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A NOTE ON MUTIPLICATION OPERATORS ON KÖTHE-BOCHNER SPACES

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ABSTRACT. Let $(\Omega, \mathcal{A}, \mu)$ is a finite measure space, E an order continuous Banach function space over μ, X a Banach space and $E(X)$ the Köthe-Bochner space. A new simple proof is given of the result that a continuous linear operator $T\colon E(X) \to E(X)$ is a multiplication operator (by a function in L^∞) iff $T(g\langle f, x^*\rangle x) = g\langle T(f), x^*\rangle x$ for every $g \in L^\infty,\ f \in E(X), x \in X, x^* \in X^*$.

1. Introduction and Notation

In this paper all vector spaces are taken over the real field R. $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and $L^\infty(\mu) = L^\infty,\ L^1(\mu) = L^1$ have their usual meanings. E is an ideal in the vector lattice L^1, $E \supset L^\infty$, and has the norm $\|.\|_E$ so that $(E, \|.\|_E)$ is a Banach lattice and is called Köthe function space relative to the measure μ ([**3**]). The order in E is the natural order of functions in L^1. Also the inclusions $L^\infty \subset E \subset L^1$ are continuous. $(X, \|.\|_X)$ is another Banach space such that the Banach space $(E(X), \|.\|_{E(X)})$ is the associated Köthe-Bochner function space relative to E. Thus $E(X)$ consists of all strongly measurable functions $f\colon \Omega \to X$ for which the real functions $\omega \to \|f(\omega)\|$ belongs to E and $\|f\|_{E(X)} = \|\|f(.)\|_X\|_E$ ([**3**]). For measure theory we refer to [**1**]. If Y is a Banach space, Y^* will denote its dual and for a $y \in Y,\ y^* \in Y^*,\ \langle y, y^*\rangle$ will also be used for $y^*(y)$.

In ([**2**]) a result is proved about the mutiplication operators in Köthe-Bochner spaces. The proof is quite sophisticated and, besides several lemmas, makes use of Markushevich bases. In this note we give a simple elementary proof.

2. Main Theorem

Now we come to the main theorem

Theorem 1. *Suppose E an order continuous Köthe function space over μ, X a Banach space and $E(X)$ the associated Köthe-Bochner space. Let $T\colon E(X) \to E(X)$ be a continuous linear operator. The following statements are equivalent:*

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(i) There is a $g_0 \in L^\infty$ *such that* $T(f) = g_0 f$ *for all* $f \in E(X)$.

(ii) $T(g\langle f, x^*\rangle x) = g\langle T(f), x^*\rangle x$ *for every* $g \in L^\infty$, $f \in E(X)$, $x \in X$, *and* $x^* \in X^*$.

Proof. (i) $\Longrightarrow$ (ii): Obvious.

(ii) $\Longrightarrow$ (i): For an $h \in E$, $x \in X$, $g \in L^\infty$, we have $T((gh\langle x, x^*\rangle)x) = gh\langle T(x), x^*\rangle x$; take any $x^* \in X^*$ with $\langle x, x^*\rangle = 1$. We get $T(ghx) = ghpx$ for some $p \in E$ (note since $|\langle T(x)(.), x^*\rangle| \leq \|T(x)(.)\|$, we have $\langle T(x)(.), x^*\rangle \ \in E$) and so $ghpx \in E(X)$. p may depend on x. Suppose $T(x_1) = p_1 x_1$ and $T(x_2) = p_2 x_2$. We claim $p_1 = p_2$. If x_1, x_2 are linearly dependent, there is nothing to prove; otherwise $x_1,\ x_1 - x_2$ are linearly independent. Take an $x^* \in X^*$ such that $\langle x_1, x^*\rangle = 1, : \langle x_1 - x_2, x^*\rangle = 0$. This means $0 = T(\langle x_1 - x_2, x^*\rangle z) = \langle T(x_1 - x_2), x^*\rangle z = \langle p_1 x_1 - p_2 x_2, x^*\rangle z = (p_1 - p_2) z$, for all $z \in E$. From this it follows that $p_1 = p_2$.

Now we want to prove that p is bounded. Suppose this is not true. Select a strictly increasing sequence $\{c_n\}$ of positive real numbers such that (i) $c_n \rangle n^3$, (ii) $\mu(Q_n)\rangle 0$ where $Q_n = |p|^{-1}(c_n, c_{n+1})$. For each n, choose positive α_n so that, for the functions $f_n = \alpha_n \chi_{Q_n}$, $\|f_n\|_E = 1$. Fix a $y \in X$ with $\|y\|_X = 1$. The function $f = \sum_{n=1}^\infty \frac{1}{n^2} f_n$ is in E and $f \geq \frac{1}{n^2} f_n$. This gives $f|p| \geq \frac{1}{n^2} f_n |p| \geq \frac{1}{n^2} f_n n^3$ and so $\|f|p|\|_E \geq n$ for all n. Now $\|T(fy)\|_{E(X)} = \|fpy\|_{E(X)} = \|f|p|\|_E \geq n$ for all n, which is a contradiction. So $p \in L^\infty$. We put $g_0 = p$. Thus $T(gx) = gg_0 x$, for all $x \in X$, $g \in L^\infty$ and so $T(h) = g_0 h$ for all simple functions $h \in E(X)$. Since E is order continuous, simple functions are dense, and so the result follows. □

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