

BLOW UP OF SOLUTIONS FOR A SYSTEM OF STRONGLY DAMPED PETROVSKY EQUATIONS WITH VARIABLE EXPONENTS

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ABSTRACT. In this work, we deal with the system of strongly damped Petrovsky equations with variable exponents. Under suitable conditions on variable exponents, we prove the blow up of solutions with negative initial energy.

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1. INTRODUCTION

1.1. Setting of the problem:

In this paper, we consider the following system of strongly damped Petrovsky equations with variable exponents

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m(x)-1} u_t = f_1(u, v), & \text{in } \Omega, t > 0, \\ v_{tt} + \Delta^2 v - \Delta v_t + |v_t|^{r(x)-1} v_t = f_2(u, v), & \text{in } \Omega, t > 0, \\ u(x, t) = v(x, t) = \partial_\nu u(x, t) = \partial_\nu v(x, t) = 0, & \text{on } \partial\Omega, t \geq 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & \text{in } \Omega, \\ u_t(x, 0) = u_1(x), v_t(x, 0) = v_1(x), & \end{cases} \quad (1)$$

where Ω be a bounded and regular domain of R^n , $n \geq 1$, with a smooth boundary $\partial\Omega$, and ν is the unit outer normal to $\partial\Omega$. The source terms $f_1(u, v)$ and $f_2(u, v)$ as follows

$$\begin{cases} f_1(u, v) = a|u+v|^{2(p(\cdot)+1)}(u+v) + b|u|^{p(\cdot)}u|v|^{p(\cdot)+2}, \\ f_2(u, v) = a|u+v|^{2(p(\cdot)+1)}(u+v) + b|u|^{p(\cdot)+2}|v|^{p(\cdot)}v, \end{cases} \quad (2)$$

with a, b positive constants; and $m(\cdot), r(\cdot), p(\cdot)$ are given continuous functions on $\bar{\Omega}$ satisfying some conditions to be specified later and the log-Hölder continuity condition given for any function $p(\cdot)$ by

$$|p(x) - p(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta, \quad (3)$$

where $0 < \delta < 1$ and $A > 0$. By the definition of $f_1(u, v)$ and $f_2(u, v)$, one can easily verify that

$$uf_1(u, v) + vf_2(u, v) = 2(p(x) + 2)F(u, v), \quad \forall (u, v) \in \mathbb{R}^2, \quad (4)$$

where

$$F(u, v) = \frac{1}{2(p(x) + 2)} \left[a|u+v|^{2(p(x)+2)} + 2b|uv|^{p(x)+2} \right].$$

The exponents $m(\cdot), r(\cdot)$ and $p(\cdot)$ are measurable functions on Ω satisfying

$$\begin{cases} 2 \leq m_1 \leq m(x) \leq m_2 \leq m^*, \\ 2 \leq r_1 \leq r(x) \leq r_2 \leq r^*, \\ 2 \leq p_1 \leq p(x) \leq p_2 \leq p^*, \end{cases}$$

where

$$\begin{cases} m_1 = \text{ess inf}_{x \in \Omega} m(x), \quad m_2 = \text{ess sup}_{x \in \Omega} m(x), \\ r_1 = \text{ess inf}_{x \in \Omega} r(x), \quad r_2 = \text{ess sup}_{x \in \Omega} r(x), \\ p_1 = \text{ess inf}_{x \in \Omega} p(x), \quad p_2 = \text{ess sup}_{x \in \Omega} p(x), \end{cases}$$

and

$$\begin{cases} 2 < m^*, r^*, p^* < \infty & \text{if } n \leq 4, \\ 2 < m^*, r^*, p^* < \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

1.2. Literature overview:

Antontsev et al. [4] considered the single strongly damped Petrovsky equation with variable exponents

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m(x)-1} u_t = |u|^{p(x)-1} u.$$

They proved the local existence and blow up of solutions.

In [13], Messaoudi and Talahmeh studied the following system of nonlinear wave equations with variable exponents

$$\begin{cases} u_{tt} + \Delta u + |u_t|^{m(x)-1} u_t = f_1(u, v), \\ v_{tt} + \Delta v + |v_t|^{r(x)-1} v_t = f_2(u, v). \end{cases}$$

They obtained the blow up of solutions with negative initial energy. Also, Messaoudi et al. [14] proved the existence and stability of solutions the same system.

Recently, problems with variable exponents have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability have been found ([1, 3, 5, 7, 10, 12, 15, 16, 17, 18, 19, 21, 23, 24, 25]). The problems with variable exponents arise in many branches of sciences such as image processing, electrorheological fluids and nonlinear elasticity theory [8, 9, 22].

In this paper, we prove the blow up of the solutions (1). To the best of our knowledge, there is no result the blow up of the fourth order system with variable exponents. The rest of our work is organized as follows: In section 2, we give some lemmas, definition and theorem. In section 3, we state and prove our main result.

2. PRELIMINARIES

In this part, we state some results about the variable exponents Lebesgue space and Sobolev space $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, (see [2, 6, 20]).

Let $q : \Omega \rightarrow [1, \infty]$ be a measurable function, where Ω is a domain of R^n . We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u : \Omega \rightarrow R; u \text{ measurable in } \Omega : \varrho_{q(\cdot)}(\lambda u) < \infty, \text{ for some } \lambda > 0\},$$

where

$$\varrho_{p(\cdot)}(u) = \int_{\Omega} \frac{1}{p(x)} |u(x)|^{p(x)} dx$$

is a modular. Equipped with the following Luxembourg-type norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space.

We also define the variable-exponent Sobolev space $W^{m,q(\cdot)}(\Omega)$ as

$$W^{m,q(\cdot)}(\Omega) = \left\{ u \in L^{q(\cdot)}(\Omega) : D^{\alpha} u \in L^{q(\cdot)}(\Omega), |\alpha| \leq m \right\}.$$

Lemma 1. (*Poincaré Inequality*) Let Ω be a bounded domain of R^n and $q(\cdot)$ satisfies (3), and $1 \leq q_1 \leq q(x) \leq q_2 < \infty$, where

$$q_1 = \operatorname{ess\,inf}_{x \in \Omega} q(x), \quad q_2 = \operatorname{ess\,sup}_{x \in \Omega} q(x).$$

Then

$$\|u\|_{q(\cdot)} \leq C \|\nabla u\|_{q(\cdot)}, \text{ for all } u \in W_0^{1,q(\cdot)}(\Omega),$$

where the positive constant C depending on q_1, q_2 and Ω only.

Lemma 2. *If $1 < q_1 \leq q(x) \leq q_2 < \infty$ holds, then*

$$\min \left\{ \|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2} \right\} \leq \varrho_{q(\cdot)}(w) \leq \max \left\{ \|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2} \right\},$$

for any $w \in L^{q(\cdot)}(\Omega)$.

Lemma 3. [13]. *There exist two constants c_0 and c_1 such that*

$$\frac{c_0}{2(p(x)+2)} \left[|u|^{2(p(x)+2)} + |v|^{2(p(x)+2)} \right] \leq F(u, v) \leq \frac{c_1}{2(p(x)+2)} \left[|u|^{2(p(x)+2)} + |v|^{2(p(x)+2)} \right].$$

Corollary 4. [13]. *There exist two constants c_0 and c_1 such that*

$$c_0 [\varrho(u) + \varrho(v)] \leq \int_{\Omega} F(u, v) dx \leq c_1 [\varrho(u) + \varrho(v)]. \quad (5)$$

Definition 1. (Weak solution). *A pair of functions (u, v) is said to be weak solution of (1) on $[0, T]$, $T > 0$, if*

$$\begin{aligned} (u, v) &\in L^{\infty}((0, T), H_0^2(\Omega)), \\ u_t &\in L^{\infty}((0, T), L^2(\Omega)) \cap L^{m(\cdot)+1}(\Omega \times (0, T)), \\ v_t &\in L^{\infty}((0, T), L^2(\Omega)) \cap L^{r(\cdot)+1}(\Omega \times (0, T)) \end{aligned}$$

with

$$u(\cdot, 0) = u_0, v(\cdot, 0) = v_0, u_t(\cdot, 0) = u_1, v_t(\cdot, 0) = v_1$$

and (u, v) satisfies

$$\begin{aligned} \int_{\Omega} u_t \phi - \int_{\Omega} u_1 \phi + \int_0^t \int_{\Omega} \Delta u \Delta \phi + \int_0^t \int_{\Omega} \nabla u_t \nabla \phi + \int_0^t \int_{\Omega} |u_t|^{m(\cdot)-1} u_t \phi &= \int_0^t \int_{\Omega} f_1 \phi, \\ \int_{\Omega} v_t \psi - \int_{\Omega} v_1 \psi + \int_0^t \int_{\Omega} \Delta v \Delta \psi + \int_0^t \int_{\Omega} \nabla v_t \nabla \psi + \int_0^t \int_{\Omega} |v_t|^{r(\cdot)-1} v_t \psi &= \int_0^t \int_{\Omega} f_2 \psi, \end{aligned}$$

for all $\phi \in H_0^2(\Omega) \cap L^{m(\cdot)+1}(\Omega)$, $\psi \in H_0^2(\Omega) \cap L^{r(\cdot)+1}(\Omega)$, and all $t \in [0, T]$.

We state the following theorem which can be obtained by exploiting the Faedo-Galerkin method and using the similar arguments as in [4, 11, 14].

Theorem 5. (Local existence). *Suppose that $p(\cdot)$, $m(\cdot)$, $r(\cdot) \in C(\overline{\Omega})$, satisfy (3) and, for all $x \in \Omega$,*

$$\begin{cases} p(x) \geq 3, & \text{if } n \leq 4, \\ p(x) = 3, & \text{if } n = 5, \end{cases} \quad (6)$$

$$\begin{cases} m(x) \geq 2, & \text{if } n \leq 4, \\ 2 \leq m(x) \leq 10, & \text{if } n = 5, \end{cases} \quad (7)$$

$$\begin{cases} r(x) \geq 2, & \text{if } n \leq 4, \\ 2 \leq r(x) \leq 10, & \text{if } n = 5, \end{cases} \quad (8)$$

and $(u_0, u_1), (v_0, v_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then (1) has a unique weak local solution

$$(u, v) \in L^\infty((0, T), H_0^2(\Omega)),$$

$$u_t \in L^\infty((0, T), L^2(\Omega)) \cap L^{m(\cdot)+1}(\Omega \times (0, T)),$$

$$v_t \in L^\infty((0, T), L^2(\Omega)) \cap L^{r(\cdot)+1}(\Omega \times (0, T)),$$

for $T > 0$.

3. BLOW UP RESULT

In this part we state and prove our main blow-up result. For this purpose, we define energy functional of (1) as

$$E(t) = \frac{1}{2} [\|u_t\|^2 + \|v_t\|^2] + \frac{1}{2} [\|\Delta u\|^2 + \|\Delta v\|^2] - \int_{\Omega} F(u, v) dx \quad (9)$$

Lemma 6. *$E(t)$ energy functional is nonincreasing function.*

Proof. Multiplying the first equation of (1) by u_t and the second equation by v_t , integrating over Ω , using integration by parts and summing up the product results, we get

$$E'(t) = -\|\nabla u_t\|^2 - \|\nabla v_t\|^2 - \int_{\Omega} |u_t|^{m(x)+1} dx - \int_{\Omega} |v_t|^{r(x)+1} dx \leq 0. \quad (10)$$

Lemma 7. [13]. *Assume that (6) holds. Then, we have the following inequalities:*

$$[\varrho(u) + \varrho(v)]^{\frac{s}{2(p_1+2)}} \leq C [\|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v)], \quad (11)$$

$$\|u\|_{2(p_1+2)}^s \leq C [\|\Delta u\|^2 + \|\Delta v\|^2 + \|u_t\|_{2(p_1+2)}^{2(p_1+2)} + \|v_t\|_{2(p_1+2)}^{2(p_1+2)}] \quad (12)$$

$$\|v\|_{2(p_1+2)}^s \leq C [\|\Delta u\|^2 + \|\Delta v\|^2 + \|u_t\|_{2(p_1+2)}^{2(p_1+2)} + \|v_t\|_{2(p_1+2)}^{2(p_1+2)}] \quad (13)$$

$$\int_{\Omega} |u|^{m(x)+1} dx \leq c_1 \left[(\varrho(u) + \varrho(v))^{\frac{m_1+1}{2(p_1+2)}} + (\varrho(u) + \varrho(v))^{\frac{m_2+1}{2(p_1+2)}} \right], \quad (14)$$

$$\int_{\Omega} |u|^{r(x)+1} dx \leq c_2 \left[(\varrho(u) + \varrho(v))^{\frac{r_1+1}{2(p_1+2)}} + (\varrho(u) + \varrho(v))^{\frac{r_2+1}{2(p_1+2)}} \right], \quad (15)$$

for any $u, v \in H_0^2(\Omega)$ and $2 \leq s \leq 2(p_1 + 2)$. Where $C > 1$, $c_1 > 0, c_2 > 0$ are constants and

$$H(t) = -E(t).$$

Theorem 8. Assume that (3), (6), (7) and (8) hold. Assume further that

$$2(p_1 + 1) \geq \max\{m_2 + 1, r_2 + 1\} \quad (16)$$

and

$$E(0) < 0.$$

Then the solution of problem (1) blows up in finite time.

Proof. Then $E(0) < 0$ and (10) gives $H(t) \geq H(0) > 0$. By the definition $H(t)$ and (5), we get

$$\begin{aligned} H(t) &= -\frac{1}{2} [\|u_t\|^2 + \|v_t\|^2] - \frac{1}{2} [\|\Delta u\|^2 + \|\Delta v\|^2] + \int_{\Omega} F(u, v) dx \\ &\leq \int_{\Omega} F(u, v) dx \\ &\leq c_1 [\varrho(u) + \varrho(v)]. \end{aligned} \quad (17)$$

We define

$$\Psi(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} (uu_t + vv_t) dx + \frac{\varepsilon}{2} [\|\nabla u\|^2 + \|\nabla v\|^2] \quad (18)$$

for ε small to be chosen later and

$$0 < \alpha \leq \min \left\{ \frac{p_1 + 2}{2(p_1 + 2)}, \frac{2(p_1 + 2) - (m_2 + 1)}{2m_2(p_1 + 2)}, \frac{2(p_1 + 2) - (r_2 + 1)}{2r_2(p_1 + 2)} \right\}. \quad (19)$$

Differentiating $\Psi(t)$ with respect to t , and using (1) and (4), we have

$$\begin{aligned} \Psi'(t) &= (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ &\quad - \varepsilon \left(\|\Delta u\|^2 + \|\Delta v\|^2 \right) \\ &\quad - \varepsilon \int_{\Omega} u |u_t|^{m(x)-1} u_t dx - \varepsilon \int_{\Omega} v |v_t|^{r(x)-1} v_t dx \\ &\quad + 2\varepsilon \int_{\Omega} (p(x) + 2) F(u, v) dx. \end{aligned} \quad (20)$$

By using the definition of the $H(t)$, it follows that

$$\varepsilon\eta H(t) = -\frac{\varepsilon\eta}{2} \left[\|u_t\|^2 + \|v_t\|^2 \right] - \frac{\varepsilon\eta}{2} \left[\|\Delta u\|^2 + \|\Delta v\|^2 \right] + \varepsilon\eta \int_{\Omega} F(u, v) dx,$$

where $2 < \eta < 2(p_1 + 2)$. Adding and subtracting $\varepsilon\eta H(t)$ from the right-hand side of (20), we obtain

$$\begin{aligned} \Psi'(t) &\geq (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{\eta}{2} + 1 \right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ &\quad + \varepsilon \left(\frac{\eta}{2} - 1 \right) \left(\|\Delta u\|^2 + \|\Delta v\|^2 \right) + \varepsilon\eta H(t) \\ &\quad + 2(p_1 + 2)\varepsilon \int_{\Omega} F(u, v) dx \\ &\quad - \varepsilon \int_{\Omega} \left[u |u_t|^{m(x)-1} u_t + v |v_t|^{r(x)-1} v_t \right] dx. \end{aligned} \quad (21)$$

By using (5), we have

$$\begin{aligned} \Psi'(t) &\geq (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon\beta \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 \right. \\ &\quad \left. + \|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right] \\ &\quad - \varepsilon \int_{\Omega} \left[u |u_t|^{m(x)-1} u_t + v |v_t|^{r(x)-1} v_t \right] dx \end{aligned} \quad (22)$$

where

$$\beta = \min \left\{ \eta, \left(\frac{\eta}{2} - 1 \right), \left(\frac{\eta}{2} + 1 \right), 2c_0(p_1 + 2) + \frac{\eta}{2} \right\} > 0.$$

To estimate the last term in (22), we use the Young inequality, we have

$$\int_{\Omega} |u_t|^{m(x)} |u| dx \leq \frac{1}{m_1 + 1} \int_{\Omega} \delta_1^{m(x)+1} |u|^{m(x)+1} dx + \frac{m_2}{m_1 + 1} \int_{\Omega} \delta_1^{-\frac{m(x)+1}{m(x)}} |u_t|^{m(x)+1} dx. \quad (23)$$

Similarly, we have

$$\int_{\Omega} |v_t|^{r(x)} |v| dx \leq \frac{1}{r_1 + 1} \int_{\Omega} \delta_2^{r(x)+1} |v|^{r(x)+1} dx + \frac{r_2}{r_1 + 1} \int_{\Omega} \delta_2^{-\frac{r(x)+1}{r(x)}} |v_t|^{r(x)+1} dx, \quad (24)$$

where $\delta_1, \delta_2 > 0$ are constants depending on the time t and specified later. Let us choose δ_1 and δ_2 so that

$$\delta_1^{-\frac{m(x)+1}{m(x)}} = k_1 H^{-\alpha}(t) \quad \text{and} \quad \delta_2^{-\frac{r(x)+1}{r(x)}} = k_2 H^{-\alpha}(t),$$

for a large constant k_1 and k_2 to be specified later, and substituting in (23) and (24), respectively, we get

$$\int_{\Omega} |u_t|^{m(x)} |u| dx \leq \frac{k_1^{-m_1}}{m_1 + 1} \int_{\Omega} |u|^{m(x)+1} H^{\alpha m(x)}(t) dx + \frac{m_2 k_1}{m_1 + 1} H^{-\alpha}(t) H'(t), \quad (25)$$

and

$$\int_{\Omega} |v_t|^{r(x)} |v| dx \leq \frac{k_2^{-r_1}}{r_1 + 1} \int_{\Omega} |v|^{r(x)+1} H^{\alpha r(x)}(t) dx + \frac{r_2 k_1}{r_1 + 1} H^{-\alpha}(t) H'(t). \quad (26)$$

Combining (22), (25) and (26) gives

$$\begin{aligned} \Psi'(t) &\geq \left[(1 - \alpha) - \varepsilon \frac{m_2 k_1}{m_1 + 1} - \varepsilon \frac{r_2 k_1}{r_1 + 1} \right] H^{-\alpha}(t) H'(t) \\ &\quad + \varepsilon \beta \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right] \\ &\quad - \frac{\varepsilon k_1^{-m_1}}{m_1 + 1} \int_{\Omega} |u|^{m(x)+1} H^{\alpha m(x)}(t) dx - \frac{\varepsilon k_2^{-r_1}}{r_1 + 1} \int_{\Omega} |v|^{r(x)+1} H^{\alpha r(x)}(t) dx. \end{aligned} \quad (27)$$

From (14), (15) and (17), we have

$$\int_{\Omega} |u|^{m(x)+1} H^{\alpha m(x)}(t) dx \leq C' \left[(\varrho(u) + \varrho(v))^{\frac{m_1+1}{2(p_1+2)} + \alpha m_2} + (\varrho(u) + \varrho(v))^{\frac{m_2+1}{2(p_2+2)} + \alpha m_2} \right]. \quad (28)$$

We then use Lemma 7, for

$$s = (m_2 + 1) + 2\alpha m_2 (p_1 + 2) \leq 2(p_1 + 2),$$

and

$$s = (m_1 + 1) + 2\alpha m_2 (p_1 + 2) \leq 2(p_1 + 2),$$

to do deduce, from (28), that

$$\int_{\Omega} |u|^{m(x)+1} H^{\alpha m(x)}(t) dx \leq C \left[\|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right]. \quad (29)$$

Similarly

$$\int_{\Omega} |v|^{r(x)+1} H^{\alpha r(x)}(t) dx \leq C \left[\|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right]. \quad (30)$$

Combining (29), (30) and (27), give

$$\begin{aligned} \Psi'(t) &\geq \left[(1 - \alpha) - \varepsilon \frac{m_2 k_1}{m_1 + 1} - \varepsilon \frac{r_2 k_2}{r_1 + 1} \right] H^{-\alpha}(t) H'(t) \\ &\quad + \varepsilon \left(\beta - \frac{k_1^{-m_1}}{m_1 + 1} C - \frac{k_2^{-r_2}}{r_2 + 1} C \right) \\ &\quad \times \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right]. \end{aligned} \quad (31)$$

Let us choose k_1, k_2 large enough so that

$$\gamma = \beta - \frac{k_1^{-m_1}}{m_1 + 1}C - \frac{k_2^{-r_2}}{r_2 + 1}C > 0,$$

and picking ε small enough such that

$$(1 - \alpha) - \varepsilon \frac{m_2 k_1}{m_1 + 1} - \varepsilon \frac{r_2 k_2}{r_1 + 1} \geq 0,$$

and

$$\Psi(0) = H^{1-\alpha}(0) + \varepsilon \int_{\Omega} (u_0 u_1 + v_0 v_1) dx + \frac{\varepsilon}{2} [\|\nabla u_0\|^2 + \|\nabla v_0\|^2] > 0.$$

Hence (31) takes the form

$$\begin{aligned} \Psi'(t) &\geq \gamma \varepsilon \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right] \\ &\geq \gamma \varepsilon \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|^2 + \|\Delta v\|^2 + \|u\|_{2(p_1+2)}^{2(p_1+2)} + \|v\|_{2(p_1+2)}^{2(p_1+2)} \right] \end{aligned} \quad (32)$$

Consequently, we get

$$\Psi(t) \geq \Psi(0) > 0, \text{ for all } t \geq 0.$$

On the other hand, thanks to the Hölder inequality and the embedding $L^{2(p_1+2)}(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned} \left| \int_{\Omega} uu_t dx \right| &\leq \|u\|_2 \|u_t\|_2 \\ &\leq C \|u\|_{2(p_1+2)} \|u_t\|_2, \end{aligned}$$

which implies

$$\left| \int_{\Omega} uu_t dx \right|^{\frac{1}{1-\alpha}} \leq C \|u\|_{2(p_1+2)}^{\frac{1}{1-\alpha}} \|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Similarly

$$\left| \int_{\Omega} vv_t dx \right|^{\frac{1}{1-\alpha}} \leq C \|v\|_{2(p_1+2)}^{\frac{1}{1-\alpha}} \|v_t\|_2^{\frac{1}{1-\alpha}}.$$

Young's inequality gives

$$\begin{aligned} &\left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{1-\alpha}} \\ &\leq C \left[\|u\|_{2(p_1+2)}^{\frac{\mu}{1-\alpha}} \|u_t\|_2^{\frac{\theta}{1-\alpha}} + \|v\|_{2(p_1+2)}^{\frac{\mu}{1-\alpha}} \|v_t\|_2^{\frac{\theta}{1-\alpha}} \right], \end{aligned} \quad (33)$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1 - \alpha)$, to get $\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(p_1 + 2)$, by (28).
Therefore (33), becomes

$$\begin{aligned} & \left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{1-\alpha}} \\ & \leq C \left[\|u\|_{2(p_1+2)}^s + \|v\|_{2(p_1+2)}^{\frac{\mu}{1-\alpha}} + \|u_t\|_2^2 + \|v_t\|_2^2 \right], \end{aligned}$$

where

$$s = \frac{2}{1-2\alpha} \leq 2(p_1 + 2).$$

By using (12) and (13), we have

$$\begin{aligned} & \left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{1-\alpha}} \\ & \leq C \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right], \quad (34) \end{aligned}$$

for all $t \geq 0$. Thus,

$$\begin{aligned} \Psi^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t + vv_t dx \right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}} \left[H(t) + \left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{1-\alpha}} \right] \\ &\leq C \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|^2 + \|\Delta v\|^2 + \varrho(u) + \varrho(v) \right] \quad (35) \end{aligned}$$

where $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ is used. By combining of (32) and (35), we arrive

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \quad (36)$$

where ξ is a positive constant. A simple integration of (36) over $(0, t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}},$$

which implies that the solution blows up in a finite time T^* , with

$$T^* \leq \frac{1 - \sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

This completes the proof of the theorem.

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