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MAJORIZATION PROPERTIES FOR SUBCLASSES OF $p-$VALENT MEROMORPHIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. The object of this paper is to investigate a majorization problem for certain subclasses of $p-$valent meromorphic functions defined in the punctured unit disc $\mathbb{U}^*$ having a pole of order p at origin. The subclasses under investigation are defined by convolution between any analytic functions. Several results in form of corollaries are also pointed out.

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1. Introduction and Definition

Let $f(z)$ and $g(z)$ be analytic in the open unit disc $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

The function f is majorized by g in $\mathbb{U}$ (see [18]) and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \tag{1}$$

if there exists a function $\psi(z)$, analytic in $\mathbb{U}$ satisfying $|\psi(z)| \leq 1$ and

$$f(z) = \psi(z)g(z) \quad (z \in \mathbb{U}). \tag{2}$$

It may be known that (1) is closely related to the concept of quasi subordination between analytic functions.

Definition 1. *[20, p.4] For analytic functions f and F, the function $f(z)$ is subordinate to $F(z)$ if there exists a Schwarz function w, that (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < |z|$ $(z \in \mathbb{U})$ such that*

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}). \tag{3}$$

This subordination denoted by

$$f(z) \prec F(z) \quad (z \in \mathbb{U}). \tag{4}$$

It follows from definition

$$f(z) \prec F(z) \Longrightarrow f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}).$$

In particular, if F is univalent in $\mathbb{U}$, then we have the following equivalence (see [8, 17, 21]).

$$f(z) \prec F(z) \ \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}).$$

Definition 2. *[1] The function $f(z)$ is said to be quasi subordinate to $F(z)$ if there exists an analytic function $\omega(z)$ $(|\omega(z)| \le 1)$ such that $\frac{f(z)}{\omega(z)}$ is analytic in $\mathbb{U}$ and*

$$\frac{f(z)}{\omega(z)} \prec F(z) \quad (z \in \mathbb{U}). \tag{5}$$

By definition of subordination, (5) is equivalent to

$$f(z) = \omega(z)F(\phi(z)) \quad (|\phi(z)| \le |z|, \ z \in \mathbb{U}). \tag{6}$$

Quasi subordination denoted by

$$f(z) \prec_q F(z) \quad (z \in \mathbb{U}). \tag{7}$$

The quasi subordination becomes subordination (4), if take $\omega(z) \equiv 1$ in (6). The quasi subordination (7) becomes majorization (1), if set $\phi(z) = z$ in (6).

Let $\sum_p$ mention to the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, ...\}) \tag{8}$$

that are analytic and p-valent in the punctured unit disc $\mathbb{U}^* := \mathbb{U} \setminus \{0\}$ having a pole of order p at the origin. We note that $\sum_1 = \sum$.

For the functions $f_i \in \sum_p$ given by

$$f_i(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p,i} z^{k-p} \quad (i = 1, 2; z \in \mathbb{U}^*),$$

we set the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p,1} a_{k-p,2} z^{k-p} = (f_2 * f_1)(z). \tag{9}$$

Now, by the Hadmard product of two functions, we introduce a subclass of function $f \in \sum_p$ as follows.

Definition 3. *Let* $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $\ell \in \mathbb{N}_0, \tau \in \mathbb{C}^*$, $\hbar(z)$ *given by*

$$\hbar(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} h_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, ...\}) \tag{10}$$

and $\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)} + |B|\right) < 1$. *A function* $f \in \sum_p$ *is said to be in the class* $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$ *of* $p-$*valent meromorphic functions of complex order* $\tau \neq 0$ *in* $\mathbb{U}^*$ *if and only if*

$$1 - \frac{1}{\tau}\left(\frac{z\left(f * \hbar\right)^{(\ell+1)}(z)}{\left(f * \hbar\right)^{(\ell)}(z)} + p + \ell\right) - \alpha\left|-\frac{1}{\tau}\left(\frac{z\left(f * \hbar\right)^{(\ell+1)}(z)}{\left(f * \hbar\right)^{(\ell)}(z)} + p + \ell\right)\right| \prec \frac{1 + Az}{1 + Bz}. \tag{11}$$

We note that,

- for

$$\hbar(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left[\frac{(\lambda + p)_k (c)_k}{(a)_k (1)_k}\right]^n \left[\frac{p - kt}{p}\right]^m z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{12}$$

  where $a,\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \lambda > -p$ and $t > 0$, the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B) = \mathcal{T}_{p,\ell}^{n,m}(a, c, t, \alpha, \tau; A, B)$ of multivalent meromorphic functions of complex order τ (see [26]);

- for $\alpha = 0$ and

$$\hbar(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{\Gamma(\gamma + \beta)\Gamma(k + \beta)}{\Gamma(\beta)\Gamma(k + \gamma + \beta)} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, ...\},\ \gamma > 0,\ \beta > -1),$$

  the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B) = \mathcal{M}_{\gamma,\beta}^{p,\ell}(\tau; A, B)$ of multivalent meromorphic functions of complex order τ (see [16]);

- for $\alpha = 0$ and

$$\hbar(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left[\frac{\lambda k + p + l}{p + l}\right]^m z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, ...\},\ l > -p), \tag{13}$$

the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B) = \mathcal{R}_p^{m,\ell}(\lambda,l,\tau;A,B)$ of multivalent meromorphic functions of complex order τ (see [25]);

- for $\alpha=0,\ p=1$, the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B)=\Sigma^{\ell}(\hbar,\tau;A,B)$ of meromorphic starlike functions of complex order $\tau\in\mathbb{C}^*$ in $\mathbb{U}^*$ (see [10]);

- for $A=1,\ B=-1$ and $\alpha=0$, we denote the class

$$\mathcal{K}_p^{\ell}(\hbar,0,\tau;1,-1)=\mathcal{K}_p^{\ell}(\hbar;\tau)$$
$$=\left\{f\in\sum_p:\Re\left[1-\frac{1}{\tau}\left(\frac{z\,(f*\hbar)^{(\ell+1)}(z)}{(f*\hbar)^{(\ell)}(z)}+p+\ell\right)\right]>0\right\}. \tag{14}$$

- for $A=1,\ B=-1,\ \alpha=0$ and $p=1$, the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B)=\Sigma^{\ell}(\hbar,\tau)$ of meromorphic starlike functions of complex order $\tau\in\mathbb{C}^*$ in $\mathbb{U}^*$ (see [10]);

- for $A=1,\ B=-1,\ \alpha=0,\ p=1$ and

$$\hbar(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left[\frac{\beta}{\beta+k+1}\right]^{\gamma}z^k\quad(p\in\mathbb{N}:=\{1,2,3,...\},\ \gamma>0,\ \beta>0), \tag{15}$$

the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B)=\mathcal{S}_{\beta}^{\gamma,\ell}(\tau)$ of meromorphic starlike functions of complex order $\tau\in\mathbb{C}^*$ in $\mathbb{U}^*$ (see [15]);

- for $A=1,\ B=-1,\ \alpha=0,\ \tau=(p-\vartheta)\cos\theta\ e^{-i\theta}\quad(|\theta|\le\frac{\pi}{2},\ 0\le\vartheta<p)$ and $\hbar(z)$ given by Eq. (12), the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B)=\mathcal{T}_{p,\ell}^{n,m}(a,c,t,\vartheta,\theta)$ is generalized class of θ- spiral- like functions of order ϑ if (see [26])

$$\mathrm{Re}\left\{e^{i\theta}\left[\frac{z(f*\hbar)^{(\ell+1)}(z)}{(f*\hbar)^{(\ell)}}+\ell\right]\right\}<-\vartheta\cos\theta;$$

- for $A=1,\ B=-1,\ \alpha=0,\ \ell=0$ and $\hbar(z)=\frac{1}{z^p(1-z)}$, the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B)$ reduces to

$$\Sigma(p;\tau)=\left\{f\in\Sigma_p:\ \mathrm{Re}\left[1-\frac{1}{\tau}\left(\frac{zf'(z)}{f(z)}+p\right)\right]>0,\ \tau\in\mathbb{C}^*\right\},$$

which is $p-$ valent meromorphic starlike function of complex order τ;

- for $A=1,\ B=-1,\ \alpha=0,\ \ell=0,\ p=1$ and $\hbar(z)=\frac{1}{z(1-z)}$, the class $\mathcal{K}_p^{\ell}(\hbar,\alpha,\tau;A,B)$ reduces to $\mathcal{S}(\tau)$ which is meromorphic starlike univalent function of complex order τ;

- for $A = 1,\ B = -1,\ \alpha = 0,\ \ell = 0,\ p = 1,\ \tau = 1 - \eta\ (0 \leq \eta < 1)$ and $\hbar(z) = \frac{1}{z(1-z)}$, the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$ reduces to $\Sigma(\eta)$ which is meromorphic starlike univalent function of order $\eta\ (0 \leq \eta < 1)$. This class studies by Pommerenke [27], Miller [22], Mogra et al. [23] (see also [4, 5, 6, 7] and [14]);
- for $A = 1,\ B = -1,\ \alpha = 0,\ \ell = 1$ and $\hbar(z) = \frac{1}{z^p(1-z)}$, the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$ reduces to

$$\mathcal{K}(p;\tau) = \left\{ f \in \Sigma_p :\ \mathrm{Re}\left[1 - \frac{1}{\tau}\left(\frac{zf''(z)}{f'(z)} + p + 1\right)\right] > 0,\ \tau \in \mathbb{C}^* \right\},$$

  which is $p-$ valent meromorphic convex function of complex order τ;
- for $A = 1,\ B = -1,\ \alpha = 0,\ \ell = 1,\ p = 1$ and $\hbar(z) = \frac{1}{z(1-z)}$, the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$ reduces to $\mathcal{K}(\tau)$ which is the class of meromorphic convex univalent function of complex order τ;
- for $A = 1,\ B = -1,\ \alpha = 0,\ \ell = 1,\ p = 1,\ \ell = 1$ and $\hbar(z) = \frac{1}{z(1-z)}$, the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$ reduces to $\Sigma_k(\eta)$ which is the class of meromorphic convex univalent function of order $\eta\ (0 \leq \eta < 1)$ (see [14]).

Also, there is many literature of majorization problems for univalent and multivalent functions discussed by various researchers. A majorization problem for the normalized classes of starlike functions has been investigated by Altintas et al. [2](also see [3, 9, 11, 12, 13]) and MacGreogor [18]. For recent expository work on majorization problems for meromorphic univalent and p-valent functions, see [28].

Motivated by aforementioned works, in this paper the authors investigate majorization problem for the class of $p-$valent meromorphic functions using convolution.

2. Majorization problem for the class $\mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$

Unless otherwise mentioned we shall assume throughout the sequel that $-1 \leq B < A \leq 1,\ \ p \in \mathbb{N},\ \ \ell \in \mathbb{N}_0, \tau \in \mathbb{C}^*; z \in \mathbb{U}^*$, and $\hbar(z)$ is given by (10).

Theorem 1. *Let the function $f(z) \in \sum_p$ and suppose that $g(z) \in \mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$. If $(f * \hbar)^{(\ell)}(z)$ is majorized by $(g * \hbar)^{(\ell)}(z)$ in $\mathbb{U}^*$, then*

$$|\,(f * \hbar)^{(\ell+1)}(z)| \leq |\,(g * \hbar)^{(\ell+1)}(z)| \quad (|z| < r_0), \tag{16}$$

where $r_0 = r_0(p, \ell, \alpha, \tau; A, B)$ *is the smallest positive root of the equation*

$$\begin{aligned} (p+\ell)\left[\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right] r^3-(2|B|+p+\ell)r^2 \\ -\left[2+(p+\ell)\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)\right] r+(p+\ell)=0 \end{aligned} \tag{17}$$

Proof. Let $g(z) \in \mathcal{K}_p^{\ell}(\hbar, \alpha, \tau; A, B)$. we get from (11) that

$$1-\frac{1}{\tau}\left(\frac{z\,(g*\hbar)^{(\ell+1)}(z)}{(g*\hbar)^{(\ell)}(z)}+p+\ell\right)-\alpha\left|-\frac{1}{\tau}\left(\frac{z\,(g*\hbar)^{(\ell+1)}(z)}{(g*\hbar)^{(\ell)}(z)}+p+\ell\right)\right|=\frac{1+Aw(z)}{1+Bw(z)}, \tag{18}$$

where $w(z) = b_1 z + b_2 z^2 + ...$, $w \in \mathcal{P}$, $\mathcal{P}$ indicate to the well- known class of the bounded analytic functions in $\mathbb{U}$ and satisfies the conditions $w(0) = 0$ and $w(z) < |z| \quad (z \in \mathbb{U})$.

Let

$$\aleph = 1-\frac{1}{\tau}\left(\frac{z\,(g*\hbar)^{(\ell+1)}(z)}{(g*\hbar)^{(\ell)}(z)}+p+\ell\right), \tag{19}$$

then by substituting in (18), we obtain

$$\aleph-\alpha|\aleph-1|=\frac{1+Aw(z)}{1+Bw(z)}, \tag{20}$$

which give

$$\aleph=\frac{1+\left(\frac{A-B\alpha e^{-i\theta}}{1-\alpha e^{-i\theta}}w(z)\right)}{1+Bw(z)}. \tag{21}$$

From Eqs. (19) and (21), we show

$$\frac{z\,(g*\hbar)^{(\ell+1)}(z)}{(g*\hbar)^{(\ell)}(z)}=-\frac{(p+\ell)+\left[\frac{(A-B)\tau}{1-\alpha e^{-i\theta}}+(p+\ell)B\right]w(z)}{1+Bw(z)}. \tag{22}$$

Since $|w(z)| \le |z| \quad (z \in \mathbb{U})$, the formula (22) gives

$$\begin{aligned} \left|(g*\hbar)^{(\ell)}(z)\right| &\le \frac{1+|B||z|}{(p+\ell)-\left|\frac{(A-B)\tau}{1-\alpha e^{-i\theta}}+(p+\ell)B\right||z|}\left|(g*\hbar)^{(\ell+1)}(z)\right| \\ &\le \frac{1+|B||z|}{(p+\ell)-\left|\frac{(A-B)\tau}{1-\alpha}+(p+\ell)B\right||z|}\left|(g*\hbar)^{(\ell+1)}(z)\right| \end{aligned} \tag{23}$$

Further, since that $(f*\hbar)^{(\ell)}(z)$ is majorized by $(g*\hbar)^{(\ell)}(z)$ in the punctured unit disc $\mathbb{U}^*$, from (2), we have

$$(f*\hbar)^{(\ell)}(z)=\psi(z)\,(g*\hbar)^{(\ell)}(z) \tag{24}$$

Differentiating (24) on both sides with respect to z, we get

$$(f*\hbar)^{(\ell+1)}(z)=\psi'(z)\,(g*\hbar)^{(\ell)}(z)+\psi(z)\,(g*\hbar)^{(\ell+1)}(z). \tag{25}$$

Next, noting that $\psi\in\mathcal{P}$ satisfies the inequality (see [24])

$$|\psi'(z)|\le\frac{1-|\psi(z)|^2}{1-|z|^2}, \tag{26}$$

and use (23) and (26) in (25), given

$$\left|(f*\hbar)^{(\ell+1)}(z)\right|\le\left(|\psi(z)|+\frac{(1-|\psi(z)|^2)(1+|B||z|)|z|}{(p+\ell)(1-|z|^2)\left[1-\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)|z|\right]}\right)\left|(g*\hbar)^{(\ell+1)}(z)\right|,$$

that, upon putting

$$|z|=r\ \text{ and }\ |\psi(z)|=\epsilon\quad(0\le\epsilon\le1),$$

leads to

$$|\,(f*\hbar)^{(\ell+1)}(z)|\le\Phi(\epsilon,r)\left|(g*\hbar)^{(\ell+1)}\right|$$

where

$$\Phi(\epsilon,r)=\frac{-r(1+|B|r)\epsilon^2+(p+\ell)(1-r^2)\left[1-\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)r\right]\epsilon+r(1+|B|r)}{(p+\ell)(1-r^2)\left[1-\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)r\right]}. \tag{27}$$

We note that,

$$\begin{aligned}r_0&=\max\{0\le r\le1:\Phi(\epsilon,r)\le1,\ \text{for all }0\le\epsilon\le1\}\\&=\max\{0\le r\le1:\Omega(\epsilon,r)\ge0,\ \text{for all }0\le\epsilon\le1\},\end{aligned}$$

where

$$\begin{aligned}\Omega(\epsilon,r)&=(p+\ell)(1-r^2)\left[1-\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)r\right]\\&-(1-\epsilon^2)(1+|B|r)r-(p+\ell)(1-r^2)\left[1-\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)r\right]\epsilon.\end{aligned}$$

The inequality $\Omega(\epsilon,r)\ge0$ is equivalent to

$$(p+\ell)(1-r^2)\left[1-\left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)}+|B|\right)r\right]-(1+\epsilon)(1+|B|r)r\ge0.$$

Takes its minimum value at $\epsilon = 1$ with $r_0 = r_0(p, \alpha, \tau; A, B)$ where r_0 is the smallest positive root of the equation (17). In fact that, as one case can see easily, either $\left[1 - \left(\frac{(A-B)|\tau|}{(p+\ell)(1-\alpha)} + |B|\right)\right] \neq 0$, or if it is equal to zero, the Eq. (17) has a unique root in the interval $(0, 1)$ and this is the smallest positive root of Eq. (17). The proof of Theorem 1 is complete.

3. Corollaries and concluding remarks

1. By setting $\hbar(z)$ as in Eq. (12) in Theorem 1, we obtain the result obtained by Panigraphi and El-Ashwah [26, Theorem 2.1].
2. By setting $\alpha = 0$ and $\hbar(z)$ as in Eq. (13) in Theorem 1, we obtain the result obtained by Panigraphi [25, Theorem 2.1].
3. By setting $p = 1$ and $\alpha = 0$ in Theorem 1, we obtain the result obtained by El-Ashwah and Aouf [10, Theorem 1].
4. By setting $p = 1$, $\alpha = 0$, $A = 1$, $B = -1$ and $\hbar(z)$ as in Eq. (15) in Theorem 1, we obtain the result obtained by Goyal and Goswami [15, Theorem 2.1].

By setting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the function $f \in \sum_p$ and suppose that $g \in \mathcal{K}_p^{\ell}(\hbar, \alpha, \tau)$. If $(f * \hbar)^{(\ell)}(z)$ is majorized by $(g * \hbar)^{(\ell)}(z)$ in $\mathbb{U}^*$, then*

$$|(f * \hbar)^{(\ell+1)}(z)| \leq |(g * \hbar)^{(\ell+1)}(z)| \quad (|z| < r_0),$$

where $r_1 = r_1(p, \ell, \alpha, \tau)$ is the smallest positive root of the equation

$$\left[\frac{2|\tau|}{(1-\alpha)} + (p+\ell)\right] r^3 - (2+p+\ell)r^2 - \left[2 + \left(\frac{2|\tau|}{(1-\alpha)} + (p+\ell)\right)\right] r + (p+\ell) = 0$$

given by $r_1 = \dfrac{k_1 - \sqrt{k_1^2 - (p+\ell)\left((p+\ell) + \frac{2|\tau|}{1-\alpha}\right)}}{(p+\ell) + \frac{2|\tau|}{1-\alpha}}$ *and* $k_1 = 1 + (p+\ell) + \frac{|\tau|}{1-\alpha}$.

Putting $\alpha = 0$ in Corollary 2, we obtain the following:

Corollary 3. *Let the function $f \in \sum_p$ and suppose that $g \in \mathcal{K}_p^{\ell}(\hbar, \tau)$. If $(f * \hbar)^{(\ell)}(z)$ is majorized by $(g * \hbar)^{(\ell)}(z)$ in $\mathbb{U}^*$, then*

$$|(f * \hbar)^{(\ell+1)}(z)| \leq |(g * \hbar)^{(\ell+1)}(z)| \quad (|z| < r_2),$$

where $r_2 = r_2(p, \ell, \tau)$ *is the smallest positive root of the equation*

$$(2|\tau| + p + \ell)\, r^3 - (2 + p + \ell) r^2 - [2 + (2|\tau| + p + \ell)]\, r + (p + \ell) = 0$$

given by

$$r_2 = \frac{k_2 - \sqrt{k_2^2 - (p+\ell)\,(p+\ell+2|\tau|)}}{p+\ell+2|\tau|} \text{ and } k_2 = 1 + (p+\ell) + |\tau|. \qquad (28)$$

Putting $\hbar(z) = \frac{1}{z^p(1-z)}$ in Corollary 3, we obtain the following:

Corollary 4. *Let the function* $f \in \sum_p$ *and suppose that* $g \in \mathcal{K}_p^{\ell}(\tau)$. *If* $f^{(\ell)}(z)$ *is majorized by* $g^{(\ell)}(z)$ *in* $\mathbb{U}^*$, *then*

$$|f^{(\ell+1)}(z)| \le |g^{(\ell+1)}(z)| \quad (|z| < r_2),$$

where $r_2 = r_2(p, \ell, \tau)$ *given by (28).*

Putting $\hbar(z) = \frac{1}{z^p(1-z)}$ and $\ell = 0$ in Corollary 4, we obtain the following:

Corollary 5. *Let the function* $f \in \sum_p$ *and suppose that* $g \in \Sigma_p(\tau)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* $\mathbb{U}^*$, *then*

$$|f'(z)| \le |g'(z)| \quad (|z| < r_3),$$

where $r_3 = r_3(p, \tau)$ *given by* $r_3 = \frac{k_3 - \sqrt{k_3^2 - p(p+2|\tau|)}}{p+2|\tau|}$ *and* $k_3 = 1 + p + |\tau|$.

Letting $p = 1$ and $\tau = 1$ in Corollary 5 leads to the following result [26]:

Corollary 6. *Let the functions* $f \in \sum$ *and* $g \in \sum_1(1) = \mathcal{S}(1)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* $\mathbb{U}^*$, *then*

$$|zf'(z)| \le |zg'(z)| \;\; \text{for} \;\; |z| \le \frac{3-\sqrt{6}}{3}.$$

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