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SLOWLY VARYING SOLUTIONS OF A CLASS OF FIRST ORDER SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. We analyze positive solutions of the two-dimensional systems of nonlinear differential equations

(A) $$x' + p(t)y^{\alpha} = 0, \qquad y' + q(t)x^{\beta} = 0,$$

(B) $$x' = p(t)y^{\alpha}, \qquad y' = q(t)x^{\beta},$$

in the framework of regular variation and indicate the situation in which system (A) (resp. (B)) possesses decaying solutions (resp. growing solutions) with precise asymptotic behavior as $t \to \infty$.

1. Introduction

This paper is devoted to the study of the existence and precise asymptotic behavior of positive solutions of two simple classes of first order systems of nonlinear differential equations the form

(A) $$x' + p(t)y^{\alpha} = 0, \qquad y' + q(t)x^{\beta} = 0,$$

(B) $$x' = p(t)y^{\alpha}, \qquad y' = q(t)x^{\beta},$$

where the following assumptions are always assumed to hold:
(a) α and β are positive constants such that $\alpha\beta < 1$;
(b) $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$.

The aim of this paper is to show how an application of the theory of regular variation gives the possibility to acquire as detailed information as possible about the asymptotic behavior at infinity of solutions $(x(t), y(t))$ of (A) such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, which are referred to as *decaying solutions* of (A), and solutions $(x(t), y(t))$ of (B) such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \infty$, which are referred to as *growing solutions* of (B). Such solutions will be constructed as solutions of the integral equations

$$x(t) = \int_t^{\infty} p(s)y(s)^{\alpha}\mathrm{d}s, \quad y(t) = \int_t^{\infty} q(s)x(s)^{\beta}\mathrm{d}s, \qquad t \geq T,$$

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and

$$x(t) = x_0 + \int_T^t p(s)x(s)^\alpha \mathrm{d}s, \quad y(t) = y_0 + \int_T^t q(s)x(s)^\beta \mathrm{d}s, \qquad t \geq T,$$

$x_0 > 0$, $y_0 > 0$ and $T > a$ being constants in the class of nearly regularly varying functions with specific asymptotic behavior at infinity. The Schauder-Tychonoff fixed point theorem is employed for this purpose.

For the in-depth analysis of oscillation and asymptotic behavior for systems of nonlinear differential equations the reader is referred to the book of Mirzov [**7**]. As for systems (A) and (B) under consideration, it is very difficult to characterize the existence of decaying (resp. growing) positive solutions and determine their precise asymptotic behavior at infinity in the case $p(t)$ and $q(t)$ are general positive continuous functions. However, if we limit ourselves to systems (A) and (B) with *regularly varying coefficients* $p(t)$ and $q(t)$, then fairly detailed and precise information can be acquired about the existence and precise asymptotic behavior of regularly varying solutions of (A) or (B). For related results see [**2**]–[**5**].

It is hoped that the present work will be a step towards deep and systematic investigations of positive solutions of general multi-dimensional systems of differential equations based on the use of the theory of regular variation combined with fixed point techniques.

2. Regularly varying functions

For the reader's convenience we recall here the definition of regularly varying functions, notations and some of basic properties including Karamata's integration theorem which will play an important role in establishing the main results of this paper.

Definition 2.1. A measurable function $f\colon (0,\infty) \to (0,\infty)$ is said to be *regularly varying of index* $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \qquad \text{for all} \quad \lambda > 0,$$

or equivalently it is expressed in the form

$$f(t) = c(t) \exp\left\{ \int_{t_0}^t \frac{\delta(s)}{s} \mathrm{d}s \right\}, \qquad t \geq t_0,$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If $c(t) \equiv c_0$, then $f(t)$ is called a *normalized* regularly varying function of index ρ. The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV instead of $\mathrm{RV}(0)$ and call members of SV *slowly varying functions*. By definition any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$. So, the class SV of slowly varying functions is of fundamental importance in theory of regular variation. Typical examples of slowly varying functions are all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \qquad \text{and} \qquad \exp\left\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\right\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\left\{(\log t)^{\frac{1}{3}} \cos\,(\log t)^{\frac{1}{3}}\right\}$$

is a slowly varying function which oscillates in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \qquad \text{and} \qquad \liminf_{t\to\infty} L(t) = 0.$$

A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^\rho L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t\to\infty} L(t) =$ const > 0. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) is used to denote the set of all trivial RV(ρ)-functions (or the set of all nontrivial RV(ρ)-functions).

The regularity of differentiable positive functions can be decided by the following simple criterion (see [**6**]).

Proposition 2.1. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t\frac{f'(t)}{f(t)} = \rho.$$

The following proposition, known as Karamata's integration theorem, is particularly useful in handling slowly and regularly varying functions analytically and is extensively used throughout the paper. Here and throughout the symbol $\sim$ is used to denote the asymptotic equivalence, that is,

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

Proposition 2.2. *Let $L(t) \in \mathrm{SV}$. Then,*

(i) *if $\alpha > -1$,*

$$\int_a^t s^\alpha L(s)\mathrm{d}s \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if $\alpha < -1$,*

$$\int_t^\infty s^\alpha L(s)\mathrm{d}s \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s}\mathrm{d}s \in SV \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^\infty \frac{L(s)}{s} \mathrm{d}s \in SV \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

A measurable function $f\colon (0,\infty) \to (0,\infty)$ is called *regularly bounded* if for any $\lambda_0 > 1$, there exist positive constants m and M such that

$$1 < \lambda < \lambda_0 \quad \Longrightarrow \quad m \le \frac{f(\lambda t)}{f(t)} \le M \quad \text{for all large } t.$$

The totality of regularly bounded functions is denoted by RO. It is clear that $\mathrm{RV}(\rho) \subset \mathrm{RO}$ for any $\rho \in \mathbb{R}$. Any function which is bounded both from above and from below by positive constants is regularly bounded. For example, $2 + \sin t$ and $2 + \sin(\log t)$ are regularly bounded. Note that $2 + \sin t$ and $2 + \sin(\log t)$ are not slowly varying, whereas $2 + \sin(\log_n t)$, $n \ge 2$, are slowly varying.

We now define the class of nearly regularly varying functions which is a useful subclass of RO including all regularly varying functions. To this end it is convenient to introduce the following notation.

Notation 2.1. Let $f(t)$ and $g(t)$ be two positive continuous functions defined in a neighborhood of infinity, say for $t \ge T$. We use the notation $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants m and M such that

$$mg(t) \le f(t) \le Mg(t) \quad \text{for} \qquad t \ge T.$$

Clearly, $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely. It is easy to see that if $f(t) \asymp g(t)$, $t \to \infty$, and if $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$.

Definition 2.2. If $f(t)$ satisfies $f(t) \asymp g(t)$, $t \to \infty$, for some $g(t)$ which is regularly varying of index ρ, then $f(t)$ is called a *nearly regularly varying function of index ρ.*

Since $2 + \sin t \asymp 2 + \sin(\log_n t)$, $t \to \infty$, for all $n \ge 2$, the function $2 + \sin t$ is nearly slowly varying, and the same is true of $2 + \sin(\log t)$. If $g(t) \in \mathrm{RV}(\rho)$, then the functions $(2 + \sin t)g(t)$ and $(2 + \sin(\log t))g(t)$ are nearly regularly varying of index ρ, but not regularly varying of index ρ.

A vector function $(x(t), y(t))$ defined on some interval $[T,\infty)$ is called *positive* if both $x(t) > 0$ and $y(t) > 0$ for $t \ge T$ and is called *regularly varying* (or *nearly regularly varying*) of index (ρ,σ) if $x(t)$ and $y(t)$ are regularly varying (or nearly regularly varying) of indices ρ and σ, respectively.

The reader is referred to Bingham et al [**1**] for the most complete exposition of theory of regular variation and its applications and to Marić [**6**] for the comprehensive survey of results up to 2000 on the asymptotic analysis of second order linear and nonlinear ordinary differential equations in the framework of regular variation.

3. Decaying slowly varying solutions of (A)

In order to have slowly varying solutions $(x(t), y(t))$ for (A) (or (B)) it is necessary that $p(t) \in \mathrm{RV}(-1)$ and $q(t) \in \mathrm{RV}(-1)$, that is, $p(t) = t^{-1}l(t)$ and $q(t) = t^{-1}m(t)$ for some $l, m \in \mathrm{SV}$. This means that $p(t) \sim L(t)q(t),\ t \to \infty$, for some $L \in \mathrm{SV}$. The simplest choice of $L(t)$ is $L(t) \equiv 1$, which we are concerned with in this paper hoping to have an insight into the structure of slowly varying solutions of (A) (or (B)) with general admissible $p(t)$ and $q(t)$.

We recall that a decaying solution $(x(t), y(t))$ of (A) defined in $[T, \infty)$ satisfies the system of integral equations

$$\text{(IA)} \qquad x(t) = \int_t^\infty p(s)y(s)^\alpha \mathrm{d}s, \quad y(t) = \int_t^\infty q(s)x(s)^\beta \mathrm{d}s, \quad t \geq T.$$

We associate with (IA) the system of asymptotic relations

$$\text{(AR)} \qquad x(t) \sim \int_t^\infty p(s)y(s)^\alpha \mathrm{d}s, \quad y(t) \sim \int_t^\infty q(s)x(s)^\beta \mathrm{d}s, \quad t \to \infty$$

which may be regarded as an approximation of the system (IA) at infinity. The following result concerning (AR) will be crucial in constructing decaying slowly varying solutions of (IA).

Theorem 3.1. *Suppose that $p(t)$ and $q(t)$ are regularly varying functions of index -1 such that $p(t) \sim q(t)$ as $t \to \infty$. Then, the asymptotic system* (AR) *possesses decaying slowly varying solutions if and only if $p(t)$ and $q(t)$ are integrable on $[a, \infty)$ are in such case the asymptotic behavior of all such solutions $(x(t), y(t))$ of* (AR) *is governed by the formulas*

$$(3.1) \qquad x(t) \sim \Big[\frac{1-\alpha\beta}{\alpha+1}\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} \int_t^\infty p(s)\mathrm{d}s\Big]^{\frac{\alpha+1}{1-\alpha\beta}}, \qquad t \to \infty,$$

$$(3.2) \qquad y(t) \sim \Big[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}} \int_t^\infty q(s)\mathrm{d}s\Big]^{\frac{\beta+1}{1-\alpha\beta}}, \qquad t \to \infty.$$

Proof. We begin with the proof of the "only if" part of the theorem. Let $(x(t), y(t))$ be a decaying slowly varying solution of (AR) on $[T, \infty)$. Put

$$(3.3) \qquad \xi(t) = \int_t^\infty p(s)y(s)^\alpha \mathrm{d}s, \quad \eta(t) = \int_t^\infty q(s)x(s)^\beta \mathrm{d}s.$$

Notice that $\xi(t)$ and $\eta(t)$ are slowly varying because $p(t)$ and $q(t)$ are in $\mathrm{RV}(-1)$. Then, using (3.3) we have

$$q(t)x(t)^\beta \xi'(t) = -p(t)q(t)x(t)^\beta y(t)^\alpha = p(t)y(t)^\alpha \eta'(t), \qquad t \geq T,$$

which in view of $p(t) \sim q(t)$, $t \to \infty$, implies that

$$\xi(t)^\beta \xi'(t) \sim \eta(t)^\alpha \eta'(t), \quad \text{i.e.,} \quad \left(\frac{\xi(t)^{\beta+1}}{\beta+1}\right)' \sim \left(\frac{\eta(t)^{\alpha+1}}{\alpha+1}\right)', \quad t \to \infty.$$

Integrating the above from t to ∞ yields

$$\frac{\xi(t)^{\beta+1}}{\beta+1} \sim \frac{\eta(t)^{\alpha+1}}{\alpha+1}, \qquad t \to \infty, \tag{3.4}$$

whence it follows that

$$\xi(t) \sim \left(\frac{\beta+1}{\alpha+1}\right)^{\frac{1}{\beta+1}} \eta(t)^{\frac{\alpha+1}{\beta+1}}, \quad \eta(t) \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{\alpha+1}} \xi(t)^{\frac{\beta+1}{\alpha+1}}, \quad t \to \infty,$$

which is clearly equivalent to

$$\begin{aligned} x(t) &\sim \left(\frac{\beta+1}{\alpha+1}\right)^{\frac{1}{\beta+1}} y(t)^{\frac{\alpha+1}{\beta+1}}, \\ y(t) &\sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{\alpha+1}} x(t)^{\frac{\beta+1}{\alpha+1}}, \qquad t \to \infty. \end{aligned} \tag{3.5}$$

Using (3.5) in (3.3), we obtain the following system of asymptotic relations for $x(t)$ and $y(t)$ for $t \to \infty$

$$\begin{aligned} x(t) &\sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} \int_t^\infty p(s)x(s)^{\frac{\alpha(\beta+1)}{\alpha+1}}\mathrm{d}s; \\ y(t) &\sim \left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}} \int_t^\infty q(s)y(s)^{\frac{\beta(\alpha+1)}{\beta+1}}\mathrm{d}s. \end{aligned} \tag{3.6}$$

Let $u(t)$ denote the right-hand side of the upper relation in (3.6). Then, the upper relation can be converted into the following differential asymptotic relation for $u(t)$

$$-u'(t) = \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} p(t)x(t)^{\frac{\alpha(\beta+1)}{\alpha+1}} \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} p(t)u(t)^{\frac{\alpha(\beta+1)}{\alpha+1}},$$

or

$$-u(t)^{-\frac{\alpha(\beta+1)}{\alpha+1}}u'(t) \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} p(t), \quad t \to \infty. \tag{3.7}$$

Integrating (3.7) from t to ∞ and noting that $u(t) \to 0$ as $t \to \infty$, we see that $p(t)$ is integrable on $[a, \infty)$ and obtain

$$x(t) \sim u(t) \sim \left[\frac{1-\alpha\beta}{\alpha+1}\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} \int_t^\infty p(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}, \quad t \to \infty. \tag{3.8}$$

Denoting by $v(t)$ the right-hand side of the lower relation in (3.6) and arguing as above, we conclude that $q(t)$ is integrable on $[a, \infty)$ and the asymptotic formula for $y(t)$ is given by

$$y(t) \sim v(t) \sim \left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}} \int_t^\infty q(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}, \quad t \to \infty. \tag{3.9}$$

This finishes the proof of the "only if" part, of Theorem 3.1. To prove the "if" part it suffices to show that if $p(t)$ and $q(t)$ are integrable on $[a,\infty)$, then the vector function $(X(t),Y(t))$ defined by

$$
\begin{aligned}
X(t) &= \left[\frac{1-\alpha\beta}{\alpha+1}\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}}\int_t^\infty p(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}},\\
Y(t) &= \left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}}\int_t^\infty q(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}.
\end{aligned}
\tag{3.10}
$$

satisfies the system of asymptotic relations

$$
\begin{aligned}
X(t) &\sim \int_t^\infty p(s)Y(s)^\alpha \mathrm{d}s,\\
Y(t) &\sim \int_t^\infty q(s)X(s)^\beta \mathrm{d}s, \qquad\qquad t\to\infty.
\end{aligned}
\tag{3.11}
$$

But this is a matter of straightforward calculation of rudimentary nature. For example, the validity of the first relation is confirmed as follows:

$$
\begin{aligned}
\int_t^\infty p(s)Y(s)^\alpha \mathrm{d}s &= \int_t^\infty p(s)\left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}}\int_s^\infty q(r)\mathrm{d}r\right]^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}\mathrm{d}s\\
&\sim \int_t^\infty p(s)\left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}}\int_s^\infty p(r)\mathrm{d}r\right]^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}\mathrm{d}s\\
&= \left(\frac{\alpha+1}{\beta+1}\right)^{-\frac{1}{\beta+1}}\left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}}\int_t^\infty p(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}\\
&= X(t), \qquad\qquad t\to\infty.
\end{aligned}
$$

This completes the proof. □

One of our main results in this section is the following theorem which ensures the existence of decaying solutions for (A) in the class of slowly varying functions.

Theorem 3.2. *Let $p(t)$, $q(t)$ be positive continuous functions which are integrable on $[a,\infty)$ and are nearly regularly varying of index -1. Assume that $p(t)\asymp p_0(t)\in RV(-1)$, $q(t)\asymp q_0(t)\in RV(-1)$, and $p_0(t)\sim q_0(t)$ as $t\to\infty$. Then, the system* (A) *possesses a decaying slowly varying solution $(x(t),y(t))$ such* *that for $t \to \infty$,*

$$x(t) \asymp \left[\frac{1-\alpha\beta}{\alpha+1} \left(\frac{\alpha+1}{\beta+1} \right)^{\frac{\alpha}{\alpha+1}} \int_t^\infty p_0(s)\mathrm{d}s \right]^{\frac{\alpha+1}{1-\alpha\beta}}, \tag{3.12}$$

$$y(t) \asymp \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_t^\infty q_0(s)\mathrm{d}s \right]^{\frac{\beta+1}{1-\alpha\beta}}. \tag{3.13}$$

Proof. By hypothesis there exist positive constants k and K such that

$$kp_0(t) \le p(t) \le Kp_0(t), \qquad kq_0(t) \le q(t) \le Kq_0(t), \quad t \ge a. \tag{3.14}$$

Let $X_0(t)$ and $Y_0(t)$ denote the functions defined by

$$\begin{aligned} X_0(t) &= \left[\frac{1-\alpha\beta}{\alpha+1} \left(\frac{\alpha+1}{\beta+1} \right)^{\frac{\alpha}{\alpha+1}} \int_t^\infty p_0(s)\mathrm{d}s \right]^{\frac{\alpha+1}{1-\alpha\beta}}, \\ Y_0(t) &= \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_t^\infty q_0(s)\mathrm{d}s \right]^{\frac{\beta+1}{1-\alpha\beta}}. \end{aligned} \tag{3.15}$$

As shown in Theorem 3.1, $(X_0(t), Y_0(t))$ satisfies the system of asymptotic relations

$$X_0(t) \sim \int_t^\infty p_0(s)Y_0(s)^\alpha \mathrm{d}s, \quad Y_0(t) \sim \int_t^\infty q_0(s)X_0(s)^\beta \mathrm{d}s, \quad t \to \infty, \tag{3.16}$$

from which it follows that there exists $T > a$ such that

$$\frac{1}{2}X_0(t) \le \int_t^\infty p_0(s)Y_0(s)^\alpha \mathrm{d}s \le 2X_0(t), \quad t \ge T, \tag{3.17}$$

and

$$\frac{1}{2}Y_0(t) \le \int_t^\infty q_0(s)X_0(s)^\beta \mathrm{d}s \le 2Y_0(t), \quad t \ge T. \tag{3.18}$$

Let us define W as the set of continuous vector functions $(x(t), y(t)) \in C[T,\infty) \times C[T,\infty)$ satisfying

$$aX_0(t) \le x(t) \le AX_0(t), \qquad bY_0(t) \le y(t) \le BY_0(t), \quad t \ge T, \tag{3.19}$$

where the positive constants a, A, b, B are required to satisfy the inequalities

$$a \le \frac{k}{2}b^\alpha, \quad 2KB^\alpha \le A, \quad b \le \frac{k}{2}a^\beta, \quad 2KA^\beta \le B. \tag{3.20}$$

It is easy to see that there is an infinitely many such choices of (a, A, b, B). For instance, one can choose

$$a = \left(\frac{k}{2}\right)^{\frac{\alpha+1}{1-\alpha\beta}}, \quad A = (2K)^{\frac{\alpha+1}{1-\alpha\beta}}, \quad b = \left(\frac{k}{2}\right)^{\frac{\beta+1}{1-\alpha\beta}}, \quad B = (2K)^{\frac{\beta+1}{1-\alpha\beta}}.$$

Clearly, W is a closed convex subset of the locally convex space $C[T,\infty) \times C[T,\infty)$.

Consider the integral operators

$$\begin{aligned} Fy(t) &= \int_t^\infty p(s)y(s)^\alpha \mathrm{d}s, \\ Gx(t) &= \int_t^\infty q(s)x(s)^\beta \mathrm{d}s, \qquad t \geq T, \end{aligned} \tag{3.21}$$

and define the mapping $\Phi \colon W \to C[T,\infty)\times C[T,\infty)$ by

$$\Phi(x(t),y(t)) = (Fy(t),Gx(t)), \quad t \geq T. \tag{3.22}$$

It can be proved that Φ is a continuous self-map on W and sends W into a relatively compact subset of $C[T,\infty)\times C[T,\infty)$.

(i) $\Phi(W) \subset W$. Let $(x(t),y(t)) \in W$. Then, using (3.16)–(3.22), we see that for $t \geq T$,

$$\begin{aligned} Fy(t) &\leq KB^\alpha \int_t^\infty p_0(s)Y_0(s)^\alpha \mathrm{d}s \leq 2KB^\alpha X_0(t) \leq AX_0(t), \\ Fy(t) &\geq kb^\alpha \int_t^\infty p_0(s)Y_0(s)^\alpha \mathrm{d}s \geq \frac{k}{2} b^\alpha X_0(t) \geq aX_0(t), \\ Gx(t) &\leq KA^\beta \int_t^\infty q_0(s)X_0(s)^\beta \mathrm{d}s \leq 2KA^\beta Y_0(t) \leq BY_0(t), \\ Gx(t) &\geq ka^\beta \int_t^\infty q_0(s)X_0(s)^\beta \mathrm{d}s \geq \frac{k}{2} a^\beta Y_0(t) \geq bY_0(t). \end{aligned}$$

This shows that $\Phi(x(t),y(t)) \in W$. Implying that Φ maps W into itself.

(ii) $\Phi(W)$ is relatively compact. The inclusion $\Phi(W) \subset W$ guarantees that $\Phi(W)$ is uniformly bounded on $[T,\infty)$. From the inequalities

$$\begin{aligned} 0 \geq (Fy)'(t) &= -p(t)y(t)^\alpha \geq -KB^\alpha p_0(t)Y_0(t)^\alpha, \\ 0 \geq (Gx)'(t) &= -q(t)x(t)^\beta \geq -KA^\beta q_0(t)X_0(t)^\beta, \end{aligned}$$

holding for $t \geq T$ and for all $(x(t),y(t)) \in W$, it follows that $\Phi(W)$ is equicontinuous on $[T,\infty)$. The relative compactness then follows from the Arzela-Ascoli lemma.

(iii) Φ is continuous. Let $\{(x_n(t),y_n(t))\}$ be a sequence in W converging to $(x(t),y(t)) \in W$ uniformly on any compact subinterval of $[T,\infty)$. Noting that

$$\begin{aligned} |Fy_n(t) - Fy(t)| &\leq \int_t^\infty p(s)|y_n(s)^\alpha - y(s)^\alpha| \mathrm{d}s, \\ |Gx_n(t) - Gx(t)| &\leq \int_t^\infty q(s)|x_n(s)^\beta - y(s)^\beta| \mathrm{d}s, \end{aligned}$$

for $t \geq T$ and applying the Lebesgue dominated convergence theorem to the above integrals, we find that $Fy_n(t) \to Fy(t)$ and $Gx_n(t) \to Gx(t)$ as $n \to \infty$ uniformly on $[T,\infty)$, which implies that

$$\begin{aligned} &\Phi(x_n(t),y_n(t)) = (Fy_n(t),Gx_n(t)) \\ &\to (Fy(t),Gx(t)) = \Phi(x(t),y(t)) \qquad \text{as } n \to \infty, \end{aligned}$$

the convergence being uniform on $[T,\infty)$. This establishes the continuity of Φ in the topology of $C[T,\infty)\times C[T,\infty)$.

Therefore, by the Schauder-Tychonoff fixed point theorem, there exists $(x(t),y(t))\in W$ such that $(x(t),y(t))=\Phi(x(t),y(t)$, $t\geq T$, that is,

$$x(t)=\int_t^\infty p(s)y(s)^\alpha \mathrm{d}s, \qquad y(t)=\int_t^\infty q(s)x(s)^\beta \mathrm{d}s, \quad t\geq T.$$

This means that $(x(t),y(t))$ is a solution of the differential system (A). That $x(t)$ and $y(t)$ are nearly slowly varying functions satisfying (3.12) and (3.13), respectively, is a consequence of the fact that $(x(t),y(t))$ is a member of W.

It remains to verify that $(x(t),y(t))$ is really slowly varying. For this purpose we note that

$$\begin{aligned} x(t)&=\int_t^\infty p(s)y(s)^\alpha \mathrm{d}s \asymp X_0(t), \\ y(t)&=\int_t^\infty q(s)x(s)^\beta \mathrm{d}s \asymp Y_0(t), \qquad t\to\infty, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} -x'(t)&=p(t)y(t)^\alpha \asymp p_0(t)Y_0(t)^\alpha, \\ -y'(t)&=q(t)x(t)^\beta \asymp q_0(t)X_0(t)^\beta, \qquad t\to\infty. \end{aligned} \tag{3.24}$$

According to Proposition 2.1 in order to make sure that $x(t)$ and $y(t)$ are slowly varying, it suffices to show that

$$\lim_{t\to\infty} t\frac{x'(t)}{x(t)}=0, \qquad \lim_{t\to\infty} t\frac{y'(t)}{y(t)}=0. \tag{3.25}$$

It is convenient to rewrite $X_0(t)$ and $Y_0(t)$ as

$$\begin{aligned} X_0(t)&=\left[\lambda\int_t^\infty p_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}} \sim \left[\lambda\int_t^\infty q_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}, \qquad t\to\infty, \\ Y_0(t)&=\left[\mu\int_t^\infty q_0(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}} \sim \left[\mu\int_t^\infty p_0(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}, \qquad t\to\infty. \end{aligned}$$

Using (3.23), (3.24) and the above simplified expressions for $X_0(t)$ and $Y_0(t)$, we obtain

$$\begin{aligned} -t\frac{x'(t)}{x(t)} &\asymp \frac{tp_0(t)\left[\mu\int_t^\infty p_0(s)\mathrm{d}s\right]^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}}{\left[\lambda\int_t^\infty p_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}} \\ &= \frac{(\alpha+1)tp_0(t)}{(1-\alpha\beta)\int_t^\infty p_0(s)\mathrm{d}s}, \qquad t\to\infty, \end{aligned} \tag{3.26}$$

and similarly,

$$-t\frac{y'(t)}{y(t)} \asymp \frac{(\beta+1)tq_0(t)}{(1-\alpha\beta)\int_t^\infty q_0(s)\mathrm{d}s}, \quad t\to\infty. \tag{3.27}$$

Since $p_0(t)$ and $q_0(t)$ are regularly varying of index -1, from Karamata's integration theorem it follows that

$$\lim_{t\to\infty} \frac{tp_0(t)}{\int_t^\infty p_0(s)\mathrm{d}s} = \lim_{t\to\infty} \frac{tq_0(t)}{\int_t^\infty q_0(s)\mathrm{d}s} = 0$$

which, combined with (3.26) and (3.27), ensures that (3.25) holds true as desired. Thus, both $x(t)$ and $y(t)$ are slowly varying functions. This completes the proof. □

As the next result shows, under the stronger assumption that $p(t)$ and $q(t)$ in Theorem 3.2 are regularly varying functions, the existence of decaying slowly varying solutions for (A) can be completely characterized.

Corollary 3.1. *Let $p(t)$, $q(t)$ be continuous regularly varying functions of index -1 such that $p(t) \sim q(t)$ as $t \to \infty$. Then, system* (A) *possesses decaying slowly varying solutions if and only if $p(t)$ and $q(t)$ are integrable on $[a, \infty)$, in which case the asymptotic behavior of all such solutions $(x(t), y(t))$ is governed by the formulas* (3.1) *and* (3.2).

Example 3.1. Consider the system of differential equations

$$x' + p(t)y^\alpha = 0, \quad y' + q(t)x^\beta = 0, \tag{3.28}$$

where $p(t)$ and $q(t)$ are positive continuous functions on $[\mathrm{e}, \infty)$.

(i) Suppose that $p(t)$, $q(t)$ are continuous nearly regularly varying functions of index -1 such that $p(t) \asymp r(t)$ and $q(t) \asymp r(t)$ as $t \to \infty$, where $r(t) \in \mathrm{RV}(-1)$ satisfies

$$r(t) \sim (\alpha+1)^{-\frac{\alpha}{\alpha+1}}(\beta+1)^{-\frac{\beta}{\beta+1}}t^{-1}(\log t)^{-\frac{\alpha+\beta+2}{(\alpha+1)(\beta+1)}}, \quad t \to \infty.$$

Then, system (3.28) possesses a slowly varying solution $(x(t), y(t))$ such that

$$\begin{aligned} x(t) &\asymp (\beta+1)^{\frac{1}{\beta+1}}(\log t)^{-\frac{1}{\beta+1}}, \\ y(t) &\asymp (\alpha+1)^{\frac{1}{\alpha+1}}(\log t)^{-\frac{1}{\alpha+1}}, \qquad t \to \infty. \end{aligned}$$

(ii) Suppose that $p(t)$ and $q(t)$ are continuous regularly varying functions of index -1 satisfying

$$p(t) \sim q(t) \sim (\alpha+1)^{-\frac{\alpha}{\alpha+1}}(\beta+1)^{-\frac{\beta}{\beta+1}}t^{-1}(\log t)^{-\frac{\alpha+\beta+2}{(\alpha+1)(\beta+1)}}, \quad t \to \infty.$$

Then, system (3.28) possesses slowly varying solutions $(x(t), y(t))$ and all of them enjoy the following precise asymptotic behavior at infinity:

$$\begin{aligned} x(t) &\sim (\beta+1)^{\frac{1}{\beta+1}}(\log t)^{-\frac{1}{\beta+1}}, \\ y(t) &\sim (\alpha+1)^{\frac{1}{\alpha+1}}(\log t)^{-\frac{1}{\alpha+1}}, \qquad t \to \infty. \end{aligned}$$

4. Growing slowly varying solutions of (B)

Let $(x(t), y(t))$ be a growing slowly varying solution of (B) on $[T, \infty)$, $T \geq a$. Then, it satisfies the system of integral equations

$$\text{(IB)} \qquad x(t) = x_0 + \int_T^t p(s)y(s)^\alpha \mathrm{d}s, \quad y(t) = y_0 + \int_T^t q(s)x(s)^\beta \mathrm{d}s, \quad t \geq T,$$

where $x_0 > 0$, $y_0 > 0$, and hence the system of asymptotic relations

$$\text{(BR)} \qquad x(t) \sim \int_T^t p(s)y(s)^\alpha \mathrm{d}s, \quad y(t) \sim \int_T^t q(s)x(s)^\beta \mathrm{d}s, \quad t \geq T.$$

An analogue of Theorem 3.1 is the following result.

Theorem 4.1. *Suppose that $p(t)$ and $q(t)$ are regularly varying functions of index -1 such that $p(t) \sim q(t)$ as $t \to \infty$. Then, the asymptotic system* (BR) *possesses growing slowly varying solutions if and only if*

$$\text{(4.1)} \qquad \int_a^\infty p(t)dt = \int_a^\infty q(t)\mathrm{d}t = \infty,$$

and in such case the asymptotic behavior of all such solutions $(x(t), y(t))$ of (BR) *is governed by the formulas*

$$\text{(4.2)} \qquad x(t) \sim \left[\frac{1-\alpha\beta}{\alpha+1}\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} \int_a^t p(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}, \qquad t \to \infty,$$

$$\text{(4.3)} \qquad y(t) \sim \left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}} \int_a^t q(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}, \qquad t \to \infty.$$

Proof. (The "only if" part) Let $(x(t), y(t))$ be a growing slowly varying solution of (AR) on $[T, \infty)$. Put

$$\text{(4.4)} \qquad \xi(t) = \int_T^t p(s)y(s)^\alpha \mathrm{d}s, \qquad \eta(t) = \int_T^t q(s)x(s)^\beta \mathrm{d}s.$$

Notice that $\xi(t)$ and $\eta(t)$ are slowly varying because $p(t)$ and $q(t)$ are in $\mathrm{RV}(-1)$. Then, using (4.4), we have

$$q(t)x(t)^\beta \xi'(t) = p(t)q(t)x(t)^\beta y(t)^\alpha = p(t)y(t)^\alpha \eta'(t), \quad t \geq T,$$

which in view of $p(t) \sim q(t)$, $t \to \infty$, implies that

$$\xi(t)^\beta \xi'(t) \sim \eta(t)^\alpha \eta'(t), \quad \text{i.e.,} \quad \left(\frac{\xi(t)^{\beta+1}}{\beta+1}\right)' \sim \left(\frac{\eta(t)^{\alpha+1}}{\alpha+1}\right)', \quad t \to \infty.$$

Integrating the above from T to t yields

$$\text{(4.5)} \qquad \frac{\xi(t)^{\beta+1}}{\beta+1} \sim \frac{\eta(t)^{\alpha+1}}{\alpha+1}, \quad t \to \infty,$$

whence it follows that

$$\xi(t) \sim \left(\frac{\beta+1}{\alpha+1}\right)^{\frac{1}{\beta+1}} \eta(t)^{\frac{\alpha+1}{\beta+1}}, \qquad \eta(t) \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{\alpha+1}} \xi(t)^{\frac{\beta+1}{\alpha+1}}, \quad t \to \infty,$$

which is clearly equivalent to
(4.6)

$$x(t) \sim \left(\frac{\beta+1}{\alpha+1}\right)^{\frac{1}{\beta+1}} y(t)^{\frac{\alpha+1}{\beta+1}}, \qquad y(t) \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{1}{\alpha+1}} x(t)^{\frac{\beta+1}{\alpha+1}}, \quad t \to \infty.$$

Using (4.6) in (4.4), we obtain the following system of asymptotic relations for $x(t)$ and $y(t)$ for $t \to \infty$:

$$(4.7) \qquad \begin{aligned} x(t) &\sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} \int_T^t p(s) x(s)^{\frac{\alpha(\beta+1)}{\alpha+1}} \mathrm{d}s, \\ y(t) &\sim \left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}} \int_T^t q(s) y(s)^{\frac{\beta(\alpha+1)}{\beta+1}} \mathrm{d}s. \end{aligned}$$

Let $u(t)$ denote the right-hand side of the upper relation in (4.7). Then, the upper relation can be converted into the following differential asymptotic relation for $u(t)$:

$$u'(t) = \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} p(t) x(t)^{\frac{\alpha(\beta+1)}{\alpha+1}} \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} p(t) u(t)^{\frac{\alpha(\beta+1)}{\alpha+1}},$$

or

$$(4.8) \qquad u(t)^{-\frac{\alpha(\beta+1)}{\alpha+1}} u'(t) \sim \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} p(t), \quad t \to \infty.$$

Integrating (4.8) from T to t and noting that $u(t) \to \infty$ as $t \to \infty$, we see that $p(t)$ is not integrable on $[a, \infty)$ and obtain

$$(4.9) \quad x(t) \sim u(t) \sim \left[\frac{1-\alpha\beta}{\alpha+1} \left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}} \int_T^t p(s) \mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}, \quad t \to \infty.$$

Let $v(t)$ denote the right-hand side of the lower relation in (4.7). Arguing as above, we conclude that $q(t)$ is not integrable on $[a, \infty)$ and the asymptotic formula for $y(t)$ is given by

$$(4.10) \qquad y(t) \sim v(t) \sim \left[\tfrac{1-\alpha\beta}{\beta+1} \left(\tfrac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}} \int_T^t q(s) \mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}, \quad t \to \infty.$$

This finishes the proof of the “only if” part of Theorem 4.1.

(The "if" part) It suffices to show that if $p(t)$ and $q(t)$ are not integrable on $[a, \infty)$, then the vector function $(X(t), Y(t))$ defined by

$$\begin{aligned} X(t) &= \left[\frac{1-\alpha\beta}{\alpha+1} \left(\frac{\alpha+1}{\beta+1} \right)^{\frac{\alpha}{\alpha+1}} \int_a^t p(s)\mathrm{d}s \right]^{\frac{\alpha+1}{1-\alpha\beta}}, \\ Y(t) &= \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_a^t q(s)\mathrm{d}s \right]^{\frac{\beta+1}{1-\alpha\beta}}. \end{aligned} \tag{4.11}$$

satisfies the system of asymptotic relations

$$X(t) \sim \int_a^t p(s)Y(s)^\alpha \mathrm{d}s, \qquad Y(t) \sim \int_a^t q(s)X(s)^\beta \mathrm{d}s, \quad t \to \infty. \tag{4.12}$$

But this is a matter of straightforward calculation of rudimentary nature. For example, the validity of the first relation is confirmed as follows:

$$\begin{aligned} \int_a^t p(s)Y(s)^\alpha \mathrm{d}s &= \int_a^t p(s) \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_a^s q(r)\mathrm{d}r \right]^{\frac{\alpha(\beta+1)}{1-\alpha\beta}} \mathrm{d}s \\ &\sim \int_a^t p(s) \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_a^s p(r)\mathrm{d}r \right]^{\frac{\alpha(\beta+1)}{1-\alpha\beta}} \mathrm{d}s \\ &\sim \left(\frac{\alpha+1}{\beta+1} \right)^{-\frac{1}{\beta+1}} \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_a^t p(s)\mathrm{d}s \right]^{\frac{\alpha+1}{1-\alpha\beta}} \\ &= X(t), \qquad\qquad t \to \infty. \end{aligned}$$

This completes the proof. □

A dual result to Theorem 3.2 is the following theorem.

Theorem 4.2. *Let $p(t)$ and $q(t)$ be positive continuous functions which satisfy* (4.1). *Assume that $p(t) \asymp p_0(t) \in \mathrm{RV}(-1), q(t) \asymp q_0(t) \in \mathrm{RV}(-1)$ and $p_0(t) \sim q_0(t)$ as $t \to \infty$. Then, the system* (B) *possesses a growing slowly varying solution $(x(t), y(t))$ such that for $t \to \infty$,*

$$x(t) \asymp \left[\frac{1-\alpha\beta}{\alpha+1} \left(\frac{\alpha+1}{\beta+1} \right)^{\frac{\alpha}{\alpha+1}} \int_a^t p_0(s)\mathrm{d}s \right]^{\frac{\alpha+1}{1-\alpha\beta}}, \tag{4.13}$$

$$y(t) \asymp \left[\frac{1-\alpha\beta}{\beta+1} \left(\frac{\beta+1}{\alpha+1} \right)^{\frac{\beta}{\beta+1}} \int_a^t q_0(s)\mathrm{d}s \right]^{\frac{\beta+1}{1-\alpha\beta}}. \tag{4.14}$$

Proof. Since $p(t) \asymp p_0(t)$ and $q(t) \asymp q_0(t)$, there exist positive constants k and K such that

$$kp_0(t) \le p(t) \le Kp_0(t), \quad kq_0(t) \le q(t) \le Kq_0(t), \quad t \ge a. \tag{4.15}$$

Let $X_0(t)$ and $Y_0(t)$ denote the functions defined by

$$
\begin{aligned}
X_0(t) &= \left[\frac{1-\alpha\beta}{\alpha+1}\left(\frac{\alpha+1}{\beta+1}\right)^{\frac{\alpha}{\alpha+1}}\int_a^t p_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}, \\
Y_0(t) &= \left[\frac{1-\alpha\beta}{\beta+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\frac{\beta}{\beta+1}}\int_a^t q_0(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}.
\end{aligned}
\tag{4.16}
$$

It is known from the proof of Theorem 4.1 that $(X_0(t), Y_0(t))$ satisfies

$$
\begin{aligned}
X_0(t) &\sim \int_b^t p_0(s)Y_0(s)^{\alpha}\mathrm{d}s, \\
Y_0(t) &\sim \int_b^t q_0(s)X_0(s)^{\beta}\mathrm{d}s, \qquad t\to\infty,
\end{aligned}
\tag{4.17}
$$

for any $b \geq a$, from which it follows that there exists $T_0 \geq a$ such that

$$
\begin{aligned}
&\int_{T_0}^t p_0(s)Y_0(s)^{\alpha}\mathrm{d}s \leq 2X_0(t), \\
&\int_{T_0}^t q_0(s)X_0(s)^{\beta}\mathrm{d}s \leq 2Y_0(t), \qquad t \geq T_0.
\end{aligned}
\tag{4.18}
$$

Since (4.17) holds for $b = T_0$, there exists $T_1 > T_0$ such that

$$
\begin{aligned}
&\int_{T_0}^t p_0(s)Y_0(s)^{\alpha}\mathrm{d}s \geq \frac{1}{2}X_0(t), \\
&\int_{T_0}^t q_0(s)X_0(s)^{\beta}\mathrm{d}s \geq \frac{1}{2}Y_0(t), \qquad t \geq T_1.
\end{aligned}
\tag{4.19}
$$

Choose positive constants a, A, b and B so that $a < A$, $b < B$ and the inequalities

$$
a \leq \frac{1}{2}kb^{\alpha}, \quad b \leq \frac{1}{2}ka^{\beta}, \quad 4KB^{\alpha} \leq A, \quad 4KA^{\beta} \leq B,
\tag{4.20}
$$

and

$$
aX_0(T_1) \leq \frac{1}{2}AX_0(T_0), \quad bY_0(T_1) \leq \frac{1}{2}BY_0(T_0)
\tag{4.21}
$$

hold. It is easy to check that such choices of a, A, b and B are indeed possible by taking, if necessary, k sufficiently small and K sufficiently large. With these constants we define W as the set of continuous functions $(x(t), y(t))$ on $[T_0, \infty)$ such that

$$
\begin{aligned}
aX_0(t) &\leq x(t) \leq AX_0(t), \\
Y_0(t) &\leq y(t) \leq BY_0(t), \qquad t \geq T_0.
\end{aligned}
\tag{4.22}
$$

It is clear that W is closed and convex in $\mathrm{C}[T_0,\infty)\times\mathrm{C}[T_0,\infty)$. Consider the mapping $\Phi\colon W \to \mathrm{C}[T_0,\infty)\times\mathrm{C}[T_0,\infty)$ defined by

$$
\Phi(x(t), y(t)) = (\mathcal{F}y(t), \mathcal{G}x(t)), \quad t \geq T_0,
\tag{4.23}
$$

where

$$\mathcal{F}y(t) = x_0 + \int_{T_0}^{t} p(s)y(s)^{\alpha}\mathrm{d}s,$$

(4.24)

$$\mathcal{G}x(t) = y_0 + \int_{T_0}^{t} q(s)x(s)^{\beta}\mathrm{d}s, \qquad t \geq T_0.$$

Here x_0 and y_0 are positive constants satisfying

$$aX_0(T_1) \leq x_0 \leq \frac{1}{2}AX_0(T_0), \quad bY_0(T_1) \leq y_0 \leq \frac{1}{2}BY_0(T_0). \tag{4.25}$$

It is proved without difficulty that Φ is a continuous self-map of W with the property that $\Phi(W)$ is relatively compact in $\mathrm{C}[T_0,\infty) \times \mathrm{C}[T_0,\infty)$.

(i) $\Phi(W) \subset W$. Let $(x(t), y(t)) \in W$. Using (4.18)–(4.25), we see that

$$\mathcal{F}y(t) \geq x_0 \geq aX_0(T_1) \geq aX_0(t), \quad T_0 \leq t \leq T_1,$$

and

$$\begin{aligned}\mathcal{F}y(t) &\geq \int_{T_0}^{t} p(s)y(s)^{\alpha}\mathrm{d}s \geq \int_{T_0}^{t} kp_0(s)\big(bY_0(s)\big)^{\alpha}\mathrm{d}s \\ &\geq \frac{1}{2}kb^{\alpha}X_0(t) \geq aX_0(t), \qquad t \geq T_1.\end{aligned}$$

On the other hand, for $t \geq T_0$ we have

$$\begin{aligned}\mathcal{F}y(t) &\leq \frac{1}{2}AX_0(T_0) + \int_{T_0}^{t} Kp_0(s)\big(BY_0(s)\big)^{\alpha}\mathrm{d}s \\ &\leq \frac{1}{2}AX_0(t) + 2KB^{\alpha}X_0(t) \leq \frac{1}{2}AX_0(t) + \frac{1}{2}AX_0(t) = AX_0(t).\end{aligned}$$

This implies that $aX_0(t) \leq \mathcal{F}y(t) \leq AX_0(t)$ for $t \geq T_0$. An analogous computation applies to $\mathcal{G}$, showing that $bY_0(t) \leq \mathcal{G}x(t) \leq BY_0(t)$ for $t \geq T_0$. It follows that $\Phi\big(x(t), y(t)\big) \in W$.

(ii) The relative compactness of $\Phi(W)$ follows from the inclusion $\Phi(W) \subset W$ which guarantees that $\Phi(W)$ is uniformly bounded on $[T_0,\infty)$ and the inequalities

$$\begin{aligned}&0 \leq \big(\mathcal{F}y\big)'(t) \leq B^{\alpha}p(t)Y_0(t)^{\alpha},\\ &0 \leq \big(\mathcal{G}x\big)'(t) \leq A^{\beta}q(t)X_0(t)^{\beta}, \qquad t \geq T_0,\end{aligned}$$

imply that $\Phi(W)$ is equicontinuous on $[T_0,\infty)$.

(iii) To prove the continuity of Φ, it suffices to consider a sequence $\{(x_n(t), y_n(t))\}$ in W converging to $(x(t), y(t)) \in W$ uniformly on compact subintervals of $[T_0,\infty)$ and to show that $\mathcal{F}y_n(t) \to \mathcal{F}y(t)$ and $\mathcal{G}x_n(t) \to \mathcal{G}x(t)$ uniformly on compact subintervals of $[T_0,\infty)$ by applying the Lebesgue dominated convergence theorem to the integrals

$$|\mathcal{F}y_n(t) - \mathcal{F}y(t)| \leq \int_{T_0}^{t} p(s)|y_n(s)^{\alpha} - y(s)^{\alpha}|\mathrm{d}s,$$

and

$$|\mathcal{G}x_n(t) - \mathcal{G}x(t)| \leq \int_{T_0}^{t} q(s)|x_n(s)^{\beta} - x(s)^{\beta}|\mathrm{d}s, \quad t \geq T_0.$$

Consequently by the Schauder-Tychonoff fixed point theorem Φ has a fixed point $\big(x(t), y(t)\big) \in W$, which satisfies the system of integral equations

$$x(t) = x_0 + \int_{T_0}^{t} p(s)y(s)^{\alpha}\mathrm{d}s, \quad y(t) = y_0 + \int_{T_0}^{t} q(s)x(s)^{\beta}\mathrm{d}s, \quad t \geq T_0.$$

That $(x(t), y(t))$ provides a growing solution of (B) which is nearly slowly varying is a consequence of the fact that $(x(t), y(t))$ is a member of W.

It remains to verify that $(x(t), y(t))$ is really a slowly varying vector function. For this purpose we note that

$$\begin{aligned} x(t) &= x_0 + \int_{T_0}^{t} p(s)y(s)^{\alpha}\mathrm{d}s \asymp X_0(t), \\ y(t) &= y_0 + \int_{T_0}^{t} q(s)x(s)^{\beta}\mathrm{d}s \asymp Y_0(t), \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} x'(t) &= p(t)y(t)^{\alpha} \asymp p_0(t)Y_0(t)^{\alpha}, \\ y'(t) &= q(t)x(t)^{\beta} \asymp q_0(t)X_0(t)^{\beta}, \end{aligned} \tag{4.27}$$

as $t \to \infty$. In order to make sure that $x(t)$ and $y(t)$ are slowly varying it suffices to show that

$$\lim_{t\to\infty} \frac{tx'(t)}{x(t)} = 0, \quad \lim_{t\to\infty} \frac{ty'(t)}{y(t)} = 0 \tag{4.28}$$

(see Proposition 2.1). It is convenient to rewrite $X_0(t)$ and $Y_0(t)$ as

$$\begin{aligned} X_0(t) &= \left[\lambda \int_a^t p_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}} \sim \left[\lambda \int_a^t q_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}, \quad t \to \infty, \\ Y_0(t) &= \left[\mu \int_a^t q_0(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}} \sim \left[\mu \int_a^t p_0(s)\mathrm{d}s\right]^{\frac{\beta+1}{1-\alpha\beta}}, \quad t \to \infty. \end{aligned}$$

Using (4.26), (4.27) and the above simplified expressions for $X_0(t)$ and $Y_0(t)$, we obtain

$$\frac{tx'(t)}{x(t)} \asymp \frac{tp_0(t)\left[\mu \int_a^t p_0(s)\mathrm{d}s\right]^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}}{\left[\lambda \int_a^t p_0(s)\mathrm{d}s\right]^{\frac{\alpha+1}{1-\alpha\beta}}} \asymp \frac{(\alpha+1)tp_0(t)}{(1-\alpha\beta)\int_a^t p_0(s)\mathrm{d}s}, \quad t \to \infty, \tag{4.29}$$

and similarly

$$\frac{ty'(t)}{y(t)} \asymp \frac{(\beta+1)tq_0(t)}{(1-\alpha\beta)\int_a^t q_0(s)\mathrm{d}s}, \quad t \to \infty, \tag{4.30}$$

Since $p_0(t)$ and $q_0(t)$ are regularly varying of index -1, from Karamata's integration theorem it follows that

$$\lim_{t\to\infty} \frac{tp_0(t)}{\int_a^t p_0(s)\mathrm{d}s} = \lim_{t\to\infty} \frac{tq_0(t)}{\int_a^t q_0(s)\mathrm{d}s} = 0$$

which combined with (4.29) and (4.30), ensures that (4.28) holds true as desired. Therefore, both $x(t)$ and $y(t)$ are slowly varying functions. This completes the proof. □

Corollary 4.1. *Let $p(t)$ and $q(t)$ be continuous regularly varying functions of index -1 such that $p(t) \sim q(t)$ as $t \to \infty$. Then, system* (B) *possesses growing slowly varying solutions if and only if* (4.1) *holds, in which case the asymptotic behavior of all such solutions $(x(t), y(t))$ is governed by the formulas* (4.2) *and* (4.3).

Example 4.1. Consider the system of differential equations

$$x' = p(t)y^{\alpha}, \quad y' = q(t)x^{\beta}, \tag{4.31}$$

where $p(t)$ and $q(t)$ are positive continuous functions on $[\mathrm{e}, \infty)$.

(i) Suppose that $p(t)$ and $q(t)$ are nearly regularly varying functions of index -1 such that $p(t) \asymp r(t)$ and $q(t) \asymp r(t)$ as $t \to \infty$, where $r(t) \in \mathrm{RV}(-1)$ satisfies

$$r(t) \sim (\alpha+1)^{-\frac{\alpha}{\alpha+1}}(\beta+1)^{-\frac{\beta}{\beta+1}}t^{-1}(\log t)^{-\frac{2\alpha\beta+\alpha+\beta}{(\alpha+1)(\beta+1)}}, \quad t \to \infty.$$

Then, system (4.31) possesses a nearly varying solution $(x(t), y(t))$ such that

$$x(t) \asymp (\beta+1)^{\frac{1}{\beta+1}}(\log t)^{\frac{1}{\beta+1}}, \quad y(t) \asymp (\alpha+1)^{\frac{1}{\alpha+1}}(\log t)^{\frac{1}{\alpha+1}}, \quad t \to \infty.$$

(ii) Suppose that $p(t)$ and $q(t)$ are regularly varying of index -1 satisfying

$$p(t) \sim q(t) \sim (\alpha+1)^{-\frac{\alpha}{\alpha+1}}(\beta+1)^{-\frac{\beta}{\beta+1}}t^{-1}(\log t)^{-\frac{2\alpha\beta+\alpha+\beta}{(\alpha+1)(\beta+1)}}, \quad t \to \infty.$$

Then, system (4.31) possesses slowly varying solutions $(x(t), y(t))$ and all of them enjoy the following precise asymptotic behavior at infinity:

$$x(t) \sim (\beta+1)^{\frac{1}{\beta+1}}(\log t)^{\frac{1}{\beta+1}}, \quad y(t) \sim (\alpha+1)^{\frac{1}{\alpha+1}}(\log t)^{\frac{1}{\alpha+1}}, \quad t \to \infty.$$

5. Application to generalized Thomas-Fermi equations

We conclude this paper with a remark that our results for the systems (A) and (B) can be applied to provide new results on strongly monotone positive solutions of the second order nonlinear differential equation of Thomas-Fermi type

$$(p(t)|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x, \tag{5.1}$$

where α and β are positive constants such that $\alpha > \beta$ and $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$.

A positive solution $x(t)$ of (5.1) is said to be *strongly decreasing* if

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} p(t)|x'(t)|^{\alpha-1}x'(t) = 0$$

and *strongly increasing* if

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} p(t)|x'(t)|^{\alpha-1}x'(t) = \infty.$$

It is easy to see that if $x(t)$ is a strongly decreasing solution of (5.1), then by putting

$$y(t) = -p(t)|x'(t)|^{\alpha-1}x'(t) = p(t)(-x'(t))^{\alpha},$$

equation (5.1) is converted into the system of first order equations

$$x' + p(t)^{-\frac{1}{\alpha}} y^{\frac{1}{\alpha}} = 0, \qquad y' + q(t)x^{\beta} = 0. \tag{5.2}$$

Likewise if $x(t)$ is a strongly increasing solution of (5.1), then by putting

$$y(t) = p(t)|x'(t)|^{\alpha-1}x'(t) = p(t)x'(t)^{\alpha},$$

equation (5.1) is converted into the system of first order equations

$$x' = p(t)^{-\frac{1}{\alpha}} y^{\frac{1}{\alpha}}, \qquad y' = q(t)x^{\beta}. \tag{5.3}$$

Conversely, if $(x(t), y(t))$ is a strongly decreasing solution of system (5.2) [resp. strongly increasing solution of system (5.3)], then $x(t)$ is a strongly decreasing [resp. strongly increasing] solution of equation (5.1).

We now assume that $p \in \mathrm{RV}(\alpha)$ and $q \in \mathrm{RV}(-1)$ and seek positive solutions $x(t)$ of (5.1) such that $x(t)$ and $p(t)|x'(t)|^{\alpha}$ are slowly varying. Then, by applying Corollary 3.1 and Corollary 4.1 to (5.2) and (5.3), respectively, we obtain the following proposition concerning the existence and asymptotic behavior of strongly monotone solutions of equation (5.1) in the framework of slowly varying functions.

Proposition 5.1. *Let $p \in \mathrm{RV}(\alpha)$ and $q \in RV(-1)$ and suppose that $p(t)^{-1/\alpha} \sim q(t)$ as $t \to \infty$.*

(i) *Equation* (5.1) *possesses strongly decreasing solutions which are slowly varying if and only if $p(t)^{-1/\alpha}$ and $q(t)$ are integrable on $[a, \infty)$ and in that case the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$x(t) \sim \left[\frac{(\alpha-\beta)^{\alpha+1}}{\alpha(\alpha+1)^{\alpha}(\beta+1)}\left(\int_t^{\infty} p(s)^{-\frac{1}{\alpha}}\mathrm{d}s\right)^{\alpha+1}\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{5.4}$$

(ii) *Equation* (5.1) *possesses strongly increasing solutions which are slowly varying if and only if $p(t)^{-1/\alpha}$ and $q(t)$ are non-integrable on $[a, \infty)$ and in that case the asymptotic behavior of any such solution $x(t)$ is governed by the unique formula*

$$x(t) \sim \left[\frac{(\alpha-\beta)^{\alpha+1}}{\alpha(\alpha+1)^{\alpha}(\beta+1)}\left(\int_a^{t} p(s)^{-\frac{1}{\alpha}}\mathrm{d}s\right)^{\alpha+1}\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{5.5}$$

We notice that the asymptotic analysis of equation (5.1) in the framework of regularly varying functions has just begun. See, for example, the paper [**5**] in which the special case with $p(t) \equiv 1$ of (5.1) was studied. Nothing seems to be known about regularly varying solutions of (5.1) with general positive $p(t)$, and our observation could be a clue to a comprehensive study of generalized Thomas-Fermi differential equations by means of regular variation.

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