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CHARACTERIZATION OF SPACING SHIFTS WITH POSITIVE TOPOLOGICAL ENTROPY

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ABSTRACT. Suppose $P \subseteq \mathbb{N}$ and let (Σ_P, σ_P) be the spacing shift defined by P. We show that if the topological entropy $h(\sigma_P)$ of a spacing shift is equal zero, then (Σ_P, σ_P) is proximal. Also $h(\sigma_P) = 0$ if and only if $P = \mathbb{N} \setminus E$ where E is an intersective set. Moreover, we show that $h(\sigma_P) > 0$ implies that P is a Δ^*-set; and by giving a class of examples, we show that this is not a sufficient condition. Using these results we solve question 5 given in [J. Banks et al., *Dynamics of Spacing Shifts*, Discrete Contin. Dyn. Syst., to appear].

1. Introduction and Definitions

In this paper we give a characterization of spacing shifts with positive topological entropy by the combinatorial property of the set $P \subseteq \mathbb{N}$ which defines a spacing shift. A detailed study of spacing shifts can be found in [**1**], so here we only consider the basic definitions and notions needed for our task.

A topological dynamical system (TDS) is a pair (X, T) such that X is a compact metric space with metric d and T is a continuous surjective self map. The *orbit closure* of a point x in (X, T) is the set $\overline{\mathcal{O}}(x) = \overline{\{T^n(x) : n \in \mathbb{N}\}}$. A system (X, T) is *transitive* if it has a point x such that $\overline{\mathcal{O}}(x) = X$. Also a point x is *recurrent* if for every neighborhood U of x there exists $n \neq 0$ such that $T^n(x) \in U$. We let $N(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ and $N(U, V) = \{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}$ where U and V are non-empty open sets.

Let $x_1, x_2 \in X$. One says that $(x_1, x_2) \in X \times X$ is a *proximal pair* if

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0;$$

and a TDS is called *proximal* if all $(x_1, x_2) \in X \times X$ are proximal pairs.

A set $D \subset \mathbb{N}$ is called Δ*-set* if there exists an increasing sequence of natural numbers $S = (s_n)_{n\in\mathbb{N}}$ such that the difference set $\Delta(S) = \{s_i - s_j : \ i > j\} \subset D$. Denote by $\boldsymbol{\Delta}$ the set of all Δ-sets.

Let $A = \{a_n\}_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers. Then $s = a_{i_1} + a_{i_2} + \ldots + a_{i_n}$, $\ i_j < i_{j+1}$ is called a *partial finite sum* of A. The *finite sums* set of A denoted by $FS(A)$ is the set of all partial finite sums. A set $F \subset \mathbb{N}$ is

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called *IP-set* if it contains the finite sums set of an increasing sequence of natural numbers. Let $\mathcal{IP}$ be the set of all IP-sets. A subset of natural numbers is called an *$(IP-IP)$-set* if it contains $\Delta(F)$ where $F \in \mathcal{IP}$. Any IP-set is a Δ-set; for set $S = \{a_1, a_1+a_2, a_1+a_2+a_3, \ldots\}$, and trivially any $(IP-IP)$-set is an IP-set.

A collection $\mathcal{F}$ of non-empty subsets of $\mathbb{N}$ is called a *family* if it is hereditary upward: if $F \in \mathcal{F}$ and $F \subset F'$, then $F' \in \mathcal{F}$. The dual family $\mathcal{F}^*$ is defined to be the family of all subsets of $\mathbb{N}$ that meet all sets in $\mathcal{F}$. That is

$$\mathcal{F}^* = \{G \subset \mathbb{N} : \quad G \cap F \neq \emptyset, \ \forall F \in \mathcal{F}\}.$$

Hence $\mathcal{IP}^*$ and $\boldsymbol{\Delta}^*$ are the dual families of $\mathcal{IP}$ and $\boldsymbol{\Delta}$, respectively.

The notions for a subset of natural numbers such as Δ or IP are structural notions. For instance, an IP-set is more structured than a Δ-set. Other structures are also defined [**7**] and [**3**]. There are also notions for largeness which are defined by means of densities on subsets of natural numbers. See [**7**], [**2**] for a rather complete treatment for both of these notions. Let $A \subseteq \mathbb{N}$. Then

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

is called the *upper density* of A. Also the *lower density* is defined as

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}.$$

When $\overline{d}(A) = \underline{d}(A)$, then this common value is called the *density* of A and is denoted by $d(A)$. The *upper Banach density* of A is denoted by $d^*(A)$ and is defined as

$$d^*(A) = \limsup_{N_i - M_i \to\infty} \frac{|A \cap \{M_i, M_i+1, \ldots, N_i\}|}{N_i - M_i + 1}.$$

When there is $k \in \mathbb{N}$ such that all the intervals contained in $\mathbb{N} \setminus A$ have length less than k, then A is called *syndetic*. The length of the largest of such intervals will be called the *gap* of A. Clearly, $\underline{d}(A) > 0$ for any syndetic set A. The dual of syndetic sets are *thick* sets; a set is thick if and only if $d^*(A) = 1$. We say A is *thickly syndetic* if for every N the positions where consecutive elements of length N begin form a syndetic set.

Note that Δ^*-sets are highly structured and are syndetic [**3**]. Another kind of large and structured subsets of $\mathbb{N}$ are *Bohr sets*. We say that a subset $A \subset \mathbb{N}$ is a Bohr set if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ and open set $U \subset \mathbb{T}^m$ such that the set

$$\{n \in \mathbb{N} : \ n\alpha \in U\}$$

is contained in A. In particular, every $k\mathbb{N}$ is a Bohr set.

Let (Σ, σ) denote the one-sided full shift over $\{0, 1\}$. We are dealing with one-sided shifts, but all results hold mutatis mutandis, in the two-sided case.

Definition 1.1. For any set $P \subset \mathbb{N}$, define a *spacing shift* to be the set

$$\Sigma_P = \{s \in \Sigma : \ s_i = s_j = 1 \ \Rightarrow |i-j| \in P \cup \{0\}\}.$$

By σ_P we denote the shift map restricted to Σ_P.

With any $y \in \Sigma_P$ we associate a set $A_y = \{i : y_i = 1\}$. Therefore, any notion of largeness and structure for A_y gives analogous notion for points in the shift. That is we set

$$d(y) := d(A_y) = \lim_{n\to\infty} \frac{\sum_1^n y_i}{n} = \lim_{n\to\infty} \frac{|A_y \cap \{1, \dots, n\}|}{n}.$$

Similarly, $\overline{d}(y)$, $\underline{d}(y)$ and $d^*(y)$ can be defined.

By Definition 1.1, it is clear that $A_y - A_y \subset P$.

Let (Σ, σ) be a shift space on a set of finite alphabet A. A *word* of length n in $x = x_1x_2 \ldots \in \Sigma$ is an element of A^n of the form $x_ix_{i+1} \ldots x_{i+n-1}$ for some $i \in \mathbb{N}$. The *topological entropy* of Σ denoted by $h(\sigma)$ is defined as

$$h(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log |L_n(\Sigma)| \tag{1.1}$$

where $L_n(X)$ is the set of all words of length n and $|I|$ denotes the cardinality of I.

2. Zero entropy gives proximality

The following questions arise in [**1**, Question 5].

1. Is there P such that $\mathbb{N} \setminus P$ does not contain IP-set but Σ_P is proximal?
2. Is there P such that $\mathbb{N} \setminus P$ does not contain IP-set but $h(\sigma_P) > 0$?
3. Are proximality and zero entropy essentially different in the context of spacing shifts?

We give positive answer to the first question but negative to the second. In fact for the second question we will show that if $\mathbb{N} \setminus P$ contains Δ-set (and hence IP-set), then the entropy is zero. For the last question, we will show that zero entropy in spacing shifts implies proximality.

For any $x, y \in \Sigma_P$, let

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n} |\{0 \le m \le n-1 : d(\sigma^m(x), \sigma^m(y)) < t\}|.$$

Remark 2.1. In [**1**] the authors show that if there are $x, y \in \Sigma_P$, $t > 0$ such that $F_{xy}(t) < 1$, then $h(\sigma_P) > 0$. If such x, y and t exist, then there exists $y' \in \Sigma_P$ such that $\overline{d}(y') > 0$. To see it, let $t = 2^{-l}$, then there exists an increasing sequence $\{q_i\}_{i=1}^{\infty}$ and $\varepsilon > 0$ such that either $|\{0 \le j \le q_i : x_j \ne 0\}| > \frac{q_i\varepsilon}{l+1}$, or $|\{0 \le j \le t_i : y_j \ne 0\}| > \frac{q_i\varepsilon}{l+1}$. Hence $\overline{d}(x)$ or $\overline{d}(y)$ is positive.

In [**1**, Lemma 3.5], it was proved that if $\mathbb{N} \setminus P$ contains an IP-set, then $d(y) = 0$ for $y \in \Sigma_P$. We give a stronger result with a simpler proof.

Theorem 2.2. *If $\mathbb{N} \setminus P$ contains a Δ-set, then $d^*(y) = 0$ for all $y \in \Sigma_P$.*

Proof. If $y \in \Sigma_P$, then $A_y - A_y \subset P$. But if there is y such that $d^*(y) > 0$ then $A_y - A_y$ is a Δ^*-set [**5**] and $\mathbb{N} \setminus P$ cannot have a Δ-set. □

The following result is a reformulation of two results in [**1**].

Theorem 2.3. *If for all $y \in \Sigma_P$, $d(y) = 0$, then*
1. $h(\sigma_P) = 0$,
2. σ_P *is proximal.*

Proof. (1) and (2) are proved in [**1**, Theorem 3.6] and [**1**, Theorem 3.11] respectively for the case when $\mathbb{N} \setminus P$ contains an IP-set. The proof of these theorems are based on the fact that if $\mathbb{N} \setminus P$ contains an IP-set then $d(y) = 0$, for any $y \in \Sigma_P$. Then this last result will lead to the both conclusions. □

Again the proof of the next Theorem is a minor alteration of the proof of [**1**, Theorem 3.18].

Theorem 2.4. *There exists $y \in \Sigma_P$ with $d^*(y) > 0$ if and only if $h(\sigma_P) > 0$.*

Proof. First suppose there exists a point $y \in \Sigma_P$ such that $d^*(y) > 0$, so there exist two increasing sequences $\{M_i\}_{i=1}^{\infty}$, $\{N_i\}_{i=1}^{\infty}$ and $\gamma > 0$ such that

$$|\{M_i \leq j \leq N_i : \ y_j = 1\}| \geq (N_i - M_i)\gamma.$$

Then by definition of topological entropy for shift spaces (1.1), we have

$$h(\sigma_P) \geq \lim_{N_i - M_i \to \infty} \frac{1}{N_i - M_i} \log(2^{(N_i - M_i)\gamma}) > 0.$$

Conversely, if for any $y \in \Sigma_P$, $d^*(y) = 0$, then $d(y) = 0$ and the proof follows from Theorem 2.3. □

An immediate consequence of the above theorem is that if P is not in Δ^*, then $h(\sigma_P) = 0$. In particular, this sorts out the second question.

By Theorem 2.3, if $h(\sigma_P) > 0$, then there is a point $y \in \Sigma_P$ such that $d(y) > 0$. Combining this with the results of the above Theorem, we have the following.

Corollary 2.5. *There is a point $y \in \Sigma_P$ with $d(y) > 0$ if and only if for some $y' \in \Sigma_P$ we have $d^*(y') > 0$.*

The following gives an answer to the third question. Moreover, this result and the fact that when P misses an IP-set, then it is not Δ^* and so has zero entropy, are an answer for the first question as well.

Theorem 2.6. *If $h(\sigma_P) = 0$, then Σ_P is proximal.*

Proof. Suppose $h(\sigma_P) = 0$. Then by Theorem 2.4, for any $y \in \Sigma_P$, we have $d^*(y) = 0$ which implies that $d(\{i : y_i = 0\}) = 1$. Hence for any two points $x, y \in \Sigma_P$, $d(\{i : x_i = 0\} \cap \{i : y_i = 0\}) = 1$ and this in turn implies that Σ_P is proximal. □

3. A necessary condition for transitivity

The question how to characterize such set P that define transitive spacing shifts remains open [**1**, Question 1]. Nevertheless, we offer here some remarks. A necessarity is the following.

Theorem 3.1. *Suppose Σ_P is transitive. Then P is an $(IP - IP)$-set.*

Proof. For any TDS such as (X, T), the return times of a recurrence point x to any non-empty open set U, that is, $N(x,U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ is an IP-set [**6**, Theorem 2.17]. Now let y be a transitive point. Then y is a recurrence point and $N(y, [1])$ is an IP-set. But $N(y, [1]) = \{y_i : \ y_i = 1\} = A_y$ and so $A_y - A_y \subset P$ and as a result P is an $(IP - IP)$-set. □

An application of the above theorem is that any thick subset of natural numbers is an $(IP - IP)$-set. This is because (Σ_P, σ_P) is weakly mixing if and only if P is thick (see [**8**, Theorem 2.2]), and it is well known that any weakly mixing TDS is transitive.

It is not hard to see that for any infinite subset A of $\mathbb{N}$, the spacing shift (Σ_P, σ_P) where $P = FS(A) - FS(A)$ is a transitive system. On the other hand, let $k \geq 3$, $p_2 > p_1$ and $p_2 - p_1 \neq kn$ for any $n \in \mathbb{N}$. Now if $P = k\mathbb{N} \cup \{p_1, p_2\}$, then Σ_P is not transitive, since for open sets $U = [10^{p_1 - 1}1]$ and $V = [10^{p_2 - 1}1]$, the set $N(U, V)$ is empty, however P is clearly an $(IP - IP)$-set.

By now we understand that this is the structure in P and not density which gives interesting dynamics to our spacing shifts systems. For instance, if P is not a Δ-set, then for all $y \in \Sigma_P$, $\sum_{i=1}^{\infty} y_i < \infty$. This gives a very simple dynamics to Σ_P. In fact, the orbit of any point is a finite set, as any point is mapped eventually onto the fixed point 0^∞. So one may choose P to have high density and yet (Σ_P, σ_P) with simple dynamics. As an example, for any $\varepsilon > 0$, let $\frac{1}{k} < \varepsilon$ and set $P = \mathbb{N} \setminus k\mathbb{N}$. Then $d(P) \geq 1 - \varepsilon$, and P is not a Δ-set since $k\mathbb{N}$ is a Δ^*-set.

4. Combinatorial characterization for zero entropy

In section 2, we showed that P must be at least Δ^*-set, that is a highly structured and large set to have positive entropy. Here we show that even if P is a Δ^*-set, it is not guaranteed that $h(\sigma_P) > 0$.

One calls $E \subset \mathbb{N}$ a *density intersective* set if for any $A \subset \mathbb{N}$ with positive upper Banach density, we have $E \cap (A - A) \neq \emptyset$. For instance, any IP-set is a density intersective set. In fact, if $R \subset \mathbb{N}$ is an IP-set and $p(\cdot)$ is a polynomial such that $p(\mathbb{N}) \subset \mathbb{N}$, then $E = \{p(n) : \ n \in R\}$ is a density intersective set [**4**].

Theorem 4.1. *The topological entropy of a spacing shift (Σ_P, σ_P) is equal to zero if and only if $P = \mathbb{N} \setminus E$ where E is a density intersective set.*

Proof. Suppose $h(\sigma_P) = 0$. If $E = \mathbb{N} \setminus P$ is not density intersective, then there must be a set A with positive upper Banach density such that $A - A \subseteq P$. Choose $y \in \Pi_{i=0}^{\infty}\{0, 1\}$ such that $y_i = 1$ if and only if $i \in A$. Then $y \in \Sigma_P$ and $A = A_y$. But this is absurd by Theorem 2.4.

For the other side, if E is density intersective, then P does not contain any $A - A$ where A is as above. Therefore, for all $y \in \Sigma_P$, $d^*(y) = 0$ which implies $h(\sigma_P) = 0$. □

It is an easy exercise to show that $\{n^2 : \ n \in \mathbb{N}\}$ does not contain any Δ-set. So $P = \mathbb{N} \setminus E$ is a Δ^*-set and by the above theorem, $h(\sigma_P) = 0$.

4.1. Positive entropy with no non-zero periodic points

Any spacing shift has 0^∞ as its periodic point. But a spacing shift has a non-zero periodic point of period k if and only if P contains $k\mathbb{N}$ [**1**, Lemma 2.6]. This implies there is a point y with $d(y) \geq \frac{1}{k}$ and so by Theorem 2.4, we have positive entropy.

Theorem 4.2. *There is P such that Σ_P has positive entropy with no non-zero periodic points.*

Proof. A theorem of Kříž [**9**] states that there is a set A with positive upper Banach density whose difference set contains no Bohr set. So let $y = \{y_i\}_{i\in\mathbb{N}}$ be defined by $y_i = 1$ if and only if $i \in A$. Set $P = A - A$. Then $y \in \Sigma_P$, $A_y = A$ and $\overline{d}(y) = \overline{d}(A) > 0$. Therefore, $h(\sigma_P) > 0$ and since P does not contain any Bohr set, it does not contain any $k\mathbb{N}$ and the proof is complete. $\square$

Note that more can be said on the dynamical properties of the spacing shift produced with the set P given by Kříž. For instance, D. Kwietniak [**10**] used that and gave an example of a proximal spacing shift with positive entropy.

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