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A SE-REPRESENTATION OF A Γ-SEMIGROUP WITH APARTNESS BY Γ-SE-TRANSFORMATIONS OF A SET WITH APARTNESS

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Abstract. The notion of the S-acts and its properties as well as the concept of the S_Γ-acts over a Γ-semigroup have been the subject of study for many years in classical algebra. From 2019, the focus of this author's interest is on Γ-semigroups with apartness into the Bishop's constructive framework. In this paper, the concept of the S_Γ-act with apartness over Γ-semigroup with apartness is introduced and study some important properties of such acts like co-subact and co-congruence on it. In addition to the previous, the paper describes some properties of S_Γ-act with apartness in terms of a se-representation of a Γ-semigroup with apartness by the family of se-transformations of a set with apartness.

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1. Introduction

The classical theory of Γ-semigroups has been around for more than three decades and counts more than a few hundred of research's papers and many defended PhD theses (for example [4, 11, 12, 13, 31, 32, 33]). Along with Γ-semigroups, other structures such as ordered Γ-semigroups ([4, 11, 13]) and free Γ-semigroups ([34]) have been studied in recent years.

Semigroups with appartness, within the Bishop's constructive framework (in terms of the following books [5, 6, 14, 15, 35]), have been the subject of study for more than two decades (see, for example [7, 8, 9, 10, 20, 21, 22]).

The material presented in this article is an extension of this author's papers [23, 25, 26, 27, 28] about Γ-semigroups with apartness into the Bishop's constructive algebra (for example, [14, 15, 24] and the Chapter 'Algebra' in the book [35]). The paper [24] gives an overview of some algebraic structures with apartness such as semigroups with apartness, groups with apartness and commutative rings with apartness. In [23], the concept of Γ-semigroup with apartness was introduced and some of its important features were described. In articles [27, 28], the concepts of co-filters and co-ideals are introduced and analyzed in a Γ-semigroup with apartness ordered under a co-order relation. In the papers [26, 29], the author discusses constructive versions of the first and third theorems on the embedding se-isomorphisms between (co-ordered) Γ-semigroups with apartness. Papir [30] deals with a construction of free Γ-semigroup with apartness.

In this paper, the concept of the S_Γ-act with apartness over Γ-semigroup with apartness is introduced and study some important properties of such acts. In addition to the previous, the paper describes some properties of S_Γ-act with apartness in terms of a se-representation of a Γ-semigroup with apartness by the family of Γ-transformations of a set with apartness.

The article is designed as follows: The Preliminaries section, which follows the Introduction section, has two subsections. In the first subsection, a reader is allowed to be familiar with specific concepts and procedures in Bishop's constructive algebra. The second subsection gives the notion of Γ-semigroups with apartness (Definition 1) as well as some other terms necessary to understand the material presented in Section 3. In this subsection, the structure $F_{se}(\Gamma, F_{se}(A, A))$ is designed (Example 3), which is shown to be a Γ-semigroup with apartness. This semigroup is an important subject in the Section 3. Section 3 is the central part of this report and it contains five subsections. It designs the concepts of S_Γ-acts (Definition 3), S_Γ-subact (Definition 4) and S_Γ-cosubacts (Definition 5). Finally, a se-representation of Γ-semigroup with apartness (Subsection 3.4) is described (Theorem 3 and Theorem 4) using the previously mentioned structure $F_{se}(\Gamma, F_{se}(A, A))$. In Subsection 3.5 is discussed about the concept of co-congruences on S_Γ-act with apartness. Two S_Γ-acts with apartness $M/(q^\lhd, q)$ and $[M : q]$, from which the second one don't have its counterpart in the classical theory, can be designed for a given co-congruence q on S_Γ-act M.

2. Preliminaries

2.1. Constructive environment

We look at the carrier S of algebraic systems as a relational system $(S, =, \neq)$, where $' = '$ is the standard equality, and $' \neq '$ is an apartness:

$(\forall x, y \in S)(x \neq y \implies \neg(x = y))$ (consistency);
$(\forall x, y \in S)(x \neq y \implies y \neq x)$ (symmetry);
$(\forall x, y, z \in S)(x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity);

This relation is extensive with respect to the equality in the standard way

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq$$

where $' \circ '$ is the standard composition of relations. A relation that is consistent and symmetric is recognized as a difference relation. The apartness is tight with the equality if holds

$$(\forall x, y \in S)(\neg(x \neq y) \implies x = y).$$

Let us note that the apartness does not have to be tight with the equality, in the general case.

The set $(S, =, \neq)$ is said to be discrete if

$$(\forall x, y \in S)(x = y \vee x \neq y)$$

holds. Such are, for example, the sets $\mathbb{N}$ (the semi-ring of natural numbers) and $\mathbb{Z}$ (the ring of integers) where tight apartness is determined by $x \neq y \iff \neg(x = y)$.

For example (see [5, 6, 15]), in the field of real numbers $\mathbb{R}$, the apartness relation is present, determined as follows

$$a \neq b \iff (\exists k \in \mathbb{N})(|a - b| > \frac{1}{k}).$$

For the element x and a subset A of set S we write $x \lhd A$ if and only if $(\forall a \in A)(x \neq a)$. A subset B of the set S is said to be detachable if the following

$$(\forall x \in S)(x \in B \vee x \lhd B)$$

holds. If A and B are subsets of a set with apartness S, then the relation $=$ and $\neq$, defined by

$A = B \iff A \subseteq B \wedge B \subseteq A$ and

$A \neq B \iff ((\exists a \in A)(a \lhd B) \vee (\exists b \in B)(b \lhd A))$,

are compatible an equality and a diversity between them in the class $\mathcal{P}(S)$. It should be noted here that the relation $\neq$ is not an apartness relation on $\mathcal{P}(S)$ but only a difference.

In the product $S \times T$ of sets $(S, =_S, \neq_S)$ and $(T, =_T, \neq_T)$ an apartness is determined as follows

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) \neq (y, v) \iff (x \neq_S y \vee u \neq_T v)).$$

An apartness on the product $S_1 \times ... \times S_n$ can be determined by induction:

$$(\forall (x_i)_i, (y_i)_i \in S_1 \times ... \times S_n)((x_i)_i \neq (y_i)_i \iff (\exists j \in \{1, ..., n\})(x_i \neq_i y_i)).$$

In addition, any relation R on S, any functions f between such sets and any operation w in S appearing in this article are strongly extensional relative to the apartness (see, for example: [24]). For example:

- If $f : (S, =_S, \neq_S) \longrightarrow (T, =_T, \neq_T)$ is a function from S into T, then it is strongly extensional (se-function / se-mapping, for short) if holds

$$(\forall x, y \in S)(f(x) \neq_T f(y) \implies x \neq_S y).$$

If f and g are two functions from S into T, it is assumed that the equality and the difference between them are determined as follows:

$$f = g \iff (D := Dom(f) = Dom(g) \wedge (\forall x \in D)(f(x) =_T g(x))), \text{ and}$$

$$f \neq f \iff (Dom(f) \neq Dom(g) \vee (\exists\, x \in Dom(f) \cap Dom(g))(f(x) \neq_T g(x))).$$

In light of the above, the family of $F_{se}(A, B)$ of all se-mappings from A into B can be viewed as a set with apartness. In addition, the set $F_{se}(A, A)$ is a (noncommutative) monoid with apartness under standard composition of mappings as the internal operation in it and with the identity Id_A as a neutral element.

- The function $f : S \longrightarrow T$ is said to be an embedding if the following holds

$$(\forall x, y \in S)(x \neq_S y \implies f(x) \neq_T f(y)).$$

- If $w : S \times S \longrightarrow S$ is a total se-function, then the system $((S, =_S, \neq_S), w)$ is a semigroup with apartness if holds

$$(\forall x, y, z \in S)(w(x, w(y, z)) =_S w(w(x, y), z)).$$

Let us point out that the following applies

$$(\forall x, y, u, v \in S)(w(x, u) \neq_S w(y, v) \implies (x \neq_S y \vee u \neq_S v)).$$

In what follows, we will write $x \cdot y$, or xy for short, instead of $w(x, y)$. Thus, in the semigroup with apartness $(S, \cdot) := ((S, =_S, \neq_S), \cdot)$ except

$$(\forall x, y, u, v \in S)((x =_S y \wedge u =_S v) \implies xu =_S yv)$$

is also valid

$$(\forall x, y, u, v \in S)(xu \neq_S uv \implies (x \neq_S y \vee u \neq_S v)).$$

Of course, in this text we use standard classical mapping terms such as are, for example, injective, surjective, and bijective mapping.

The choice of an intuitionistic logical environment it allows identification and determination of two connected lines of classes of substructures in a semigroup with apartness $(S, =_S, \neq_S, \cdot)$:

The first line of classes is the recognizable standard substructures of subsemigroup, left, right and two-sided ideal in a semigroup. The second family of substructures in a semigroup with apartness S is introduced as follows:

(i) A subset T of S is a co-subsemigroup in S if holds

$$(\forall x, y \in S)(xy \in T \implies x \in T \vee y \in T);$$

(ii) A subset K of S is a left co-ideal in S if holds

$$(\forall x, y \in S)(xy \in K \implies x \in K);$$

(iii) A subset K of S is a right co-ideal in S if holds

$$(\forall x, y \in S)(xy \in K \implies y \in K);$$

(iv) A subset K of S is a co-ideal in S if holds

$$(\forall x, y \in S)(xy \in K \implies x \in K \wedge y \in K).$$

It can be shown without much difficulty that, for example, if K is a co-ideal in S, then the set $K^{\lhd} := \{x \in S : x \lhd K\}$ is an ideal in S but the vice versa does not have to hold.

Also, for the concepts of equivalences and congruences on a semigroup with apartness S, their constructive duals are determined as follows:

(v) A relation $q \subseteq S \times S$ is a co-equivalence on S if holds:
$(\forall x, y \in S)((x, y) \in q \implies x \neq_S)$ (consistency),
$(\forall x, y \in S)((x, y) \in q \implies (y, x) \in q)$ (symmetry),
$(\forall x, y, z \in S)((x, z) \in q \implies ((x, y) \in q \vee (y, z) \in q))$ (co-transitivity).

(vi) A co-equality relation q on a semigroup with apartness S is a co-congruence on S if the following holds

$$(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v) \in q)).$$

It can be proved ([24], Theorem 2.3) that if q is a co-congruence on S, then the relation $q^{\lhd} := \{(x, y) \in S \times S : (x, y) \lhd q\}$ is a congruence on S but the opposite does not have to be true. The presence of this relation on a semigroup with apartness S allows ([24], Theorem 2.4 and Theorem 2.5) to design the semigroups $S/(q^{\lhd}, q) := \{aq^{\lhd} : a \in S\}$ and $[S : q] := \{aq : a \in S\}$. In doing so, compatible equality and apartness in them are determined as follows:

$$(\forall x, y \in S)((xq^{\lhd} =_1 yq^{\lhd} \iff (x, y) \lhd q) \text{ and } (xq^{\lhd} \neq_1 yq^{\lhd} \iff (x, y) \in q)),$$

$$(\forall x, y \in S)((xq =_2 yq \iff (x, y) \lhd q) \text{ and } (xq \neq_2 yq \iff (x, y) \in q)).$$

2.2. Γ-semigroup with apartness

Definition 1 ([23], Definition 2.1). *Let $(S, =, \neq)$ and $(\Gamma, =, \neq)$ be two non-empty sets with apartness. S is called a Γ-semigroup with apartness if there exist a strongly extensional total mapping*

$$w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto w_S(x, a, y) := xay \in S$$

satisfying the condition

$$w_S(w_S(x, a, y), b, z) =: (xay)bz =_S xa(ybz) := w_S(x, a, w_S(y, b, z))$$

for any $x, y, z \in S$ and $a, b \in \Gamma$.

We recognize immediately that the following implications

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq_S ubv \implies (x \neq_S u \vee a \neq_\Gamma b \vee y \neq_S v)),$$

and

$$(\forall x, y \in S)(\forall a, b \in \Gamma)(xay \neq_S xby \implies a \neq_\Gamma b)$$

are valid, because w_S is a strongly extensional function.

We will write (S, Γ, w_S) if S is a Γ-semigroup with apartness with the operation $w_S : S \times \Gamma \times S \longrightarrow S$.

In this subsection, the determination of the concept of Γ-homomorphism between Γ-semigroups with apartness is taken from article [23].

Definition 2 ([23], Definition 2.7). *Let (S, Γ, w_S) is a Γ-semigroup and (T, Λ, w_T) a Λ-semigroups with apartness. A pair (h, θ) of strongly extensional functions $h : S \longrightarrow T$ and $\theta : \Gamma \longrightarrow \Lambda$ is called a se-homomorphism from Γ-semigroup S to Λ-semigroup T if the following holds*

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \theta)(xay) :=_T h(x)\theta(a)h(y)).$$

Particularly, for $\Gamma = \Lambda$, we have

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \iota_\Gamma)(xay) :=_T h(x)\iota_\Gamma(a)h(y)) =_T h(x)ah(y)$$

where $\iota_\Gamma : \Gamma \longrightarrow \Gamma$ is the identical mapping on Γ. So, the following

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \theta)(w_S(xay)) =_T w_T(h(x), \theta(a), h(y)))$$

must hold. In particular, for $\Gamma = \Lambda$ must hold

$$(h, \iota_\Gamma) \circ w_S = w_T \circ (h, \iota_\Gamma, h).$$

Example 1. *Let $S := \mathbb{Q}^-$ be a set of all negative rational numbers. It is clear that $\mathbb{Q}^-$ is not $\mathbb{Q}$-semigroup under usual product of rational numbers. Let $\Gamma := \{-1\}$. If for $x, y \in \mathbb{Q}^-$ and $\alpha \in \Gamma$, the product $x\alpha y$ is a common product of rational numbers, then $\mathbb{Q}^-$ is a Γ-semigroup with apartness.*

Example 2. *Let $S = \{i, 0, -i\} \subseteq \mathbb{C}$ and $\Gamma = S$. Then S is a Γ-semigroup with apartness under the multiplication in the field $\mathbb{C}$ of complex numbers, and in doing so, S is not a semigroup with respect to multiplication.*

In what follows, we will use the following writing method: If α and β are relations on the set S, we will write

$$\alpha \circ \beta := \{(a, c) \in S \times S : (\exists b \in S)((a, b) \in \alpha \wedge (b, c) \in \beta)\}.$$

So, we will write $(\alpha \circ \beta)(a) = \beta(\alpha(a))$.

Example 3. *Let $(A, =_A, \neq_A)$ and $(\Gamma, =_\Gamma, \neq_\Gamma)$ be inhabited sets with apartness. Then the set $S := F_{se}(\Gamma, F_{se}(A, A))$ is a Γ-semigroup with apartness under the operation $w_S : S \times \Gamma \times S \longrightarrow S$ defined by*

$$(\forall f, g \in F_{se}(\Gamma, F_{se}(A, A)))(\forall \alpha, \in \Gamma)((\forall \beta \in \Gamma)(w_S(f\alpha g)(\beta) := g(\beta) \circ f(\alpha))).$$

Let us show, first, that w_S is well-defined. Let $f, f', g, g' \in S$ and $\alpha, \alpha' \in \Gamma$ such that $f =_S f'$, $g =_S g'$ and $\alpha =_\Gamma \alpha'$. Then, for every $\beta \in \Gamma$ and every $a \in A$ the following is valid

$$w_S(f, \alpha, g)(\beta)(a) =_A (g(\beta) \circ (\alpha))(a) =_A (g'(\beta') \circ (\alpha'))(a) =_A w_S(f', \alpha', g')(\beta)(a).$$

Thus, $w_S(f, \alpha, g) =_S w_S(f', \alpha', g')$, which shows that the operation w_S is correct defined.

Second, we will show that w_S is a se-mapping. Suppose that

$$w_S(f, \alpha, g) \neq_S w_S(f', \alpha', g').$$

This means that for some $\beta \in \Gamma$ and some $a \in A$ the following is valid

$$w_S(f, \alpha, g)(\beta)(a) \neq_A w_S(f', \alpha', g')(\beta')(a)$$

that is, to be valid

$$(g(\beta) \circ f(\alpha))(a) \neq_A (g'(\beta) \circ f(\alpha'))(a).$$

Then

$$g(\beta)((f(\alpha)(a)) \neq_A g'(\beta)((f'(\alpha')(a)).$$

Thus

$$g(\beta)((f(\alpha)(a)) \neq_A g(\beta)((f'(\alpha')(a)) \vee g(\beta)((f'(\alpha')(a)) \neq_A g'(\beta)((f'(\alpha')(a)).$$

From the second case we immediately conclude that $g(\beta) \neq g'(\beta)$*, and, therefore, hence* $g \neq_S g'$*. Let the first case* $g(\beta)((f(\alpha)(a)) \neq_A g(\beta)((f'(\alpha')(a))$ *be satisfied. Then,* $f(\alpha)(a) \neq_A f'(\alpha')(a)$ *since* $g(\beta)$ *is a se-mapping. Thus*

$$f(\alpha)(a) \neq_A f(\alpha')(a) \vee f(\alpha')(a) \neq_A f'(\alpha')(a)$$

by co-transitivity of the relation $\neq_A$*. We have:*

- $f(\alpha)(a) \neq_A f(\alpha')(a) \implies f(\alpha) \neq f(\alpha')$
  $\implies \alpha \neq_\Gamma \alpha'$ *since* f *us a se-mapping;*
- $f(\alpha')(a) \neq_A f'(\alpha')(a) \implies f(\alpha') \neq f'(\alpha')$
  $\implies f \neq_S f'$.

This proves the validity of the implication

$$w_S(f, \alpha, g) \neq_S w_S(f', \alpha', g') \implies (f \neq_S f' \vee \alpha \neq_\Gamma \alpha' \vee g \neq_S g').$$

It remains to prove that w_S *satisfies the condition of the Definition 1. Let* $f, g, h \in S$ *and* $\alpha, \beta \in \Gamma$ *arbitrary elements. For any* $\gamma \in \Gamma$*, we have*

$$\begin{aligned}((f\alpha g)\beta h)(\gamma) &= h(\gamma) \circ (f\alpha g)(\gamma) = h(\gamma) \circ (g(\beta) \circ f(\alpha)) = (h(\gamma) \circ g(\beta)) \circ f(\alpha) \\ &= (g\beta h)(\gamma) \circ f(\alpha) = (f\alpha(g\beta h))(\gamma).\end{aligned}$$

Thus, it is shown that w_S *is a well-defined se-function on* $S \times \Gamma \times S$*. Therefore, the structure* (S, Γ, w_S) *is a* Γ*-semigroup.*

An inhabited subset J of a Γ-semigroup with apartness (S, Γ, w_S) is called right ideal of S if $J\Gamma S \subseteq J$. An inhabited subset J of a Γ-semigroup (S, Γ, w_S) is called a left ideal of S if $S\Gamma J \subseteq J$. A subset J is called ideal of S if it is both a left and a right ideal of S. Their constructive duals were introduced ([23], Definition 2.3, Definition 2.4,) as follows:

(vii) A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a right Γ-coideal of S if the following implication holds

$$(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in B \implies y \in B);$$

(viii) A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a left Γ-coideal of S if the following implication is valid

$$(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in B \implies x \in B);$$

(ix) A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a (two side) Γ-coideal of S if the following implication is valid

$$(\forall x, y \in S)(\forall \alpha \in \Gamma)(x\alpha y \in B \implies (x \in B \wedge y \in B)).$$

In [23], Propositions 2.4, 2.5 and 2.6, it is shown that if B is a left (right, two-sided) co-ideal in a Γ-semigroup with apartness S, then the set $B^{\lhd}$ is a left (right, two-sided, res.) ideal in S.

3. The Main Results

3.1. Γ-act over Γ-semigroup within the classical framework

The concept of S-act has been introduced as follows (see, for example, [1, 2, 3]): if S is a semigroup, a nonempty set M is called a left S-act if there is a total mapping λ_S from $S \times M$ into M such that the following holds

$$(\forall x, y \in S)(\forall a \in M)(\lambda_S(x, \lambda_S(y, a)) = \lambda_S((xy), a)).$$

Every semigroup can be consider as act over itself. By a similar way we define right S-act. The S-act theory is a generalization of R-module theory (where R is a ring). In this subsection we recall (see for example [1, 2, 3, 16]) how the notion of Γ-acts over a fixed Γ-semigroup S is determined.

Let (S, Γ, w_S) be a Γ-semigroup. A nonempty set M is a called left S_Γ-act (denoted by $S_\Gamma M$) if there is a total mapping λ_S from $S \times \Gamma \times M$ into M such that the following condition is satisfied

$$(\forall x, y \in S)(\forall \alpha, \beta \in \Gamma)(\forall a \in M)(\lambda_S(w_S(x, \alpha, y), \beta, a) = \lambda_S(x, \alpha, \lambda_S(y, \beta, a))).$$

Similarity one can define a right Γ acts.

3.2. Concept of S_Γ-acts over Γ-semigroup with apartness

Let $(S, =_S, \neq_S)$ be a Γ-semigroup with apartness over the set with apartness $(\Gamma, =_\Gamma, \neq_\Gamma)$ and with the internal operation $w_S : S \times \Gamma \times S \longrightarrow S$. The following definition introduces the concept of left S_Γ-acts over a set with apartness $(M, =_M, \neq_M)$ using a se-mapping $\lambda : S \times \Gamma \times M \longrightarrow M$:

Definition 3. *Let (S, Γ, w_S) be a Γ-semigroup with apartness. An inhabited set with apartness $(M, =_M, \neq_M)$ is called a left S_Γ-act (denoted by $S_\Gamma M$) if there a total se-mapping*

$$\lambda_S : S \times \Gamma \times M \ni (x, \alpha, a) \longmapsto \lambda_S(x, \alpha, a) := x\alpha a \in M$$

such that the following holds

$$(\forall x,y\in S)(\forall\alpha,\beta\in\Gamma)(\forall a\in M)(\lambda_S(w_S(x,\alpha,y),\beta,a)=_M\lambda_S(x,\alpha,\lambda_S(y,\beta,a))). \tag{1}$$

Similarity one can define a right S_Γ-acts.

Example 4. (i) *Any* Γ*-semigroup with apartness* (S,Γ,w_S) *is a* S_Γ*-act over itself because there is a total mapping* $\lambda_S := w_S : S\times\Gamma\times S\longrightarrow S$ *that satisfies required condition.*

(ii) *If* (S,Γ,w_S) *is a* Γ*-semigroup with apartness and* M *an inhabited set with apartness. Any fixed element* $m\in M$ *gives rise on* S_Γ*-act structure of* M *by the mapping* $\lambda_S : S\times\Gamma\times M\longrightarrow M$ *define by*

$$(\forall x\in S)(\forall\alpha\in\Gamma)(\forall a\in M)(\lambda_S(x,\alpha,a):=m).$$

This example shows that any inhabited set with apartness can be concider as S_Γ*-act for any* Γ*-semigroup with apartness* (S,Γ,w_S)*. In particular, every singleton set is a one-element* S_Γ*-act.*

(i) Using a shorter way of writing, the above condition can be written in the form

$$(\forall x,y\in S)(\forall\alpha,\beta\in\Gamma)(\forall a\in M)((x\alpha y)\beta a=_M xa(y\beta a)).$$

(ii) As can be seen from the definitions of S_Γ-acts, the differences in the determination of this concept between the classical case and the case in Bishop's constructive environment are in the carriers (sets) of this algebraic structure but also in the processes with them (operations in them).

Remark 1. *Since the operation* λ_S *is a strongly extensional function, the implications*

$$(\forall x,y\in S)(\forall\alpha,\beta\in\Gamma)(\forall a,b\in M)((x=_S y\wedge\alpha=_\Gamma\wedge a=_M b)\Longrightarrow x\alpha a=_M y\beta b),$$

$$(\forall x,y\in S)(\forall\alpha,\beta\in\Gamma)(\forall a,b\in M)(x\alpha a\neq_M y\beta b\Longrightarrow(x\neq_S y\vee\alpha\neq_\Gamma\vee a\neq_M b))$$

are valid formulas.

Example 5. *Let* $S:=[0,1]\subseteq\mathbb{R}$*,* $\Gamma:=\{\frac{1}{n}:n\in\mathbb{N}\}$ *and* $M:=S$*. Then* M *is an* S_Γ*-act under usual multiplication of real numbers.*

Example 6. *Let* $(G,=,\neq,\cdot)$ *be a group with apartness (see, for example [17]). Then* G *is* $G_\mathbb{N}$*-act with operation* $\lambda_G : G\times\mathbb{N}\times G\longrightarrow G$ *defined by* $\lambda_G(x,n,a):=xna$ *for any* $x,a\in G$ *and* $n\in\mathbb{N}$*.*

Example 7. *Let $((R, =, \neq), +, 0, \cdot, 1)$ be a ring with apartness (in sense of articles [18, 19]) and let $((M, =, \neq), +, 0)$ be an R-module with apartness (see, for example [19]). A mapping $\lambda_R : R \times R \times M \longrightarrow M$ can be defined by $\lambda_R(x, \alpha, a) \longmapsto (x\alpha)a$ since M is a R-module. Then M is an R_R-act.*

Example 8. *Let S be a Γ-semigroup with apartness and J be a left ideal of S. Then J is a left S_Γ-act under the mapping $\lambda_S : S \times \Gamma \times J \ni (s, \alpha, a) \longmapsto \lambda_S(s, \alpha, s) := s\alpha a \in J$.*

Example 9. *Let S be as in the Example 2. If $S = \Gamma = M$, then M is S_Γ-act under the multiplication in $\mathbb{C}$.*

Example 10. *It can be easily verified that $\mathbb{R}^n$ is $\mathbb{R}_\mathbb{R}$-act with apartness with respect to the mapping $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined as follows*

$$(x, y, (a_1, ..., a_n)) \longmapsto (xya_1, ..., xya_n).$$

Specifically, for $n = 2$, $\mathbb{R}^2$ is a $\mathbb{R}_\mathbb{R}$-act with apartness.

3.3. Concepts of Γ-subacts and Γ-cosubacts

In the classical case, the notion Γ-subacts was introduced in standard manner (see, for example, [1, 2]): Let M be an S_Γ-act. A nonempty subset N of M is called S_Γ-subact if $S\Gamma N \subseteq N$. The concept of S_Γ-subacts of a S_Γ-act over a Γ-semigroup with apartness can be introduced in an analogous way:

Definition 4. *Let (S, Γ, w_S) be a Γ-semigroup with apartness and M be a S_Γ-act with apartness. An inhabited subset N of M is a S_Γ-subact with apartness if holds*

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a \in M)(a \in N \implies x\alpha a \in N).$$

Example 11. *It is clear that $\mathbb{Z}$ is $\mathbb{N}$-semigroup with apartness. Then $\mathbb{Z}$ is a $\mathbb{Z}_\mathbb{N}$-act and $2\mathbb{Z}$ is a $\mathbb{Z}_\mathbb{N}$-subact.*

The constructive dual of this notion is introduced as follows:

Definition 5. *Let (S, Γ, w_S) be a Γ-semigroup with apartness and M be a S_Γ-act with apartness. An inhabited subset N of M is a S_Γ-cosubact with apartness if the following holds*

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a \in M)(x\alpha a \in N \implies a \in N).$$

Obviously, if we look at the Γ-semigroup with apartness (S, Γ, w_S) as S_Γ-act, then every left co-ideal in S is a S_Γ-subact.

Example 12. *In a $\mathbb{N}$-semigroup with apartness $\mathbb{Z}$, the set $\mathbb{Z}$ is $\mathbb{Z}_{\mathbb{N}}$-act (Example 11). The subset $2\mathbb{N}+1 \subseteq \mathbb{Z}$ is an $\mathbb{Z}_{\mathbb{N}}$-cosubact. Indeed, if the product $x\alpha a$ is an odd number, then x is an odd number also (but also α and a).*

Example 13. *Let $M := \mathbb{R}^2$ be as in Example 10. Then the subsets $N_1 = \{(0, y) : y \in \mathbb{R}\}$ and $N_2 = \{(x, 0) : x \in \mathbb{R}\}$ are $\mathbb{R}_{\mathbb{R}}$-subacts with apartness of $\mathbb{R}_{\mathbb{R}}$-act with apartness $\mathbb{R}^2$.*

Since the family $(\mathfrak{S}_\Gamma)_{SA}(M)$ of all S_Γ-cosubacts with apartness of S_Γ-act with apartness M is not empty, we have:

Theorem 1. *The family $(\mathfrak{S}_\Gamma)_{SA}(M)$ is a complete semi-lattice.*

Proof. Let $\{N_i\}_{i\in I}$ be a family of S_Γ-cosubacts with apartness of a S_Γ-act with apartness M. Since $N_i \neq \emptyset$ for any $i \in I$, we conclude that $\cup_{i\in I} N_i \neq \emptyset$. Now, let $a \in \cup_{i\in I} N_i$, $x \in S$ and $\alpha \in \Gamma$ be arbitrary elements such that $x\alpha a \in \cup_{i\in I} N_i$. Then there an index $j \in I$ such that $x\alpha a \in N_j$. Thus $a \in N_j \subseteq \cup_{i\in I} N_i$. So, the subset $\cup_{i\in I} N_i$ is a S_Γ-cosubact of M.

Theorem 2. *Let $\{N_i\}_{i\in I}$ be a family of S_Γ-cosubacts with apartness of a S_Γ-act with apartness M such that $\cap_{i\in I} N_i \neq \emptyset$. Then the subset $\cap_{i\in I} N_i$ is a S_Γ-cosubact with apartness of M.*

Proof. Let $x \in S$, $\alpha \in \Gamma$ and $a \in \cap_{i\in U} N_i$ such that $x\alpha a \in \cap_{i\in I} N_i$. Then for any index $i \in I$, $x\alpha a \in N_i$ holds. Thus $(\forall \in I)(a \in N_i)$ because N_i is a S_Γ-cosubact of M. So, $a \in \cap_{i\in I} N_i$.

3.4. se-representation of Γ-semigroup with apartness

The concept of se-representation of a Γ-semigroup with apartness S by Γ-se-transformations of an inhabited set with apartness A is introduced by the following determination:

Definition 6. *Let (S, Γ, w_S) be a Γ-semigroup with apartness over set $(\Gamma, =_\Gamma, \neq_\Gamma)$. A se-representation of a Γ-semigroup S by Γ-se-transformations of an inhabited set with apartness $(A, =_A, \neq_A)$ is a se-homomorphism*

$$(\Phi, \iota_\Gamma) : (S, \Gamma, w_S) \longrightarrow F_{se}(\Gamma, F_{se}(A, A)).$$

It is understood that

$$(\forall x, y \in S)(\forall \alpha, \beta \in \Gamma)(((\Phi, \iota_\Gamma)(x\alpha y))(\beta) = (\Phi(x)\alpha\Phi(y))(\beta) = \Phi(y)_\beta \circ \Phi(x)_\alpha)$$

holds and the following implications

$$(x, \alpha, y) =_S (u, \beta, v) \implies (\Phi, \iota_\Gamma)(x, \alpha, y) = (\Phi, \iota_\Gamma)(u, \beta, v),$$

$$(\Phi, \iota_\Gamma)(x, \alpha, y) \neq (\Phi, \iota_\Gamma)(u, \beta, v) \implies (x, \alpha, y) \neq_S (u, \beta, v)$$

are valid formulas for arbitrary variables $x, y, u, v \in S$ and $\alpha, \beta \in \Gamma$.

The following theorem gives one characteristic property of these representations.

Theorem 3. *Let (S, Γ, w_S) be Γ-semigroup with apartness and let A be a set with apartness. The presence of a representation of the Γ-semigroup with apartness S by means of the set with apartness A, transforms the set A into a S_Γ-act.*

Proof. Let $A := (A, =_A, \neq_A)$ be an inhabited set with apartness and $S := (S, \Gamma, w_S)$ be a Γ-semigroup with apartness over an inhabited set with apartness $\Gamma := (\Gamma, =_\Gamma, \neq_\Gamma)$. If $(\Phi, \iota_\Gamma) : S \longrightarrow F_{se}(\Gamma, F_{se}(A, A))$ is a se-representation, we can define the operation $\lambda_S : S \times \Gamma \times A \longrightarrow A$ as follows

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a \in A)(\lambda_S(x, \alpha, a) := ((\Phi(x))(\alpha))(a)).$$

For ease of writing, we put $T := F_{se}(\Gamma, F_{se}(A, A))$. Then for $f, g \in T$, we have

$$\begin{aligned} f =_T g &\iff (\forall \alpha \in \Gamma)(f_\alpha := f(\alpha) = g(\alpha) =: g_\alpha \in F_{se}(A, A)) \\ &\iff (\forall \alpha \in \Gamma)(\forall a \in A)(f_\alpha(a) =_A g_\alpha(a) \in A) \end{aligned}$$

and

$$\begin{aligned} f \neq_T g &\iff (\exists \alpha \in \Gamma)(f_\alpha := f(\alpha) \neq f(\alpha) =: g_\alpha \in F_{se}(A, A)) \\ &\iff (\exists \alpha \in \Gamma)(\exists a \in A)(f_\alpha(a) \neq_A g_\alpha \in A). \end{aligned}$$

First, let us show that λ_S is well-defined. Let $x, y \in S$, $\alpha, \beta \in \Gamma$ and $a, b \in A$ be such that $x =_S y$, $\alpha =_\Gamma \beta$ and $a =_A b$. Then $f_\alpha(a) =_A f_\beta(b)$ any $f \in F_{se}(\Gamma, F_{se}(A, A))$. Thus

$$\begin{aligned} \lambda_S(x, \alpha, a) &:= ((\Phi(x))(\alpha)(a) =_A \Phi(x)_\alpha(a) \\ &=_A \Phi(y)_\beta(b) =_A (\Phi(y))(\beta)(b) =: \lambda_S(y, \beta, b). \end{aligned}$$

Second, let us show that λ_S is a se-mapping. Let $x, y \in S$, $\alpha, \beta \in \Gamma$ and $a, b \in A$ be such that $\lambda_S(x, \alpha, a) \neq_A \lambda_S(y, \beta, b)$. This means

$$((\Phi(x))(\alpha)(a) \neq_A ((\Phi(y))(\beta)(b).$$

Then $a \neq_A b$ or $((\Phi(x))(\alpha) \neq ((\Phi(y))(\beta)$ because $\Phi(x)_\alpha = (\Phi(x))(\alpha)$ and $\Phi(y)_\beta = (\Phi(y))(\beta)$ are se-mappings. If we assume that the second case is valid, we have

$$\alpha \neq_\Gamma \beta \text{ or } \Phi(x) \neq_T \Phi(y)$$

because $\Phi(x)$ and $\Phi(y)$ are se-mappings. If we assume again that the second case is valid, we have $x \neq_S y$ since Φ is a se-mapping. This shows that it is valid

$$\lambda_S(x, \alpha, a) \neq_A \lambda_S(y, \beta, b) \implies (x \neq_A y \vee \alpha \neq_\Gamma \beta \vee a \neq_A b)$$

which proves that λ_S is a se-mapping.

It remains to be seen whether the λ_S thus determined satisfies the condition in Definition 3. Let $x, y \in S$, $\alpha, \beta \in \Gamma$ and $a \in A$ be arbitrary elements. Then

$$\begin{aligned}\lambda_S(w_S(x,\alpha,y),\beta,a) &=_A ((\Phi(w_S(x,\alpha,y))(\beta))(a)\\ &=_A ((\Phi(x)\alpha\Phi(y))(\beta))(a)\\ &=_A (\Phi(y)_\beta \circ \Phi(x)_\alpha)(a)\\ &=_A (\Phi(x)_\alpha)(\Phi(y)_\beta(a))\\ &=_A \lambda_S(x,\alpha,\lambda_S(y,\beta,a)).\end{aligned}$$

This proves that A is Γ_S-act.

Theorem 4. *Let (S, Γ, w_S) be a Γ-semigroup with apartness and A be a S_Γ-act with apartness. Then there exists an associated se-representation of S by the Γ-semigroup with apartness $F_{se}(\Gamma, F_{se}(A, A))$.*

Proof. Let $(A, =_A, \neq_A)$ be an inhabited set with apartness and (S, Γ, w_S) be a Γ-semigroup with apartness. Suppose A is a S_Γ-act with apartness. This means that there is a total se-mapping $\lambda_S : S \times \Gamma \times A \longrightarrow A$ such that

$$(\forall x, y \in S)(\forall \alpha, \beta \in \Gamma)(\forall a \in M)(\lambda_S(w_S(x,\alpha,y),\beta,a) =_M \lambda_S(x,\alpha,\lambda_S(y,\beta,a))).$$

Let us define $(\Phi, \iota_\Gamma) : S \longrightarrow F_{se}(\Gamma, F_{se}(A, A))$ as follows $\Phi(x)(\alpha) := \Phi(x)_\alpha$ for any $x \in S$ and $\alpha \in \Gamma$ such that

$$(\forall a \in A)(\Phi(x)_\alpha(a) := x\alpha a =: \lambda_S(x,\alpha,a)).$$

It have to be shown that Φ is a se-homomorphism between Γ-semigroups with apartness.

It is clear that Φ is well-defined: For any $x \in S$ and any $\alpha \in \Gamma$, the product $x\alpha := x_\alpha : A \longrightarrow A$ is a se-mapping from A into A. So, $x_\alpha \in F_{se}(A, A)$. Therefore, $\Phi(x) : \Gamma \longrightarrow F_{se}(A, A)$ is a se-mapping from Γ into $F_{se}(A, A)$.

Let $x, y \in S$, $\alpha, \beta \in \Gamma$ and $a, b \in A$ be arbitrary elements such that $x\alpha a, y\beta b \in A$ and $x\alpha a \neq_A y\beta b$. This means

$$\Phi(x)_\alpha(a) := \lambda_S(x,\alpha,a) \neq_A \lambda(y,\beta,b) =: \Phi(y)_\beta(b).$$

Then $x \neq_S y \vee \alpha \neq_\Gamma \beta \vee a \neq_A b$ because λ_S is a se-mapping. This shows that Φ is a se-mapping.

It has to be shown that (Φ, ι_Γ) is a se-homomorphism between Γ-semigroups. Let $x, y \in S$, $\alpha, \beta \in \Gamma$ and $a \in A$ be elements taken arbitrarily. Then

$$(\Phi(x\alpha y))(\beta)(a) =_A \lambda_S(w_S(x,\alpha,y),\beta,a) =_A \lambda(x,\alpha,\lambda_S(y,\beta,a))$$

$$
\begin{aligned}
&=_A \lambda_S(x, \alpha, (\Phi, \iota_\Gamma)(y, \beta, a)) \\
&=_A \lambda_S(x, \alpha, \Phi(y)_\beta(a)) \\
&=_A \Phi(x)_\alpha((\Phi(y)_\beta)(a)) \\
&=_A (\Phi(y)_\beta \circ \phi(x)_\alpha)(a).
\end{aligned}
$$

Thus

$$(\Phi(x\alpha y))(\beta) = (\Phi(y)_\beta \circ \Phi(x)_\alpha) = (\phi(x)\alpha\Phi(y))(\beta)$$

since the previous equation is valid for every $a \in A$. Hence

$$\Phi(x\alpha y) = \Phi(x)\alpha\Phi(y)$$

since the previous equation is valid for every $\beta \in \Gamma$.

3.5. Co-congruence on a S_Γ-act

The concept of congruences on a S_Γ-act M is determined in a standard way (see, for example [34], Definition 3.8). The constructive dual of this relation, the co-congruence on S_Γ-act is introduced as follows:

Definition 7. *Let M be an S_Γ-act with apartness. A co-equivalence q on M is called a co-congruence on M, if holds*

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a, b \in M)((x\alpha a, x\alpha b) \in q \implies (a, b) \in q).$$

That this concept is well-defined is shown by the following proposition:

Proposition 1. *If q is a co-congruence on a S_Γ-act with apartness M, then the relation $q^\lhd$ is a congruence on M.*

Proof. It has already been said that if q is a co-equality on a set with apartness $(M, =_M, \neq_M)$, then the relation $q^\lhd$ is an equality relation on M. We need to show the compatibility of the relation $q^\lhd$ with the operation $\lambda_S : S \times \Gamma \times M \longrightarrow M$. Let $x \in S$, $\alpha \in \Gamma$ and $a, b, u, v \in M$ be such that $(a, b) \in q^\lhd$ and $(u, v) \in q$. Then

$$(u, x\alpha a) \in q \lor (x\alpha a, x\alpha b) \in q \lor (x\alpha b, v) \in q$$

by co-transitivity of q. If there were $(x\alpha a, x\alpha b) \in q$, we would have $(a, b) \in q$ which contradicts the hypothesis $(a, b) \lhd q$. Thus, it must be $(u, x\alpha a) \in q$ or $(x\alpha b, v) \in q$ hence $u \neq_M x\alpha a$ or $x\alpha b \neq_M v$ for consistency of the relation q. This means $(x\alpha a, x\alpha b) \neq (u, v) \in q$.

In addition to the previous one, it is valid:

Theorem 5. *The family $\mathfrak{Q}(S_\Gamma M)$ of all co-congruences on S_Γ-act with apartness M forms a complete lattice.*

Proof. Let $\{q_k\}_{k\in K}$ be a family of co-congruences on S_Γ-act M.

Without much difficulty it can be shown that $\bigcup_{k\in K} q_k$ is a co-equivalence on M. Let $x \in S$, $\alpha \in \Gamma$ and $a, b \in M$ be such that $x\alpha a, x\alpha b) \in \bigcup_{k\in K} q_k$. Then there is an index $k \in L$ such that $(x\alpha a, x\alpha b) \in q_k$. Thus $(a, b) \in q_k \subseteq \bigcup_{k\in K} q_k$.

Let X be the family of all co-congruences on S_Γ-act with apartness M contained in $\bigcap_{k\in K} q_k$. Then $\cup X$ is a co-congruence on M according to the first part of this proof.

If we put $\sqcup_{k\in K} q_k = \bigcup_{k\in K} q_k$ and $\sqcap_{k\in K} = \cup X$, then $(\mathfrak{Q}(S_\Gamma M), \sqcup, \sqcap)$ is a complete lattice.

Let q be a co-congruence on a S_Γ-act with apartness M. As already mentioned, we can design sets with apartness $M/(q^\triangleleft, q)$ and $[M : q]$. We can define the correspondence

$$\mu_{S1} : S \times \Gamma \times M/(q^\triangleleft, q) \longrightarrow M/(q^\triangleleft, q)$$

as follows:

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a \in M)(\mu_{S1}(x, \alpha, aq^\triangleleft) := (x\alpha a)q^\triangleleft). \tag{2}$$

Also, we can define the correspondence

$$\mu_{S2} : S \times \Gamma \times [M : q] \longrightarrow [M : q]$$

as follows:

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a \in M)(\mu_{S2}(x, \alpha, aq) := (x\alpha a)q). \tag{3}$$

Lemma 6. *These correspondences, designed as described, are well-defined se-mappings.*

Proof. Let $x \in S$, $\alpha \in \Gamma$ and $a, b, u, v \in M$ be such that $aq^\triangleleft =_1 bq^\triangleleft$ and $(u, v) \in q$. Then

$$(u, x\alpha a) \in q \vee (x\alpha a, x\alpha b) \in q \vee (x\alpha b, v) \in q$$

by co-transitivity of q. If the second option were valid, we would have $(a, b) \in q$ which is impossible. So it has to be $(u, x\alpha a) \in q \vee (x\alpha b, v) \in q$. Thus $x\alpha a \neq_M u$ or $x\alpha b \neq_M v$. This means $(x\alpha a, x\alpha b) \neq (u, v) \in q$. Hence $(x\alpha a, x\beta b) \triangleleft q$. We show that μ_{S1} is a se-mapping. Let $x \in S$, $\alpha \in \Gamma$ and $a, b \in M$ be such that $(x\alpha a)q^\triangleleft \neq_1 (y\beta b)q^\triangleleft$. This means $(x\alpha a, x\alpha b) \in q$. Then $(a, b) \in q$ by Definition 7. Therefore, $aq^\triangleleft \neq_1 bq^\triangleleft$. This shows that μ_{S1} is well-defined a se-mapping.

Analogously, it can be shown that μ_{S2} is well-defined a se-mapping. Let $x \in S$, $\alpha \in \Gamma$ and $a, b, u, v \in M$ be such that $aq =_2 bq$ and $(u, v) \in q$. Then

$$(u, x\alpha a) \in q \vee (x\alpha a, x\alpha b) \in q \vee (x\alpha b, v) \in q.$$

The second option would give $(a,b) \in q$ and it should be rejected because it is contrary to the hypothesis $(a,b) \lhd q$. Must be $(u, x\alpha a) \in q$ or $(x\alpha b, v) \in q$. It follows from here $x\alpha a \neq_M u$ or $x\alpha b \neq_M v$. This means $x\alpha a, x\alpha b) \neq (u,v) \in q$. So, $(x\alpha a)q =_2 (x\alpha b)q$. Let $x \in S$, $\alpha \in \Gamma$ and $a,b \in M$ be such $x\alpha aq \neq_2 x\alpha bq$. Then $(x\alpha a)q \neq_2 (x\alpha b)q$ and $(x\alpha a, x\alpha b) \in q$. Thus $(a,b) \in q$ by Definition 7. Hence $aq \neq_2 bq$.

We can now design the theorem:

Theorem 7. *Let q be a co-congruence on a S_Γ-act with apartness M. Then $M/(q^\lhd, q)$ and $[M:q]$ are S_Γ-acts.*

Proof. According to the analysis that precedes the statement of this theorem, the correspondences μ_{S1} and μ_{S2} are well-defined se-mappings. Let us show that each of these functions satisfies the condition (1). Let $x,y \in S$, $\alpha,\beta \in \Gamma$ and $a \in M$ be arbitrary elements. Then

$$\begin{aligned}\mu_{S1}(w_S(x,\alpha,y),\beta,aq^\lhd) &=_1 (x\alpha y)\beta aq^\lhd =_1 ((x\alpha y)\beta a)q^\lhd =_1 (x\alpha(y\beta a))q^\lhd \\ &=_1 x\alpha(y\beta a)q^\lhd =_1 x\alpha(y\beta aq^\lhd) =_1 \mu_{S1}(x\alpha y,\beta,aq^\lhd).\\ \mu_{S2}(w_S(x,\alpha,y),\beta,aq) &=_2 (x\alpha y)\beta aq) =_2 ((x\alpha y)\beta a)q =_2 (x\alpha(y\beta a))q \\ &=_2 x\alpha(y\beta a)q =_2 x\alpha(y\beta aq) =_2 \mu_{S2}(x\alpha y,\beta,aq).\end{aligned}$$

$M/(q^\lhd, q)$ is called quotient S_Γ-act under the co-congruence q on M. S_Γ-act $[M:q]$ is a specificity of this logical framework and has no a counterpart in the classical theory.

4. Final comments

Let M be S_Γ-act with apartness and let N be S_Λ-act with apartness. A pair (h,θ) of strongly extensional functions $h: S \longrightarrow T$ and $\theta: \Gamma \longrightarrow \Lambda$ is called a se-homomorphism from S_Γ-act M to S_Λ-act T if the following holds

$$(\forall x \in S)(\forall \alpha \in \Gamma)(\forall a \in M)((h,\theta)(x\alpha a) :=_T x\theta(\alpha)h(a)).$$

In particular, for $\Gamma = \Lambda$ it can be shown without major difficulties that the relation $Ker(h)$ is a congruence on M and that the relation $Cokeer(h) := \{(a,b) \in M \times M : h(a) \neq_N h(b)\}$ is a co-congruence on M compatible with $Ker(h)$. Additionally, on the other hand, there is the se-epimorphism $\pi : M \longrightarrow M/(q^\lhd, q)$ and the se-epimorphism $\vartheta : M \longrightarrow [M:q]$ for a given co-congruence q na S_Γ-act M. Thus, two forms of the so-called first theorem on isomorphisms between S_Γ-acts with apartness could be designed. While one would refer to S_Γ-act $M/(q^\lhd, q)$, the other would be related to the structure $[M:q]$.

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