □

By imitating the same proof in the previous result, we can rewrite Lemma 2.5 as follows:

Lemma 2.6. *Let $\mathcal{N}$ be a normal subgroup of $\mathcal{U}(\mathcal{O}_n)$. If $\mathcal{N}$ contains an involution of type k such that k and $n-1$ are relatively primes, then $\mathcal{N}$ contains an involution of type 1.*

Therefore, we have the following theorem.

Theorem 2.1. *A non-trivial involution u normally generates the group $\mathcal{U}(\mathcal{O}_n)$ if either*

(1) *$n-1$ is a prime number, or*
(2) *the type of u is relatively prime to $n-1$.*

Proof. If $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(\mathcal{O}_n)$ that contains a non-trivial involution with hypothesis of either (1) or (2), then by either Lemma 2.5 or Lemma 2.6, $\mathcal{N}$ contains an involution of type 1, therefore by Lemma 2.4, $\mathcal{N}$ contains an involution of any given type, then by Lemma 2.1 it contains all involutions, hence by Leen's result $\mathcal{N} = \mathcal{U}(\mathcal{O}_n)$. □

3. The $\mathcal{O}_\infty$ Case

In this section, we discuss the case of the Cuntz algebra $\mathcal{O}_\infty$. We may ask, if a normal subgroup of $\mathcal{U}(\mathcal{O}_\infty)$ contains a non-trivial involution u_0, then does it contain all the involutions of $\mathcal{O}_\infty$? Hence by using Leen's result in Theorem 1.2, $\mathcal{O}_\infty$ is normally generated by a non-trivial involution u_0. We give a positive answer to the question under some conditions on the non-trivial involution u_0.

Recall that the Cuntz algebra $\mathcal{O}_\infty$ is the universal unital C^*-algebra generated by an infinite sequence of isometries $s_1, s_2, s_3, \dots$ with mutually orthogonal projections $p_j = s_j s_j^*$. The involution $1 - 2p_j$ is denoted by u_j $(1 \leq j \leq \infty)$.

Now let us recall the following main results concerning $\mathcal{O}_\infty$.

Theorem 3.1. [**4**, 3.11]

(i) $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$.
(ii) $K_1(\mathcal{O}_\infty) \cong 0$.

Theorem 3.2. [**4**, 3.12] *In $\mathcal{O}_\infty$, every projection is equivalent to a projection either of the form $\sum_{i=1}^k s_i s_i^*$ $(1 \leq k < \infty)$ or $1 - \sum_{i=1}^k s_i s_i^*$ $(1 \leq k < \infty)$.*

In $\mathcal{O}_\infty$, the type of an involution v is $n[1]$, for some $n \in \mathbb{Z}$, and we write that v has the type $n \in \mathbb{Z}$. Recall that Lemma 2.1 is also valid for $\mathcal{O}_\infty$.

Now we start by proving the following lemma, which is similar to Lemma 2.4 in the case of $\mathcal{O}_n$, where n is a finite number.

Lemma 3.3. *If $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(\mathcal{O}_\infty)$, which contains an involution of the type 1, then $\mathcal{N}$ contains an involution of any given type.*

Proof. As $\mathcal{N}$ contains an involution of type 1, and $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(\mathcal{O}_\infty)$, we have that N contains all the involutions u_i $i = 1, 2, \dots$. Then $u_1 u_2$ is an involution in $\mathcal{N}$ of type 2 indeed, if $k \in \mathbb{Z}^+$, then $u_1 u_2 \dots u_k = 1 - 2(p_1 + p_2 + \cdots + p_k)$ is an involution in $\mathcal{N}$ of type k. Also, $\mathcal{N}$ contains an involution of type 0, as $1 \in \mathcal{N}$.

Now it is enough to prove that $\mathcal{N}$ contains an involution of any negative type. Recall that if p is a projection of $\mathcal{O}_\infty$, then by Theorem 3.2, either p is equivalent to $\sum_{i=1}^k s_i s_i^*$, hence $[p] = k[1]$ or p is equivalent to $1 - \sum_{i=1}^k s_i s_i^*$, hence $[p] = (1-k)[1]$, for some $k \in \mathbb{Z}^+$. As $\mathcal{N}$ contains involutions of type 1, then the involution -1 belongs to $\mathcal{N}$. Hence for each $k \in \mathbb{Z}^+$, $-u_1 u_2 \dots u_k \in \mathcal{N}$, and

$$\begin{aligned} -u_1 u_2 \dots u_k &= -(1 - 2(p_1 + p_2 + \cdots + p_k)) \\ &= -1 + 2(p_1 + p_2 + \cdots + p_k) \\ &= 1 - 2(1 - (p_1 + p_2 + \cdots + p_k)), \end{aligned}$$

therefore, $-u_1 u_2 \dots u_k$ is an involution of type $1 - k$ and the lemma has been checked. □

Therefore, we have the following main result

Theorem 3.4. *Any involution of type 1 normally generates the group $\mathcal{U}(\mathcal{O}_\infty)$.*

Proof. Suppose that $\mathcal{N}$ is a normal subgroup of $\mathcal{U}(\mathcal{O}_\infty)$ that contains an involution of the type 1. By using Lemma 3.3, we have that $\mathcal{N}$ contains an involution of any given type, therefore by Lemma 2.1, $\mathcal{N}$ contains all the involutions, hence by Leen's result in Theorem 1.2, $\mathcal{N} = \mathcal{U}(\mathcal{O}_\infty)$. □

Let us now prove our main result.

Theorem 3.5. *Any involution of odd type normally generates the group $\mathcal{U}(\mathcal{O}_\infty)$.*

Proof. Case 1: Suppose that $\mathcal{N}$ contains an involution of type $2k+1$, for some positive integer k. By normality of $\mathcal{N}$, we may assume that $v = 1-2\sum_{i=1}^{2k+1} p_i \in \mathcal{N}$, also $u = 1-2\sum_{i=2}^{2k+2} p_i \in \mathcal{N}$. Therefore, we have that

$$vu = (1-2\sum_{i=1}^{2k+1} p_i)(1-2\sum_{i=2}^{2k+2} p_i) = 1-2(p_1+p_{2k+2}),$$

which is an involution in $\mathcal{N}$ of type 2, hence $\mathcal{N}$ contains all involutions of the type 2. Then

$$(1-2(p_1+p_2))(1-2(p_3+p_4))\dots(1-2(p_{2k-1}+p_{2k})) = 1-2\sum_{i=1}^{2k} p_i \in \mathcal{N}.$$

Therefore $\mathcal{N}$ contains the involution

$$(1-2\sum_{i=1}^{2k+1} p_i)(1-2\sum_{i=1}^{2k} p_i) = 1-2p_{2k+1},$$

which is of the type 1, hence by Theorem 3.4, we have the desired.

Case 2: Suppose that $\mathcal{N}$ contains an involution v of the type $-k$, where $k \in \mathbb{Z}^+$, which is odd. Then by normality of $\mathcal{N}$ and Lemma 2.1, the involution $w_1 = 1-2(1-(p_1+p_2+\cdots p_{k+1}))$ belongs to $\mathcal{N}$, as its type is $-k$. In fact, $w_1 = -u_1u_2\dots u_ku_{k+1}$. Similarly, the involution $w_2 = 1-2(1-(p_2+p_3+\cdots p_{k+2}))$ belongs to $\mathcal{N}$ and $w_2 = -u_2u_3\dots u_{k+2}$. Therefore, the involution $w_1w_2 = u_1u_{k+2} \in \mathcal{N}$, hence $\mathcal{N}$ contains all involutions of type 2, by using Lemma 2.1. As $k+1$ is an even integer, we get $w_3 = (u_1u_2)(u_3u_4)\dots(u_ku_{k+1}) \in \mathcal{N}$. Therefore we have that $w_1w_3 = -1 \in \mathcal{N}$, which is an involution of type 1, hence by Theorem 3.4, the proof is completed. □

Finally, we conclude by noting that similar arguments show that a normal subgroup of $\mathcal{U}(\mathcal{O}_n)$ which contains a non-trivial involution (of any type) necessarily contains all the involutions of even type.

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