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LAPLACE TRANSFORMS AND SHOUT OPTIONS

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Abstract. We use Laplace transform methods to examine the optimal exercise boundary for shout options, which give the holder the right to lock in the profit to date while retaining the right to benefit from any further upside. The result of our analysis is an integro-differential equation for the location of this optimal exercise boundary. This equation is a nonlinear Fredholm equation, or more specifically, an Urysohn equation of the first kind. Applying an inverse Laplace transform to this equation allows us to find the behavior of the free boundary close to expiry. The results are given for both call and put shout options.

1. Introduction

In the past twenty years, the role and the complexity of financial contracts have grown tremendously, causing a dramatic change in the financial industry. Issuers, investors, and government regulators have increased their reliance on derivative instruments to augment the liquidity of markets, to reallocate financial risks among market participants, and to take advantage of differences in costs and returns between these markets. A variety of financial contracts, ranging from basic American-style vanilla options to more exotic complex contracts, have been introduced to cater to the needs of a variety of investor profiles. One such financial contract is a shout option. This is an American-style exotic option which contains an early exercise feature that enables the holder to lock in the profit to date while still retaining the right to benefit from any additional upside, and as with any option carrying early exercise rights, there is an element of uncertainty as to the actions that the holder will undertake. Shout options are frequently embedded in other contracts such as the segregated funds sold by Canadian life insurance companies[**8**] and protective floor indexes. A brief introduction to shout options can be found in [**41**].

When a vanilla American option is exercised, the holder has the right to buy (a call) or sell (a put) the underlying security at a pre-determined exercise price E. Like vanilla Americans, shouts can only be exercised when they are in-the-money, meaning that the stock price is greater than the exercise price for a call or less than the exercise price for a put. When a shout option is exercised, the holder not only

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has the right to buy or sell the underlying security at the exercise price, but also receives an at-the-money vanilla European option, an at-the-money option being one whose exercise price is equal to the price of the underlying at that particular time, with this new option having the same expiry as the original option. Thus a shout option essentially consists of a vanilla American option together with a forward start option, the strike price of which is set when the American option is exercised. In effect, the holder of a shout option has the right to reset the exercise price of the option, provided that the option is in-the-money, and receive the difference between the old and the new exercise prices in cash. This enables the buyer to lock in the profit to date while still retaining the right to benefit from any further upside.

Since shout options are American-style, their valuation involves determining whether the option should be exercised prior to expiry, which leads to a free boundary problem, with the boundary separating the region where early exercise is beneficial from that where it is not. Early exercise, or shouting, can only occur when the option is in-the-money. Once the free boundary is hit, shouting occurs and the option is exchanged for a new (plain vanilla European) option whose exercise price is the stock price at that particular time, together with a payment of the difference between the original and new exercise prices. Since this new option is European, it can be priced using the Black-Scholes formula [**7**], and it follows that on the free boundary, the value of a shout option is the value of this European option together with the value of the payment. It is worth noting that when a vanilla American option is exercised, the holder receives a payment but no new option, so the pay-off from early exercise is sweeter for a shout option than for a vanilla American, and because of this a shout is more likely to be exercised early than a vanilla American. Although some exotic contracts do exist with multiple shouting opportunities, in this analysis, we assume that the holder can shout only once so that there is only one free boundary whose position must be optimized; with multiple shouting opportunities, there would be multiple free boundaries.

Much of the work done to date on shout options is numerical, although as with other options involving a free boundary and choice on the part of an investor, some standard numerical techniques such as the forward-looking Monte Carlo method are difficult to use because they cannot effectively handle the optimization component of shout options. In [**8**], a Green's function approach was used. With this approach, it is assumed that early exercise could only occur at a limited number of fixed times $t_1 < t_2 < \cdots < t_{n-1} < t_n$ between the current time t and the expiry $T > t$, so that the option is treated as Bermudan-style or semi-American rather than American-style, and then the value of the option at the time t_m is used to compute the value at time t_{m-1}, which in turn is used to compute the value at time t_{m-2} and so on. The value at time t_{m-1} is computed using an integral involving the product of the Green's function with the value at time t_m, with this integral being evaluated numerically. More standard numerical methods, such as finite differences, have also been applied to shout options [**12, 13, 43**]. In this paper, we take an analytical rather than a numerical approach, follow [**24, 4**], and use partial Laplace transform methods [**16, 11**] to derive the integro-differential equations necessary to locate the optimal exercise boundary. The result for both call and put shout options is a nonlinear Fredholm integro-differential equation for the location of the free boundary. An inverse Laplace transform is then applied to this integro-differential equation, resulting in a second integro-differential equation which allows us to determine the asymptotic behavior of the free boundary close to expiry. The idea of using integral and integro-differential equations to tackle American-style option is of course not new, and a number of authors have used integral equations to study vanilla Americans in the past, including McKean [**26**] and Van Moerbeke [**42**] who considered American calls, with later work including the studies by Kim [**21**], Jacka [**19**] and Carr [**9**], all of whom looked at the difference between European and American prices, and several recent papers [**39, 23, 17**] on the American put. A number of these integral equation approaches have used integral transforms, notably Fourier transforms [**39**] and Laplace transforms [**24, 22**], and the use of these approaches can be traced back to much earlier work on diffusion problems in physical problems [**16, 32**], and an overview of this earlier work and other integral equation formulations of these types of problems is given in §3.5 of [**11**].

The remainder of the paper is organized as follows. Section 2 contains our analysis using Laplace transform methods to locate the free boundary for both call and put shout options on a dividend paying asset under the Black-Scholes model, and explains how the analysis for the calls differ from that for the put. Section 3 contains a discussion of our results.

2. Analysis

2.1. Formulation of the problem

It is well known that the value $V(S,t)$ of a vanilla European option with constant dividend yield obeys the Black-Scholes-Merton partial differential equation or PDE [**7, 27**],

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-D)\,S\frac{\partial V}{\partial S} - rV = 0\,, \tag{1}$$

where S is the price of the underlying and $t < T$ is the time, with T being the expiry time when the holder will receive the pay-off from the option, which is $\max(S-E,0)$ for a call with a strike price of E and $\max(E-S,0)$ for a put. This equation was originally presented by Black & Scholes [**7**] for options on stocks without dividends, and later extended by Merton [**27**] to include a constant dividend yield. The parameters in the above equation are the risk-free rate, r, the dividend yield, D, and the volatility, σ, all of which are assumed constant in the present analysis. To simplify the analysis, we will work in terms of the tenor, or remaining life of the option, $\tau = T - t$, so that (1) is replaced by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-D)\,S\frac{\partial V}{\partial S} - rV\,, \tag{2}$$

While this equation (2) is applicable for $\tau \geq 0$ for European options, it is also governs the price of options for which early exercise is permitted, but in those cases, the equation is only valid where it is optimal to hold the option, and (2) must be solved together with the appropriate conditions at the optimal exercise boundary, whose location is unknown and must be solved for. In what follows, we will label the position of the free boundary as $S = S_f(\tau)$, which we can invert to give $\tau = \tau_f(S)$ as the time at which early exercise should occur. Merton [**28**] remarked on the fact that different securities may obey the same equation, and that it is the boundary and initial conditions which differentiate them, so that shout, European and American options, along with many others, all obey (2) but with differing boundary conditions. The pay-off for a shout option held to maturity is the same as that for an American held to maturity or for a European, namely $\max(S-E, 0)$ for a call and $\max(E-S, 0)$ for a put, where E is the original strike price of the option.

Just as with American options, shout options can be exercised early at any time provided they are in-the-money, meaning that the stock price is greater than the exercise price for a call or less than the exercise price for a put. Just as with American options, this leads to the constraint that the price of the option cannot fall below the pay-off from immediate exercise, In the case of a shout call, we can only shout if $S > E$, and the possibility of shouting leads to the constraint $V > V_f$ for $S > E$ where V_f is the pay-off from shouting,

$$\begin{aligned} V_f(S,\tau) = S - E + \frac{S\,\mathrm{e}^{-D\tau}}{2}\mathrm{erfc}\left[-\frac{\left(r - D + \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right] \\ - \frac{S\,\mathrm{e}^{-r\tau}}{2}\mathrm{erfc}\left[-\frac{\left(r - D - \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right], \end{aligned} \tag{3}$$

this being the difference between the current price of the underlying and the original strike price together with the value of an at-the-money European call, which we can price using the Black-Scholes formula [**7**]. In this expression, erfc denotes the complementary error function. Similarly, for a shout put, we have the constraint $V > V_f$ for $S < E$ where

$$\begin{aligned} V_f(S,\tau) = E - S - \frac{S\,\mathrm{e}^{-D\tau}}{2}\mathrm{erfc}\left[\frac{\left(r - D + \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right] \\ + \frac{S\,\mathrm{e}^{-r\tau}}{2}\mathrm{erfc}\left[\frac{\left(r - D - \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right]. \end{aligned} \tag{4}$$

As with American options, the possibility of shouting leads to a free boundary where it is optimal to shout. Several properties of this free boundary are known. Firstly, we know the value of the option at the free boundary,where $V = V_f$ given by (3) and (4) above, and also the value of the option's delta, or derivative of its value with respect to the stock price, at the free boundary, where $(\partial V/\partial S) = (\partial V_f/\partial S)$, which for a call is

$$1 + \frac{e^{-D\tau}}{2}\operatorname{erfc}\left[-\frac{\left(r - D + \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right] - \frac{e^{-r\tau}}{2}\operatorname{erfc}\left[-\frac{\left(r - D - \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right], \tag{5}$$

while for a put it is

$$-1 - \frac{e^{-D\tau}}{2}\operatorname{erfc}\left[\frac{\left(r - D + \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right] + \frac{e^{-r\tau}}{2}\operatorname{erfc}\left[\frac{\left(r - D - \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sqrt{2}\sigma}\right]. \tag{6}$$

The condition on the delta $(\partial V/\partial S)$ comes from requiring that it be continuous across the boundary, and is essentially the high contact or smooth-pasting condition, which was first proposed by Samuelson [**37**] for American options.

Secondly, we know the location of the free boundary at expiry $\tau = 0$ is $S_f(0) = E$, which can be deduced intuitively because the pay-off for early exercise is so sweet for shout options. In our terms, $\tau_f(E) = 0$. We also know that the optimal exercise boundary moves upwards (or at worst is flat) as we move away from the expiration date for a call, and downwards (or again at worst is flat) for a put.

Thirdly, we know the location of the free boundary as $\tau = T - t \to \infty$ from the behavior of the perpetual shout option, for which the pay-off from shouting is the rebate together with an at-the-money perpetual European option, and since the value of a European tends to zero as the duration tends to infinity (assuming the dividend yield is non-zero), it follows that for a non-zero dividend yield, a perpetual shout behaves like a perpetual American, so that as $\tau \to \infty$,

$$\begin{aligned} S_f(\tau) &\to S_* = \frac{E}{1 - 1/\alpha}, \\ \alpha &= \frac{1}{2\sigma^2}\left[\sigma^2 - 2(r - D) \pm \sqrt{4(r - D)^2 + 4(r + D)\sigma^2 + \sigma^4}\right], \end{aligned} \tag{7}$$

where we take the + sign for a call and the − sign for a put, so that $S_* > E$ for a call and $S_* < E$ for a put. In our terms, $\tau_f(S) \to \infty$ as $S \to S_*$. A put without dividends also falls within the above framework, but for a call option without dividends, as $\tau \to \infty$, the value of the option at the boundary tends to $2S - E$, while the value of the option below the boundary is S, and we deduce that in the absence of dividends, the optimal exercise boundary for a call is flat and located at $S_f(\tau) \equiv E$.

For a shout call with a non-zero dividend yield, the optimal exercise boundary will lie between these two limits, $E \le S_f(\tau) \le S_*$, and of course early exercise is optimal if $S \ge S_f(\tau)$ whereas retaining the option is optimal if $S < S_f(\tau)$. Likewise for a shout put, the optimal exercise boundary will lie between $E \ge S_f(\tau) \ge S_*$, and early exercise is optimal if $S \le S_f(\tau)$.

At this point, we should say a few words about the analyticity of the free boundary, and of free boundaries arising in Stefan problems in general. The Stefan problem is concerned with the heat equation with a moving boundary which is not specified a priori; such problems are typically associated with changes of phase, such as melting and solidification. Because the Black-Scholes PDE can be transformed into the heat equation by a change of variables, the free boundary problems arising in the pricing of options with American-style early exercise features are closely related to the Stefan problem. For many years, the analyticity of the interfacial boundary arising in the Stefan problem was an issue: for example, in [**36**], this was mentioned (in 1971) as a still unsolved problem. The classical Stefan problem involves heat conduction in a material occupying the semi-infinite space $x > 0$ with an arbitrarily prescribed initial temperature $U_I(x)$ at $\tau = 0$ for $x > 0$ and an arbitrarily prescribed boundary condition $U_B(\tau)$ at $x = 0$ for $\tau > 0$. Because of the change of temperature at $x = 0$, a new phase of the material starts to appear, and the phase change occurs along a free boundary $x = x_f(\tau)$ where the temperature of both phases is $U_f(\tau)$. If we label the original phase with the subscript 1 and the new phase with the subscript 2, then the complete set of equations for the classical Stefan problem is

$$\begin{aligned} \frac{\partial U_1}{\partial \tau} &= \alpha_1 \frac{\partial^2 U_1}{\partial x^2} \quad \text{for} \quad x < x_f(\tau), \\ \frac{\partial U_2}{\partial \tau} &= \alpha_2 \frac{\partial^2 U_2}{\partial x^2} \quad \text{for} \quad x > x_f(\tau), \end{aligned} \tag{8}$$

together with the boundary and initial conditions $U_1(0,\tau) = U_B(\tau)$, $U_2(x,0) = U_I(x)$, and the condition at the free boundary $U_1(x_f(\tau),\tau) = U_2(x_f(\tau),\tau) = U_f(\tau)$ and a condition on the heat flux at the free boundary,

$$k_1 \frac{\partial U_1}{\partial x}\bigg|_{x_f(\tau)} - k_2 \frac{\partial U_2}{\partial x}\bigg|_{x_f(\tau)} = \pm \rho l x_f'(\tau)\ . \tag{9}$$

Friedman [**18**] later showed that if the boundary data $U_B(\tau)$ was analytic and the initial data $U_I(x)$ was bounded, and if the initial and boundary data were continuous at their intersection, then the interfacial boundary was analytic for $\tau > 0$; however, the analyticity at $\tau = 0$ remained unanswered, because it was unclear whether x_f was a function of τ or $\tau^{1/2}$. When x_f is a function of τ, it is analytic at $\tau = 0$, but when it is a function of $\tau^{1/2}$, not all derivatives of the function exist at $\tau = 0$. Tao [**40**] later extended this analysis by removing the restriction that the initial and boundary data be continuous at their intersection, so that he did not require $U_B(0) = U_I(0)$ as Friedman had. Tao showed that the interfacial boundary $x_f(\tau)$ was an analytic function of $\tau^{1/2}$ if U_B was an analytic function of $\tau^{1/2}$ and U_I was an analytic function of x. Tao mentioned that his study was also valid when the effect of density changes during the phase transition were included, as the convective term which this adds to the governing PDE can be removed by a change of variable.

In the present problem, we are considering the Black-Scholes-Merton PDE (1), together with the initial condition that $V(S,T)$ is specified at $t = T$, while at the free boundary we have $V(S,t) = V_f(S,t)$ and $(\partial V/\partial S) = (\partial V_f/\partial S)$. We note that early exercise is possible at any time rather than on discrete occasions like a Bermudan and that the pay-off from early exercise is smooth. If we make the transformation $V(S,t) = e^{\gamma x+\delta\tau}U(x,\tau)+V_f(S,t)$, with $x = \ln(S/E)$, $\tau = T-t$, $\gamma = 1/2-(r-D)/\sigma^2$ and $\delta = -r-\gamma^2\sigma^2/2$, then we find that U obeys a nonhomogeneous heat equation of the form

$$\frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + U_f(x,\tau)\ , \tag{10}$$

together with the condition that $U(x,0)$ is specified at $\tau = 0$ while at the free boundary we have $U(x,t) = (\partial U_f/\partial x) = 0$. This transformation enables us to carry over many of the results on the Stefan problem given in [**18, 40**], and conclude that the location of the free boundary $S_f(\tau)$ is analytic in $\tau^{1/2}$ for $\tau > 0$ while not all derivatives of $S_f(\tau)$ exist at $\tau = 0$. This result also holds for vanilla American options, and indeed in that case, it is known that close to expiry the free boundary behaves like $S_f(\tau) \sim S_0 \exp\left[x_1\sqrt{\sigma^2\tau/2}+\cdots\right]$ for the vanilla call with $D < r$ and the vanilla put with $D > r$, and like $S_f(\tau) \sim S_0 \exp\left[x_1^{(0)}\sqrt{-\tau\ln\tau}+\cdots\right]$ for the vanilla call with $D > r$ and the vanilla put with $D < r$ [**14, 6, 2, 3, 17**], so that the derivatives $S_f'(\tau)$, $S_f''(\tau)$, $\cdots$, do not exist at $\tau = 0$. Similarly, for the shout options considered here, when we come to use asymptotics to examine the behavior of the free boundary close to expiry, we will see that the slope of the boundary is infinite right at expiry; this does not affect our analysis. As discussed above, we also expect that $S_f(\tau)$ asymptote to S_* as $\tau \to \infty$, so that $S_f'(\tau) \to 0$ in that limit. We should mention that, in addition to the studies on the analyticity of the free boundary for the Stefan problem, several researchers have looked at similar issues for the free boundary for vanilla American options. Van Moerbeke [**42**], whose study was contemporaneous with [**18**] and pre-dated that of [**40**], showed that the free boundary $S_f(\tau)$ for American options was continuously differentiable; it should be possible to apply the results of [**42**] to the shout options considered here. Karatzas [**20, 15**] was able to prove the existence of an optimal exercise policy for American options and show that there was an optimal stopping time. A number of researchers, for example [**19, 21, 9**], have studied American options by decomposing them into a European option together with an early exercise premium. In a sense, shout options can be viewed as supercharged American options, since they have the same pay-off at expiry but a higher pay-off for early exercise, and therefore the same sort of decomposition should work for shout options, and therefore some of the theoretical results embedded in [**19, 21, 9**] should carry over to shout options.

In our analysis, we have also used $\tau_f(S)$, the inverse function of $S_f(\tau)$, which represents the location of the free boundary as a function of the stock price. Since $S_f(\tau)$ is analytic away from $\tau = 0$ and is a monotone function, the inverse $\tau_f(S)$ will be analytic also on the interval $E < S < S_*$ for the call and $S_* < S < E$ for the put. At the ends of this interval, we have $\tau_f \to \infty$ as $S \to S_*$, and $\tau_f \to 0$ as

$S \to E$. Once again, the behavior at the ends of this interval does not affect our analysis.

2.2. Partial Laplace transform in time

Having formulated the problem, we shall now attempt to solve it using a Laplace transform in time. This is the same technique we used for American options in [**24**], and, as in that study, since the Black-Scholes-Merton PDE only holds where it is optimal to retain the option, we will modify the usual definition

$$\mathcal{L}(G)(p) = \int_0^\infty g(\tau)\, \mathrm{e}^{-p\tau}\, \mathrm{d}\tau \tag{11}$$

somewhat, and define our version as follows for $S \leq S_*$ for the call and $S \geq S_*$ for the put,

$$\mathcal{V}(S,p) = \int_{\tau_f(S)}^\infty V(S,\tau)\, \mathrm{e}^{-p\tau}\, \mathrm{d}\tau, \tag{12}$$

so that the lower limit is $\tau = \tau_f(S)$ rather than $\tau = 0$. As mentioned in §1, the partial Laplace transform has been used to successfully tackle diffusion problems in the past, notably by [**16**]. Returning to our definition of the Laplace transform, this is of course equivalent to setting $V(S,\tau) = 0$ in the region where it is optimal to exercise. This means the inverse is only meaningful where it is optimal to retain the option. Because of this definition, the price of the option $V(S,\tau)$ will obey the equation (2) everywhere where we integrate. We require the real part of p to be positive for the integral in (12) to converge. In addition, we know from the definition that $\mathcal{V}(S,p) \to 0$ as $S \to S_*$. We can also define an inverse transform

$$V(S,\tau) = \frac{1}{2\pi\,\mathrm{i}} \int_{\gamma-\mathrm{i}\,\infty}^{\gamma+\mathrm{i}\,\infty} \mathcal{V}(S,p)\, \mathrm{e}^{p\tau}\, \mathrm{d}p. \tag{13}$$

Given our definition of the forward transform, this inverse is only meaningful where it is optimal to hold the option. From our definition, the following transforms can be derived easily,

$$\begin{aligned}
\mathcal{L}\left[\frac{\partial V}{\partial \tau}\right] &= p\mathcal{V} - \mathrm{e}^{-p\tau_f(S)}\, V_f(S,\tau_f(S)),\\
\mathcal{L}\left[\frac{\partial V}{\partial S}\right] &= \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}S} + \mathrm{e}^{-p\tau_f(S)}\, \tau_f'(S) V_f(S,\tau_f(S)),\\
\mathcal{L}\left[\frac{\partial^2 V}{\partial S^2}\right] &= \frac{\mathrm{d}}{\mathrm{d}S}\left(\mathcal{L}\left[\frac{\partial V}{\partial S}\right]\right) + \mathrm{e}^{-p\tau_f(S)}\, \tau_f'(S) \frac{\partial V_f}{\partial S}(S,\tau_f(S)).
\end{aligned} \tag{14}$$

In the above, we have adopted the convention that, for the call, $\tau_f(S)$ is the location of the free boundary for $E < S < S_*$, but for $S < E$, we set $\tau_f = 0$ since it is optimal to hold the option to expiry, while for the put $\tau_f(S)$ is the location of the free boundary for $E > S > S_*$, but for $S > E$, we set $\tau_f = 0$ since it is optimal to hold the option to expiry. Applying this partial Laplace transform to the governing PDE (2), we arrive at the following (nonhomogeneous Euler) ODE for the transform of the option price,

$$\left[\frac{\sigma^2 S^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}S^2} + (r-D)\,S\frac{\mathrm{d}}{\mathrm{d}S} - (p+r)\right]\mathcal{V} + F(S) = 0, \tag{15}$$

where the nonhomogeneous term $F(S)$ takes a different value in various regions. For the shout option, we have two separate regions:
Region (a): $0 < S < E$ for the call and $S > E$ for the put, where we have $V(S_f(\tau),\tau) = 0$, $\frac{\partial V}{\partial S}(S_f(\tau),\tau) = 0$, $\tau_f = 0$ and

$$F(S) = 0. \tag{16}$$

Region (b): $E < S < S_*$ for the call and $E > S > S_*$ for the put, where $V(S_f(\tau),\tau) = V_f(S_f(\tau),\tau)$, $\frac{\partial V}{\partial S}(S_f(\tau),\tau) = \frac{\partial V_f}{\partial S}(S_f(t),t)$, $\tau_f > 0$ and

$$\begin{aligned} F(S) =&\ \mathrm{e}^{-p\tau_f(S)}\left[F_0(S) + (p+p_0)\,F_1(S)\right], \\ F_0(S) =&\ \left[\frac{\sigma^2 S^2}{2}\left(\tau_f''(S) + p_0\tau_f'^2(S) + \tau_f'^2(S)\frac{\partial}{\partial\tau} + 2\tau_f'(S)\frac{\partial}{\partial S}\right)\right. \\ &\left. +1 + (r-D)\,S\tau_f'(S)\right] V_f(S,\tau_f(S)), \\ F_1(S) =&\ -\frac{\sigma^2 S^2}{2}\tau_f'^2(S) V_f(S,\tau_f(S)), \end{aligned} \tag{17}$$

where $V_f(S,t)$ was given in (3) for the call and in (4) for the put, and F_0 and F_1 are introduced to simplify the Laplace inversion later, and

$$p_0 = \frac{4(D-r)^2 + 4\sigma^2(D+r) + \sigma^4}{8\sigma^2}.$$

The general solution of (15) is

$$\begin{aligned} \mathcal{V} =&\ S^{\frac{1}{2\sigma^2}\left(2D-2r+\sigma^2+\lambda(p)\right)}\left[C_1(p) - \frac{2}{\lambda(p)}\int S^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2+\lambda(p)\right)}F(S)\mathrm{d}S\right] \\ &+ S^{\frac{1}{2\sigma^2}\left(2D-2r+\sigma^2-\lambda(p)\right)}\left[C_2(p) + \frac{2}{\lambda(p)}\int S^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2-\lambda(p)\right)}F(S)\mathrm{d}S\right], \end{aligned} \tag{18}$$

where $\lambda(p) = 2^{3/2}\sigma\left[p+p_0\right]^{1/2}$, and C_1 and C_2 are constants of integration, which may depend on the transform variable p. Notice that since r, D and σ are all assumed to be positive, and we assume that p has a positive real part from the definition of the Laplace transform, then the real part of the first exponent $\left(2D - 2r + \sigma^2 + \lambda(p)\right)/(2\sigma^2)$ is assumed positive, while the real part of the second exponent, $\left(2D - 2r + \sigma^2 - \lambda(p)\right)/(2\sigma^2)$ is assumed negative.

2.3. Analysis for the call

Considering first the call, applying this solution (18) to the two separate regions outlined above, we find that in region (a) we must discard the second solution in order to satisfy the boundary condition on $S = 0$ that $V(0,t) = 0$ and consequently that $\mathcal{V}(0,p) = 0$, so in this region we have

$$\mathcal{V} = C_1^{(a)}(p)\left(\frac{S}{E}\right)^{\frac{1}{2\sigma^2}\left(2D-2r+\sigma^2+\lambda(p)\right)}. \tag{19}$$

In region (b), we find the solution which satisfies the condition that $\mathcal{V}(S,p) \to 0$ as $S \to S_*$ is

$$\begin{aligned}\mathcal{V} = {} & \frac{2}{S\lambda(p)}\int\limits_S^{S_*}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \\ & \times\left[\left(\frac{\widetilde{S}}{S}\right)^{-\frac{\lambda(p)}{2\sigma^2}} - \left(\frac{\widetilde{S}}{S}\right)^{\frac{\lambda(p)}{2\sigma^2}}\right]F(\widetilde{S})\mathrm{d}\widetilde{S},\end{aligned} \tag{20}$$

where $F(S)$ is given by (17). We must now match the solutions in these two regions together. We require that $\mathcal{V}$ and $(d\mathcal{V}/dS)$ are continuous across $S = E$, which tells us that

$$C_1^{(a)}(p) = \frac{2}{E\lambda(p)}\int\limits_E^{S_*}\left(\widetilde{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2+\lambda(p)\right)}F(\widetilde{S})\mathrm{d}\widetilde{S}, \tag{21}$$

and

$$\int_E^{S_*}\widetilde{S}^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2-\lambda(p)\right)}F(\widetilde{S})d\widetilde{S} = 0, \tag{22}$$

where $F(S)$ is given by (17). This equation is of the same form as that for the American call with $D > r$ given in [**24**], but of course the nonhomogeneous term $F(S)$ is different since the pay-off at early exercise is different. For the shout options discussed here, the pay-off from shouting is given by (3) and (4), whereas for vanilla American options, the terms involving erfc in (3) and (4) would be absent.

Of course, (22) is actually the Laplace transform of an integro-differential equation in (S,τ) space, and we can obtain this latter equation by applying the inverse Laplace transform (13) to (22). To invert the transform, we first divide by $(p+p_0)^{3/2}E^{-\left(2D-2r+3\sigma^2\right)/(2\sigma^2)}(S_f(\tau))^{\lambda(p)/(2\sigma^2)}$, and rewrite (22) as

$$\begin{aligned}& \int\limits_E^{S_*}\left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\exp\left[-\frac{\sqrt{2(p+p_0)}}{\sigma}\ln\frac{S_f(\tau)}{\widetilde{S}}\right] \\ & \times\frac{\mathrm{e}^{-p\tau_f(\widetilde{S})}}{\sqrt{p+p_0}}\left[\frac{F_0(\widetilde{S})}{p+p_0} + F_1(\widetilde{S})\right]\mathrm{d}\widetilde{S} = 0,\end{aligned} \tag{23}$$

and then use the following standard inverse transforms [**35**],

$$\mathcal{L}^{-1}\left[\mathrm{e}^{-ap}\,G(p)\right] = H\left(\tau - a\right)g\left(\tau - a\right),$$

$$\mathcal{L}^{-1}\left[G(p+p_0)\right] = \mathrm{e}^{-p_0\tau}\,g(\tau),$$

$$\mathcal{L}^{-1}\left[p^{-1/2}\exp\left(-ap^{1/2}\right)\right] = \frac{1}{\sqrt{\pi}\tau^{1/2}}\exp\left[-\frac{a^2}{4\tau}\right], \tag{24}$$

$$\mathcal{L}^{-1}\left[p^{-3/2}\exp\left(-ap^{1/2}\right)\right] = \frac{2\tau^{1/2}}{\sqrt{\pi}}\exp\left[-\frac{a^2}{4\tau}\right] - a\,\mathrm{erfc}\left[\frac{a}{2\sqrt{\tau}}\right],$$

where $H(t)$ is the Heaviside step function, to obtain

$$\begin{aligned}\int_E^{S_*} & H\left(\tau - \tau_f(\widetilde{S})\right)\sqrt{\tau - \tau_f(\widetilde{S})}\left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))} \\ & \times\left[\frac{1}{\sqrt{\pi}}\left(2F_0(\widetilde{S}) + \frac{F_1(\widetilde{S})}{\tau - \tau_f(\widetilde{S})}\right)\exp\left(-\frac{\left(\ln(S_f(\tau)/\widetilde{S})\right)^2}{2\sigma^2(\tau - \tau_f(\widetilde{S}))}\right)\right. \\ & \left. - \frac{\sqrt{2}F_0(\widetilde{S})\ln(S_f(\tau)/\widetilde{S})}{\sigma(\tau - \tau_f(\widetilde{S}))}\mathrm{erfc}\left(\frac{\ln(S_f(\tau)/\widetilde{S})}{\sigma\sqrt{2(\tau - \tau_f(\widetilde{S}))}}\right)\right]\mathrm{d}\widetilde{S} = 0,\end{aligned} \tag{25}$$

or applying the step function

$$\begin{aligned}\int_E^{S_f(\tau)} & \sqrt{\tau - \tau_f(\widetilde{S})}\left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))} \\ & \times\left[\frac{1}{\sqrt{\pi}}\left(2F_0(\widetilde{S}) + \frac{F_1(\widetilde{S})}{\tau - \tau_f(\widetilde{S})}\right)\exp\left(-\frac{\left(\ln(S_f(\tau)/\widetilde{S})\right)^2}{2\sigma^2(\tau - \tau_f(\widetilde{S}))}\right)\right. \\ & \left. - \frac{\sqrt{2}F_0(\widetilde{S})\ln(S_f(\tau)/\widetilde{S})}{\sigma(\tau - \tau_f(\widetilde{S}))}\mathrm{erfc}\left(\frac{\ln(S_f(\tau)/\widetilde{S})}{\sigma\sqrt{2(\tau - \tau_f(\widetilde{S}))}}\right)\right]\mathrm{d}\widetilde{S} = 0.\end{aligned} \tag{26}$$

To evaluate this, we will change variables so that we have an integral with respect to τ,

$$\begin{aligned}\int_0^{\tau} & \sqrt{\tau - z}\left(\frac{S_f(z)}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \mathrm{e}^{-p_0(\tau-z)} \\ & \times\left[\frac{1}{\sqrt{\pi}}\left(2\widetilde{F}_0(z) + \frac{\widetilde{F}_1(z)}{\tau - z}\right)\exp\left(-\frac{\left(\ln(S_f(\tau)/S_f(z)\right)^2}{2\sigma^2(\tau - z)}\right)\right. \\ & \left. - \frac{\sqrt{2}\widetilde{F}_0(z)\ln(S_f(\tau)/S_f(z))}{\sigma(\tau - z))}\mathrm{erfc}\left(\frac{\ln(S_f(\tau)/S_f(z))}{\sigma\sqrt{2(\tau - z)}}\right)\right]\mathrm{d}z = 0,\end{aligned} \tag{27}$$

where

$$\widetilde{F}_0(\tau) = F_0(S_f(\tau))S_f'(\tau)$$

$$= \left[S_f'(\tau) + \left(r - D + \sigma^2\right) S_f(\tau) + \frac{\sigma^2 S_f^2(\tau)}{2S_f'(\tau)}\left(p_0 - \frac{S_f''(\tau)}{S_f'(\tau)}\right)\right] V_f(S_f(\tau),\tau)$$

$$+ \frac{\sigma^2 S_f^2(\tau)}{2S_f'(\tau)}\frac{\partial V_f}{\partial \tau}(S_f(\tau),\tau) + \sigma^2 S_f(\tau)E, \tag{28}$$

$$\widetilde{F}_1(\tau) = F_1(S_f(\tau))S_f'(\tau)$$

$$= -\frac{\sigma^2 S_f^2(\tau)}{2S_f'(\tau)} V_f(S_f(\tau),\tau).$$

This last equation (27) is an integro-differential equation in (S,τ) space for the location of the free boundary for the shout call. By making the substitutions $S_f(\tau) = E\,\mathrm{e}^{x_f(\tau)}$ and $z = \tau y$, we can rewrite this as

$$\int_0^1 \sqrt{1-y}\exp\left[-\frac{1}{2\sigma^2}\left(2D - 2r + 3\sigma^2\right)x_f(y\tau) - p_0\tau(1-y)\right]$$

$$\times\left[\frac{1}{\sqrt{\pi}}\left(2\widetilde{F}_0(y\tau) + \frac{\widetilde{F}_1(y\tau)}{\tau(1-y)}\right)\exp\left(-\frac{\left(x_f(\tau) - x_f(y\tau)\right)^2}{2\sigma^2\tau(1-y)}\right)\right. \tag{29}$$

$$\left. - \frac{\widetilde{F}_0(y\tau)\sqrt{2}(x_f(\tau) - x_f(y\tau))}{\sigma\tau(1-y))}\operatorname{erfc}\left(\frac{(x_f(\tau) - x_f(y\tau))}{\sigma\sqrt{2\tau(1-y)}}\right)\right]\mathrm{d}y = 0.$$

2.3.1. The free boundary close to expiry. While we have been unable to obtain a complete solution to (29), and such a solution would almost certainly need to numerical, it is possible to study the behavior of the free boundary close to expiry, in the limit $\tau \to 0$, by making several approximations and assumptions. We will assume that in this limit, the free boundary behaves like [**14**]

$$x_f(\tau) \sim x_1\sqrt{\sigma^2\tau/2} + x_2\tau + \cdots\,, \tag{30}$$

and substitute this assumed form into the integro-differential equation (29), which we expand as a series in τ. The factor $\sqrt{\sigma^2/2}$ is included to simplify the analysis at a later stage. Upon making this substitution, we will attempt to solve for the coefficient x_1 and either of two things can happen: if the equation for x_1 has no solution or the solution is clearly wrong, for example of the wrong sign, then we must conclude that our ansatz (30) is incorrect, while on the other hand if we are able to find a plausible value for x_1, we can conclude that the expansion (30) is correct with the leading coefficient x_1 as given.

The assumed form (30) was chosen because it is both the simplest and the most common form found in this kind of free boundary problem, and is the form of the boundary for the vanilla American call with $D < r$ and the vanilla American put with $D > r$. We note that in a companion paper [**5**] on another exotic, the installment option, we used a similar approach, and found that the ansatz (30) was unsuccessful for installment options, but that our second guess, an ansatz of the form $x_f(\tau) \sim x_1^{(0)}\sqrt{-\tau \ln \tau} + \cdots$, would work: this is also the form of the free boundary for for the vanilla call with $D > r$ and the vanilla put with $D < r$ [**14, 6, 2, 17**].

Returning to the shout call, in the limit $\tau \to 0$, we can show that for the assumed form (30), at leading order $\widetilde{F}_0$ and $\widetilde{F}_1$ behave like

$$\begin{aligned} \widetilde{F}_0(\tau) &\sim \frac{E^2\sigma^2}{2x_1\sqrt{\pi}}\left(1 + 3x_1\sqrt{\pi} + x_1^2 + 2x_1^3\sqrt{\pi}\right) + \mathcal{O}\left(\tau\right), \\ \widetilde{F}_1(\tau) &\sim -\frac{E^2\sigma^2}{2x_1\sqrt{\pi}}\left(1 + 2x_1\sqrt{\pi}\right)\tau + \mathcal{O}\left(\tau^2\right), \end{aligned} \tag{31}$$

and upon expanding (29) as a series in τ, at leading order we require that

$$\begin{aligned} &\int_0^1 \sqrt{1-y} \\ &\times \left[\frac{1}{\sqrt{\pi}}\left(2\left(1 + 3x_1\sqrt{\pi} + x_1^2 + 2x_1^3\sqrt{\pi}\right) - \frac{\left(1 + 2x_1\sqrt{\pi}\right)y}{1-y}\right)\right. \\ &\times \exp\left(-\frac{x_1^2\left(1-\sqrt{y}\right)^2}{1-y}\right) \\ &\left. - \frac{2\left(1 + 3x_1\sqrt{\pi} + x_1^2 + 2x_1^3\sqrt{\pi}\right)x_1(1-\sqrt{y})}{1-y}\operatorname{erfc}\left(\frac{x_1(1-\sqrt{y})}{\sqrt{1-y}}\right)\right] \mathrm{d}y = 0, \end{aligned} \tag{32}$$

which is the equation we must solve for x_1. To date, we have been unable to find a closed form expression for x_1, but evaluating (32) numerically with the computer algebra package Maple returns a value of $x_1 \approx 0.160536690345$, so for the shout call the behavior of the free boundary close to expiry is given by

$$\begin{aligned} S_f &\sim E\exp\left[0.160536690345\sqrt{\sigma^2\tau/2} + \mathcal{O}\left(\tau\right)\right] \\ &\sim E\exp\left[0.1135166\sigma\tau^{1/2} + \mathcal{O}\left(\tau\right)\right] \\ &\sim E\exp\left[0.1135166\sigma(T-t)^{1/2} + \mathcal{O}\left(T-t\right)\right]. \end{aligned} \tag{33}$$

At leading order, this does not depend on r or D but does depend on σ. It is worth recalling that the corresponding result for a vanilla American call is $x_1 = 0.903447$ [**14, 2**] if $D < r$ while is $D \geq r$, the assumed form (30) must be replaced by one involving logarithms. This means that close to expiry, the free boundary for a shout call is less steep than that for a vanilla American call, which presumably is due to early exercise being more attractive for a shout than a vanilla American because of the sweeter pay-off.

2.3.2. Laplace inversion. With regard to the value of the option itself, we can also apply an inverse Laplace transform (13) to the expressions (19), (20) for

$\mathcal{V}(S,p)$ to obtain expressions for $V(S,\tau)$. To do this, we make use of (24) together with

$$\mathcal{L}^{-1}\left[p^{1/2}\exp\left(-ap^{1/2}\right)\right] = \frac{a^2-2\tau}{4\sqrt{\pi}\tau^{5/2}}\exp\left[-\frac{a^2}{4\tau}\right]. \tag{34}$$

For $0 < S < E$, we can use (21) to invert (19), proceeding in the same manner as we did in (23–26) to obtain

$$\begin{aligned} V &= \frac{1}{\sqrt{2\pi}\sigma S}\int_E^{S_f(\tau)} \frac{\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))}}{\sqrt{\tau-\tau_f(\widetilde{S})}}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \\ &\quad\times\exp\left(-\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))}\right) \\ &\quad\times\left[F_0(\widetilde{S})+F_1(\widetilde{S})\left(\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))^2}-\frac{1}{2(\tau-\tau_f(\widetilde{S}))}\right)\right]\mathrm{d}\widetilde{S} \\ &= \frac{1}{\sqrt{2\pi}\sigma S}\int_0^{\tau}\frac{\mathrm{e}^{-p_0(\tau-z)}}{\sqrt{\tau-z}}\left(\frac{S_f(z)}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \\ &\quad\times\exp\left(-\frac{\left(\ln(S_f(z)/S)\right)^2}{2\sigma^2(\tau-z)}\right) \\ &\quad\times\left[\widetilde{F}_0(z)+\widetilde{F}_1(z)\left(\frac{\left(\ln(S_f(z)/S)\right)^2}{2\sigma^2(\tau-z)^2}-\frac{1}{2(\tau-z)}\right)\right]\mathrm{d}z, \end{aligned} \tag{35}$$

which is an expression for the value of the option for $0 < S < E$. For $E < S < S_*$, the inversion is a little more complicated. We can invert (20) directly to obtain

$$\begin{aligned} V &= -\frac{\sqrt{2}}{\sigma S}\int_S^{S_*}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} H(\tau-\tau_f(\widetilde{S}))\,\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))} \\ &\quad\times\mathcal{L}^{-1}\left(\frac{F_0(\widetilde{S})+pF_1(\widetilde{S})}{\sqrt{p}}\sinh\left[\frac{\sqrt{2}\sqrt{p}\ln(\widetilde{S}/S)}{\sigma}\right]\right)(\widetilde{S},\tau-\tau_f(\widetilde{S}))\mathrm{d}\widetilde{S} \\ &= -\frac{\sqrt{2}H(\tau-\tau_f(S))}{\sigma S}\int_S^{S_f(t)}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} H(\tau-\tau_f(\widetilde{S}))\,\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))} \\ &\quad\times\mathcal{L}^{-1}\left(\frac{F_0(\widetilde{S})+pF_1(\widetilde{S})}{\sqrt{p}}\sinh\left[\frac{\sqrt{2}\sqrt{p}\ln(\widetilde{S}/S)}{\sigma}\right]\right)(\widetilde{S},\tau-\tau_f(\widetilde{S}))\mathrm{d}\widetilde{S}, \end{aligned} \tag{36}$$

which is useful in the sense that it shows that $V(S,\tau)$ must vanish if $\tau < \tau_f(S)$, meaning in the region where exercise is optimal, which was expected given our definition of the partial Laplace transform (12). To proceed further, it is necessary to rewrite the sinh function in (36) in terms of exponentials and then use (22). For $\tau > \tau_f(S)$ and $E < S < S_*$, we then have

$$
\begin{aligned}
V = {} & \frac{1}{\sqrt{2}\sigma S}\mathcal{L}^{-1}\left(\int\limits_{S}^{S_*}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\frac{\mathrm{e}^{-p\tau_f(\widetilde{S})}}{\sqrt{p+p_0}}\right.\\
& \left.\times\exp\left[\frac{\sqrt{2}\sqrt{p+p_0}\ln(S/\widetilde{S})}{\sigma}\right]\left[F_0(\widetilde{S})+(p+p_0)\,F_1(\widetilde{S})\right]\mathrm{d}\widetilde{S}\right)\\
& +\frac{1}{\sqrt{2}\sigma S}\mathcal{L}^{-1}\left(\int\limits_{E}^{S}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\frac{\mathrm{e}^{-p\tau_f(\widetilde{S})}}{\sqrt{p+p_0}}\right.\\
& \left.\times\exp\left[\frac{\sqrt{2}\sqrt{p+p_0}\ln(\widetilde{S}/S)}{\sigma}\right]\left[F_0(\widetilde{S})+(p+p_0)\,F_1(\widetilde{S})\right]\mathrm{d}\widetilde{S}\right)\\
= {} & \frac{1}{\sqrt{2\pi}\sigma S}\int\limits_{E}^{S_*}\frac{H\left(\tau-\tau_f(\widetilde{S})\right)}{\sqrt{\tau-\tau_f(\widetilde{S})}}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))}\\
& \times\exp\left(-\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))}\right)\\
& \times\left[F_0(\widetilde{S})+F_1(\widetilde{S})\left(\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))^2}-\frac{1}{2(\tau-\tau_f(\widetilde{S}))}\right)\right]\mathrm{d}\widetilde{S}\\
= {} & \frac{1}{\sqrt{2\pi}\sigma S}\int\limits_{E}^{S_f(\tau)}\frac{\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))}}{\sqrt{\tau-\tau_f(\widetilde{S})}}\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\times\exp\left(-\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))}\right)\\
& \times\left[F_0(\widetilde{S})+F_1(\widetilde{S})\left(\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))^2}-\frac{1}{2(\tau-\tau_f(\widetilde{S}))}\right)\right]\mathrm{d}\widetilde{S}\\
= {} & \frac{1}{\sqrt{2\pi}\sigma S}\int\limits_{0}^{\tau}\frac{\mathrm{e}^{-p_0(\tau-z)}}{\sqrt{\tau-z}}\left(\frac{S_f(z)}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\times\exp\left(-\frac{(\ln(S_f(z)/S))^2}{2\sigma^2(\tau-z)}\right)\\
& \times\left[\widetilde{F}_0(z)+\widetilde{F}_1(z)\left(\frac{(\ln(S_f(z)/S))^2}{2\sigma^2(\tau-z)^2}-\frac{1}{2(\tau-z)}\right)\right]\mathrm{d}z,
\end{aligned}
\tag{37}
$$

while for $\tau < \tau_f(S)$ and $E < S < S_*$ we have $V(S,\tau) = 0$. It is worth noting that the expression (35) for $0 < S < E$ and (37) for $E < S < S_*$ are the same. This expression (37) for the option value $V(S,\tau)$ involves the location of the free boundary $S_f(\tau)$ and its derivatives. Other than very close to expiry, the evaluation of (37), would have to be implemented numerically, after first having computed $S_f(z)$ on the interval $0 < z < \tau$ using (27).

2.4. Analysis for the put

Turning now to the put, once again, we apply the solution (18) to the two separate regions defined earlier, and now we find that in region (a) we must discard the first solution in order to satisfy the boundary condition that $V \to 0$ as $S \to \infty$, so in this region we have

$$\mathcal{V} = C_2^{(a)}(p) \left(\frac{S}{E}\right)^{\frac{1}{2\sigma^2}\left(2D-2r+\sigma^2-\lambda(p)\right)}. \tag{38}$$

In region (b), we find the solution which satisfies the condition that $\mathcal{V}(S,p) \to 0$ as $S \to S_*$ is

$$\begin{aligned}\mathcal{V} = \frac{2}{S\lambda(p)} \int_S^{S_*} &\left(\frac{\widetilde{S}}{S}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \\ &\times \left[\left(\frac{\widetilde{S}}{S}\right)^{\frac{\lambda(p)}{2\sigma^2}} - \left(\frac{\widetilde{S}}{S}\right)^{-\frac{\lambda(p)}{2\sigma^2}}\right] F(\widetilde{S})\mathrm{d}\widetilde{S},\end{aligned} \tag{39}$$

where $F(S)$ is again given by (17), but of course V_f is different, being given by (3) rather than (4). If we compare these expressions to those for the call (19,20), they appear more alike than different. One of the differences is that in region (a), for the call, we discarded the second homogeneous solution given in (18), while for the put, we discarded the first. Another is that in region (b), the solutions for the call and put have opposite signs, because the region is $E < S < S_*$ for the call and $S_* < S < E$ for the put. We must now match the solutions in these two regions together, and as we did for the call, we will require that $\mathcal{V}$ and $(\mathrm{d}\mathcal{V}/\mathrm{d}S)$ be continuous across $S = E$. Proceeding as we did for the call, we find that

$$C_2^{(a)}(p) = \frac{2}{E\lambda(p)} \int_{S_*}^{E} \left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2-\lambda(p)\right)} F(\widetilde{S})\mathrm{d}\widetilde{S}, \tag{40}$$

which is similar to its counterpart (21) for the call, except that the subscripts 1 and 2 are interchanged and the sign of $\lambda(p)$ in the exponent is reversed, and

$$\int_{S_*}^{E} \widetilde{S}^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2+\lambda(p)\right)} F(\widetilde{S})\mathrm{d}\widetilde{S} = 0, \tag{41}$$

where again $F(S)$ is given by (17). This integro-differential equation for the put bears a strong resemblance to the equation for the call (22), with at first glance only the sign of $\lambda(p)$ being different, although of course the nonhomogeneous term $F(S)$ which appears in these is slightly different for the put than for the call: the definition (17) is the same for both cases, but of course V_f is given by (3) for the call and by (4) for the put. Once again, this equation is also very similar to the corresponding equation for the American put given in [**24**], with of course the nonhomogeneous term $F(S)$ again being different.

As was the case with (22), (41) is the Laplace transform of an integro-differential equation in (S,τ) space, and once again we can apply the inverse Laplace transform (13) to obtain a second integro-differential equation. To invert the transform, we again divide by $(p+p_0)^{3/2}E^{-\left(2D-2r+3\sigma^2\right)/(2\sigma^2)}(S_f(\tau))^{-\lambda(p)/(2\sigma^2)}$, and rewrite (41) as

$$\int_{S_*}^{E}\left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\exp\left[\frac{\sqrt{2(p+p_0)}}{\sigma}\ln\frac{\widetilde{S}}{S_f(\tau)}\right] \times\frac{\mathrm{e}^{-p\tau_f(\widetilde{S})}}{\sqrt{p+p_0}}\left[\frac{F_0(\widetilde{S})}{p+p_0}+F_1(\widetilde{S})\right]\mathrm{d}\widetilde{S}=0\ , \tag{42}$$

and then use the standard inverse transforms (24)to obtain

$$\int_{S_*}^{E}H\left(\tau-\tau_f(\widetilde{S})\right)\sqrt{\tau-\tau_f(\widetilde{S})}\left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))} \times\left[\frac{1}{\sqrt{\pi}}\left(2F_0(\widetilde{S})+\frac{F_1(\widetilde{S})}{\tau-\tau_f(\widetilde{S})}\right)\exp\left(-\frac{\left(\ln(\widetilde{S}/S_f(\tau))\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))}\right) -\frac{F_0(\widetilde{S})\sqrt{2}\ln(\widetilde{S}/S_f(\tau))}{\sigma(\tau-\tau_f(\widetilde{S}))}\mathrm{erfc}\left(\frac{\ln(\widetilde{S}/S_f(\tau))}{\sigma\sqrt{2(\tau-\tau_f(\widetilde{S}))}}\right)\right]\mathrm{d}\widetilde{S}=0, \tag{43}$$

or applying the step function

$$\int_{S_f(\tau)}^{E}\sqrt{\tau-\tau_f(\widetilde{S})}\left(\frac{\widetilde{S}}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)}\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))} \times\left[\frac{1}{\sqrt{\pi}}\left(2F_0(\widetilde{S})+\frac{F_1(\widetilde{S})}{\tau-\tau_f(\widetilde{S})}\right)\exp\left(-\frac{\left(\ln(\widetilde{S}/S_f(\tau))\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))}\right) -\frac{F_0(\widetilde{S})\sqrt{2}\ln(\widetilde{S}/S_f(\tau))}{\sigma(\tau-\tau_f(\widetilde{S}))}\mathrm{erfc}\left(\frac{\ln(\widetilde{S}/S_f(\tau))}{\sigma\sqrt{2(\tau-\tau_f(\widetilde{S}))}}\right)\right]\mathrm{d}\widetilde{S}=0. \tag{44}$$

To evaluate this, we will change variables so that we have an integral with respect to τ,

$$\int_0^\tau \sqrt{\tau - z}\left(\frac{S_f(z)}{E}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \mathrm{e}^{-p_0(\tau-z)} \\ \times\left[\frac{1}{\sqrt{\pi}}\left(2\widetilde{F}_0(z)+\frac{\widetilde{F}_1(z)}{\tau-z}\right)\exp\left(-\frac{\left(\ln(S_f(z)/S_f(\tau)\right)^2}{2\sigma^2(\tau-z)}\right) \right. \\ \left. -\frac{\widetilde{F}_0(z)\sqrt{2}\ln(S_f(z)/S_f(\tau))}{\sigma(\tau-z))}\operatorname{erfc}\left(\frac{\ln(S_f(\tau)/S_f(z))}{\sigma\sqrt{2(\tau-z)}}\right)\right]\mathrm{d}z = 0, \tag{45}$$

where

$$\begin{aligned}\widetilde{F}_0(\tau) &= F_0(S_f(\tau))S_f'(\tau) \\ &= \left[S_f'(\tau)+\left(r-D+\sigma^2\right)S_f(\tau)+\frac{\sigma^2 S_f^2(\tau)}{2S_f'(\tau)}\left(p_0-\frac{S_f''(\tau)}{S_f'(\tau)}\right)\right]V_f(S_f(\tau),\tau) \\ &\quad +\frac{\sigma^2 S_f^2(\tau)}{2S_f'(\tau)}\frac{\partial V_f}{\partial \tau}(S_f(\tau),\tau)-\sigma^2 S_f(\tau)E, \qquad (46)\\ \widetilde{F}_1(\tau) &= F_1(S_f(\tau))S_f'(\tau) \\ &= -\frac{\sigma^2 S_f^2(\tau)}{2S_f'(\tau)}V_f(S_f(\tau),\tau).\end{aligned}$$

As with (27), this last equation (45) is an integro-differential equation in (S,τ) space for the location of the free boundary, this time for the shout put. It differs from (27) in that the ratio $S_f(z)/S_f(\tau)$ is reversed, and also $\widetilde{F}_0$ and $\widetilde{F}_1$ will be slightly different because the pay-off at shouting for the put differs from that for the call. By making the substitutions $S_f(\tau) = E\,\mathrm{e}^{x_f(\tau)}$ and $z = \tau y$, we can rewrite (45) as

$$\int_0^1 \sqrt{1-y}\exp\left[-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)x_f(y\tau)-p_0\tau(1-y)\right] \\ \times\left[\frac{1}{\sqrt{\pi}}\left(2\widetilde{F}_0(y\tau)+\frac{\widetilde{F}_1(y\tau)}{\tau(1-y)}\right)\exp\left(-\frac{\left(x_f(y\tau)-x_f(\tau)\right)^2}{2\sigma^2\tau(1-y)}\right)\right. \\ \left. -\frac{\widetilde{F}_0(y\tau)\sqrt{2}(x_f(y\tau)-x_f(\tau))}{\sigma\tau(1-y))}\operatorname{erfc}\left(\frac{(x_f(y\tau)-x_f(\tau))}{\sigma\sqrt{2\tau(1-y)}}\right)\right]\mathrm{d}y = 0. \tag{47}$$

2.4.1. The free boundary close to expiry. Once again, we will assume that in the limit $\tau \to 0$, the free boundary has the form (30), and substitute this assumed form into the integro-differential equation (47), which we expand as a series in τ.

In this limit, we can show that at leading order,

$$
\begin{aligned}
\widetilde{F}_0(\tau) &\sim \frac{E^2\sigma^2}{2x_1\sqrt{\pi}}\left(1+x_1\sqrt{\pi}+x_1^2-2x_1^3\sqrt{\pi}\right)+\mathcal{O}\left(\tau\right)\\
\widetilde{F}_1(\tau) &\sim -\frac{E^2\sigma^2}{2x_1\sqrt{\pi}}\left(1-2x_1\sqrt{\pi}\right)\tau+\mathcal{O}\left(\tau^2\right).
\end{aligned} \tag{48}
$$

The coefficients in (48) for the put differ slightly from their counterparts for the call (31). Upon expanding (47) as a series in τ, at leading order we require that

$$
\begin{aligned}
&\int_0^1 \sqrt{1-y}\\
&\times\left[\frac{1}{\sqrt{\pi}}\left(2\left(1+x_1\sqrt{\pi}+x_1^2-2x_1^3\sqrt{\pi}\right)-\frac{\left(1-2x_1\sqrt{\pi}\right)y}{1-y}\right)\right.\\
&\times\exp\left(-\frac{x_1^2\left(1-\sqrt{y}\right)^2}{1-y}\right)\\
&\left.-\frac{2\left(1+x_1\sqrt{\pi}+x_1^2-2x_1^3\sqrt{\pi}\right)x_1(1-\sqrt{y})}{1-y}\operatorname{erfc}\left(\frac{x_1(1-\sqrt{y})}{\sqrt{1-y}}\right)\right]\mathrm{d}y=0,
\end{aligned} \tag{49}
$$

which is the equation we must solve for x_1 for the put. As with (32) for the call, we evaluated (49) numerically and found a value of $x_1 \approx -0.745457861349$, so for the shout put the behavior of the free boundary close to expiry is given by

$$
\begin{aligned}
S_f &\sim E\exp\left[-0.745457861349\sqrt{\sigma^2\tau/2}+\mathcal{O}\left(\tau\right)\right]\\
&\sim E\exp\left[-0.5271183\sigma\tau^{1/2}+\mathcal{O}\left(\tau\right)\right]\\
&\sim E\exp\left[-0.5271183\sigma(T-t)^{1/2}+\mathcal{O}\left(T-t\right)\right].
\end{aligned} \tag{50}
$$

As expected, the coefficient x_1 was positive for the call but negative for the put. Once again, we can compare the value for x_1 with that for a vanilla American. For a vanilla American put, if $D > r$ we can deduce that $x_1 = -0.90345$ using put-call symmetry [**25, 10**], while if $D \leq r$ once again the series (30) must be replaced by one including logs [**6, 23, 3**]. In either case, the boundary for the shout put is not as steep close to expiry as that for the American put, again because the shout put is more likely to be exercised early.

2.4.2. Laplace inversion. As for the call, we can apply an inverse Laplace transform to the expressions (38), (39) for $\mathcal{V}(S,p)$ to obtain $V(S,\tau)$ for the put, with the analysis extremely similar to that for the call presented earlier. For $\tau < \tau_f(S)$, once again we find that $V(S,\tau) = 0$, which again was to be expected given the definition of our partial Laplace transform (12), while for $\tau > \tau_f(S)$, we find for both $S_* < S < E$ and $S > E$ that

$$
\begin{aligned}
V = {} & \frac{1}{\sqrt{2\pi}\sigma S} \int_E^{S_f(\tau)} \frac{\mathrm{e}^{-p_0(\tau-\tau_f(\widetilde{S}))}}{\sqrt{\tau-\tau_f(\widetilde{S})}} \left(\frac{S}{\widetilde{S}}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \\
& \times \exp\left(-\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))}\right) \\
& \times \left[F_0(\widetilde{S}) + F_1(\widetilde{S})\left(\frac{\left(\ln(\widetilde{S}/S)\right)^2}{2\sigma^2(\tau-\tau_f(\widetilde{S}))^2} - \frac{1}{2(\tau-\tau_f(\widetilde{S}))}\right)\right] \mathrm{d}\widetilde{S} \qquad (51) \\
= {} & -\frac{1}{\sqrt{2\pi}\sigma S} \int_0^{\tau} \frac{\mathrm{e}^{-p_0(\tau-z)}}{\sqrt{\tau-z}} \left(\frac{S}{S_f(z)}\right)^{-\frac{1}{2\sigma^2}\left(2D-2r+3\sigma^2\right)} \\
& \times \exp\left(-\frac{\left(\ln(S_f(z)/S)\right)^2}{2\sigma^2(\tau-z)}\right) \\
& \times \left[\widetilde{F}_0(z) + \widetilde{F}_1(z)\left(\frac{\left(\ln(S_f(z)/S)\right)^2}{2\sigma^2(\tau-z)^2} - \frac{1}{2(\tau-z)}\right)\right] \mathrm{d}z\,,
\end{aligned}
$$

which is extremely similar to its counterpart (37) for the call. Other than very close to expiry, the evaluation of (51), just like that of (37), would have to be implemented numerically, after first having computed S_f on the interval $0 < z < \tau$ using (45).

3. Discussion

In this paper we have used Laplace transform methods to study shout options. The put and call shouts lead to very similar problems in spite of their well-known important differences. The integro-differential equations (22), (41) and (29), 47 presented above form the main result of this paper. Each of the first set (22), (41) is a nonlinear (Fredholm) integro-differential equation for the location of the free boundary, $\tau_f(S)$, or more specifically, an Urysohn equation of the first kind [**35**]. In this context, Fredholm simply means that the upper and lower limits of the integral are both constants [**33, 34**]. The second set of equations (29), (47) were derived by applying an inverse transform to (22), (41) and involve $S_f(\tau)$ rather than $\tau_f(S)$,

The equation for the call (22) differs slightly from that for the put (41), since the pay-offs differ. However, the differences are very small, the main difference being that $\lambda(p)$ is replaced by $-\lambda(p)$. The behavior seems symmetric between some of the intermediate equations, such as (19), (38), while in others such as (20), (39) it is anti-symmetric. Presumably there must be some sort of put-call symmetry for shout options, along the same lines as that for vanilla American options [**10, 25**], although it is not immediately obvious what form that symmetry takes.

We note that each of our integro-differential equations (22), (41) involves the transform variable p, both through the exponent of S and also through the term $e^{-p\tau_f(S)}$ in $F(S)$. Since $\tau_f(S)$ is a physical quantity, obviously it must be independent of p, and so the solution of each of these integro-differential equations involves finding a function $\tau_f(S)$ which satisfies the equation for all values of p with a positive real part. Because of this, we can think of each of these integro-differential equations as being a form of integral transform operating on $\tau_f(S)$, and inverting this integral transform would give us the location of the free boundary $\tau_f(S)$. However, this inversion would appear to be extremely difficult to do analytically because of the factor $e^{-p\tau_f(S)}$ in $F(S)$ as given in (17); if this term were absent, we could regard the equation as a form of (finite) Mellin transform [**38, 29**], although not of Naylor-type [**30, 31**]. While we may not have been able to invert this Mellin-like transform, we were able to apply a standard inverse Laplace transform (13) to the integro-differential equations (22), (41) to obtain a second set of integro-differential equations (29), (47) and use this second set to find the behavior of the free boundary close to expiry: for the call, $S_f \sim E\exp\left[0.1135166\sigma(T-t)^{1/2}\right]$ while for the put, $S_f \sim E\exp\left[-0.5271183\sigma(T-t)^{1/2}\right]$. It is interesting to compare this behavior to that of the vanilla Americans. For the shout options, the free boundary always starts from the strike price E at expiry, while for the Americans this only happens for the call with $D \geq r$ and the put with $D \leq r$. In addition, close to expiry the free boundary for the shouts always behaves like $\tau^{1/2}$, while for the Americans this form of behavior prevails for the call with $D < r$ and the put with $D > r$ but for the call with $D \geq r$ and the put with $D \leq r$ the behavior close to expiry involves logs [**6, 23, 3**]. In either case, the free boundary close to expiry for shout options seems to be less steep than that for vanilla Americans, and it would seem likely that this is because early exercise is more likely for a shout than a vanilla American on the same underlying with the same strike, simply because the rewards for early exercise are greater for a shout than an American. For an American, early exercise involves a trade-off between receiving the pay-off earlier and receiving benefits from any further upside, while with a shout early exercise results in receiving a portion of the pay-off earlier while still benefiting from further upsides. Because of this, it would appear paradoxically that although shout options are more complex contracts than vanilla Americans, the analysis of shouts is actually a little simpler than that of Americans, primarily because logs are not present for the shouts.

Moving on to the issue of the value of the option, in (19), (20), we have a series of expressions for $\mathcal{V}(p,S)$, the transform of the option price $V(S,\tau)$, with corresponding expressions in (38), (39) for the put. The constants which appear in these expressions were also given in the previous sections. We were able to apply an inverse transform (13) to these expressions, to find the value $V(S,\tau)$ of the option in the region where it is optimal to hold. We note that our expression for $V(S,\tau)$, which was given in (37) for the call and (51) for the put, involves $\tau_f(S)$, the location of the free boundary, which we know only abstractly as the solution of the applicable integro-differential equation.

In closing, we would like to make suggestions for further work on shout options. While [**8, 43**] have made great strides in valuing shouts numerically, relatively little theoretical work has been done on these options, or indeed on other exotics with American-style early exercise features, and we would suggest that studies along the lines of for instance [**17, 22, 23, 39**] might be worthwhile for these exotics. At this point, we would like to say a few words about the motivation for reformulating a free boundary problem like the present one as an integral or integro-differential equation, regardless of whether such reformulation is achieved by using Laplace transforms as in the present study or by one of the other methods mentioned above. The principal advantage of this approach is that, once an equation such as (27), (45), or those presented in [**22, 23, 39**], for the location of the free boundary is derived, the problem of finding location of the free boundary is decoupled from the problem of valuing the option.

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