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EXISTENCE AND UNIQUENESS THEOREMS FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH CAB-FRACTIONAL DERIVATIVE

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Abstract. In this study, we take into account a class of fractional Caputo-Atangana-Baleanu (CAB) integro-differential equations. We develop certain standards for determining the existence and uniqueness of solutions using the Banach contraction principle, Arzela-Ascoli theorem, and Krasnoselskii's fixed point theorem. Finally, a few instances are provided to highlight our key findings.

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1. Introduction

The area of mathematics that focuses on the integration and differentiation of real or complex orders is called fractional calculus. Even though this calculus is ancient, it has only recently become incredibly popular. This is because of its many seemingly diverse applications [1, 3, 4, 15, 31, 32, 33, 35, 36]. The most interesting speciality of the fractional operators is that there are many of these operators. This enables a researcher to choose the most suitable operator in order to describe the dynamics in a real world problem.

Researches on fractional integro-differential equations have witnessed an unprecedented boom in current years on account of the far-reaching application in various subjects, such as chemistry, physics, nuclear dynamics, biology, etc., for more details, see [2, 4, 20, 21, 22, 23, 24, 25, 26, 27, 30] and the references therein.

It is undeniable that existence and uniqueness theorems are crucial for initial value problems requiring the classical derivative operator because without them, it is impossible to correctly understand modelled systems and forecast how they will behave. Numerous mathematicians have also claimed that fractional integrals and derivatives are more practical for simulating some disorder zones and inherited characteristics of a variety of complex phenomena than integer order integrals and derivatives [19].

To reduce this problem, Atangana and Baleanu, a non-positional kernel hands a non-singularity a more general memory solution under different configuration scales. Several researchers have made their contributions to the development of FDEs related to the ABC derivative see [2, 5, 7, 14].

To describe complex problems, the concept of derivative of the fractional degree and integral partial differential equations is used. One of the difficulties in solving such equations is to predict the future behavior of a physical problem, the general model with ML and exponential laws has been proposed by Atangana and Gomez-Aguilar [10]. Koca and Atangana arrived at the results of the basic equation for ML elastic thermal conductivity.

There have been many works related to fractional differential integral equations,for more information on FDEs, the reader can see [7, 8, 11, 12, 13, 16, 17, 18, 20, 33, 35].

Motivated by the research going on in this direction, in this paper, we study existence and uniqueness of solutions for a new class of system of CAB sequential fractional integro-differential equation with the initial condition:

$$\begin{aligned}
&{}_{0}^{\mathcal{ABC}}\mathcal{D}^{\gamma}\Big[\hbar(\theta)+\lambda^{*}(\theta,\hbar(\theta))\Big]= \\
&\phi\Big(\theta,\hbar(\theta),\int_{0}^{\theta}\kappa_{1}(\theta,\zeta,\hbar(\zeta))d\zeta,\int_{0}^{\Phi}\kappa_{2}((\theta,\zeta,\hbar(\zeta))d\zeta\Big),\ \theta\in[0,1], \\
&\hbar(0)=\hbar_{0},
\end{aligned} \tag{1}$$

where ${}_{0}^{\mathcal{ABC}}\mathcal{D}^{\gamma}$ be the left CAB-derivative of fractional order $\gamma,\ \ 0<\gamma\leq 1,\theta,\zeta,\Phi\in[0,1]$. $\lambda^{*}:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $\phi:[0,1]\times\mathbb{R}^{3}\to\mathbb{R}$ are continuous functions satisfying some assumptions that will be specified later.

2. Auxiliary Results

We now gather some definitions and preliminary facts which will be used throughout this paper.

Definition 1. *[31] For $\gamma>0$, Riemann-Liouville (R-L) fractional integral of order $\gamma\in\mathbb{R}$ is defined as*

$$I^{\gamma}\hbar(\theta)=\frac{1}{\Gamma(\gamma)}\int_{0}^{\theta}(t-\theta)^{\gamma-1}d\theta,\ \ ,\ \ x>0,R(\alpha)>0 \tag{2}$$

Definition 2. *[31] For $0 < \gamma \leq 1$, the R-L fractional derivative and Caputo fractional derivative are defined as*

$$D^{\gamma}\hbar(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_0^t (t-\theta)^{-\gamma}d\theta\right)$$

$$^{c}D^{\gamma}\hbar(\theta) = \frac{1}{\Gamma(1-\gamma)}\int_0^x (t-\theta)^{-\gamma}\hbar'(\theta)dx$$

respectively.

Definition 3. *[9] Let $0 < \gamma \leq 1$ and $\hbar \in C^1[a,b], \hbar' \in L^1[a,b]$ where $0 \leq a < b$, the Caputo AB-fractional derivative and the R-L AB- fractional derivative of order γ are defined by*

$$^{\mathcal{ABC}}\mathcal{D}^{\gamma}\hbar(t) = \frac{\Psi(\gamma)}{\Gamma(1-\gamma)}\int_0^t \hbar'(\theta)\mathbb{E}_{\gamma}\left[-\gamma\frac{(t-\theta)^{\gamma}}{1-\gamma}\right]dx$$

and

$$^{\mathcal{ABC}}\mathcal{D}^{\gamma}\hbar(t) = \frac{\Psi(\gamma)}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_0^t \hbar'(\theta)\mathbb{E}_{\gamma}\left[-\gamma\frac{(t-\theta)^{\gamma}}{1-\gamma}\right]d\theta\Big)$$

respectively, where $\mathbb{E}_{\gamma}$ is called the Mittag-Leffter function and given by

$$\mathbb{E}_{\gamma}(\hbar) = \sum_{j=0}^{\infty}\frac{\hbar^j}{\Gamma(j\gamma+1)}$$

and $\Psi(\gamma)$ is a normalizing positive function satisfying $\Psi(0) = \Psi(1) = 1$.

Definition 4. *[9] Let $0 < \gamma \leq 1$ and $\hbar \in C[a,b], \hbar' \in L^1[a,b]$ where $0 \leq a < b$, the Caputo AB-fractional derivative and the R-L AB-fractional derivative of orderγ are defined by*

$$^{\mathcal{AB}}\mathcal{I}^{\gamma}\hbar(\theta) = \frac{(1-\gamma)}{\Psi(\gamma)}\hbar(\theta) + \frac{(\gamma)}{\Psi(\gamma)}\mathcal{I}^{\gamma}\hbar(\theta)$$

where $\mathcal{I}^{\gamma}$ is the R-L fractional integral defined in 1.

The following results are based on the fixed point technique for the system (1). The following assumptions are needed for establish the EU results.

Let $\Upsilon = C([0,1],\mathbb{R})$ be the Banach space of continuous functions $\hbar : [0,1] \longrightarrow \mathbb{R}$, with the norm $\|\hbar\| = sup_{\theta\in[0,1]}|\hbar(\theta)|$

- $(\mathfrak{L}_1)$ Suppose that $\phi \in ([0,1]\times\mathbb{R}^3,\mathbb{R})$ there exist positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$ such that

$$|\phi(\theta,\hbar_1,\nu_1,\mathsf{z}_1)-\phi(\theta,\hbar_2,\nu_2,\mathsf{z}_2)| \le \mathcal{M}_1(\|\hbar_1-\hbar_2\|+\|\nu_1-\nu_2\|+\|\mathsf{z}_1-\mathsf{z}_2\|)$$

$$\text{for all } \hbar_1,\nu_1,\mathsf{z}_1,\hbar_2,\nu_2,\mathsf{z}_2 \in \Upsilon, \theta\in[0,1] \text{ and } \mathcal{M}_2=\max_{\theta\in[0,1]}\|\phi(\theta,0,0,0)\|.$$

- $(\mathfrak{L}_2)$ Let $x \in [0,1]$ and $\lambda^* \in ([0,1]\times\mathbb{R},\mathbb{R})$ there exist positive constants μ_1 and μ_2 such that

$$|\lambda^*(\theta,\hbar_1)-\lambda^*(\theta,\hbar_2)| \le \mu_1(\|\hbar_1-\hbar_2\|$$

$$\text{for all } \hbar_1,\hbar_2\in\Upsilon \text{ and } \mu_2=\max_{\theta,\zeta\in[0,1]}\|\lambda^*(\theta,0)\|.$$

- $(\mathfrak{L}_3)$ If $\theta,\zeta\in[0,1]$ and $\kappa_1\in([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$ there exist positive constants ξ_1 and ξ_2 such that

$$|\kappa_1(\theta,\zeta,\hbar_1)-\kappa_1(\theta,\zeta,\hbar_2)| \le \xi_1(\|\hbar_1-\hbar_2\|$$

$$\text{for all } \hbar_1,\hbar_2\in\Upsilon \text{ and } \xi_2=\max_{\theta,\zeta\in[0,1]}\|\kappa_1(\theta,\zeta,0)\|.$$

- $(\mathfrak{L}_4)$ There exist positive constants $\Omega_1,$ and Ω_2 for $\kappa_2\in([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$ such that

$$|\kappa_2(\theta,\zeta,\hbar_1)-\kappa_2(\theta,\zeta,\hbar_2)| \le \Omega_1(\|\hbar_1-\hbar_2\|$$

$$\text{for all } \theta,\zeta\in[0,1],\ \hbar_1,\hbar_2\in\Upsilon \text{ and } \Omega_2=\max_{\theta,\zeta\in[0,1]}\|\kappa_1(\theta,\zeta,0)\|.$$

- $(\mathfrak{L}_5)$ For any positive $\overline{r}$ we take $\mathfrak{B}_{\overline{r}}=\{\hbar\in\Upsilon:\|\hbar\|\le\overline{r}\}\subset\Upsilon$. where

$$\overline{r}\ge\frac{\mathcal{Q}}{(1-\mathcal{P})}$$

where

$$\mathcal{P}=\mu_1+\mathcal{M}_1(1+\xi_1+\Phi\Omega_1)\left(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Gamma(\gamma)\Psi(\gamma)}\right)$$

and

$$\mathcal{Q} = \mu_2 + \mathcal{M}_2(1 + \xi_2 + \Phi\Omega_2)\left(\frac{(1-\gamma)}{\Psi(\gamma)} + \frac{1}{\Gamma(\gamma)\Psi(\gamma)}\right)$$

hen$\mathfrak{B}_{\overline{r}}$ is bounded, closed and convex subset in $C([0,1],\mathbb{R})$

Lemma 1. *If* $(\mathfrak{L}_3)$ *and* $(\mathfrak{L}_4)$ *are satisfied, then the estimate*

$$\|I_1\hbar(\theta)\| \leq \theta(\xi_1\|\hbar\| + \xi_2)$$

and

$$\|I_2\hbar(\theta)\| \leq \Phi(\Omega_1\|\hbar\| + \Omega_2)$$

are hold true for any $\theta \in [0,1]$ *and* $\hbar \in \Upsilon$.

Corollary 2. *[6, 14]*
If $0 < \gamma \leq 1$, *then*

$$\begin{aligned}({}_{0}^{\mathcal{AB}}\mathcal{I}^{\gamma}({}_{0}^{\mathcal{ABC}}\mathcal{D}^{\gamma})\hbar)(\theta) &= \hbar(\theta) - \hbar(0)\mathbb{E}_{\gamma}(\xi\theta\gamma) - \frac{\gamma}{1-\gamma}\hbar(0)\mathbb{E}_{\gamma,\gamma+1}(\xi\theta\gamma) \\ &= \hbar(\theta) - \hbar(0) \qquad (3)\end{aligned}$$

Theorem 3. *(Krasnoselkii's fixed point theorem) [32] Let S is a nonempty, closed, bounded and convex subset of a Banach space E. Let* A_1, A_2 *be the operators from* Ω *to E such that:*
(i) $A_1u_1 + A_2u_2 \in \Omega$ *whenever* $u_1, u_2 \in \Omega$
(ii) A_1 *continuous and compact;*
(iii) A_2 *is a contraction map. Then there exists* $z \in \Omega$ *such that* $z = A_1z + A_2z$

3. Main Results

Theorem 4. *Let* $0 < \gamma \leq 1$ *and there exists* $\phi \in ([0;1] \times \mathbb{R}^3;\mathbb{R})$ *with* $\phi^*(0,\hbar(0),0,\int_0^{\Phi}\kappa_2(0,\zeta,\hbar(\zeta))d\zeta = \lambda^*(0,\hbar(0)) = 0$. *A function* $\hbar \in C[0,1]$ *be a solution of the integral equation*

$$\begin{aligned}\hbar(\theta) &= \lambda^*(\theta,\hbar(\theta)) - \lambda^*(0,\hbar(0)) + \hbar_0 \\ &+ {}_{0}^{\mathcal{AB}}\mathcal{I}^{\gamma}\phi\Big(\theta,\hbar(\theta),\int_0^x \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big) \qquad (4)\end{aligned}$$

iff $\hbar(\theta)$ *is a solution of the* $\mathcal{ABC}-$*problem* (1)

Proof. Let $\hbar(\theta)$ satisfy (1). Applying the $\mathcal{AB}-$fractional integarl of (1) we get

$$({}_{0}^{\mathcal{AB}}\mathcal{I}^{\gamma}({}_{0}^{\mathcal{ABC}}\mathcal{D}^{\gamma})(|\hbar(t)-\lambda^{*}(\theta,\hbar(\theta))|) =$$
$$ {}_{0}^{\mathcal{AB}}\mathcal{I}^{\gamma}\phi\Big(\theta,\hbar(\theta),\int_{0}^{x}\kappa_{1}(\theta,\zeta,\hbar(\zeta))d\zeta,\int_{0}^{\Phi}\kappa_{2}(\theta,\zeta,\hbar(\zeta))d\zeta\Big)$$

By using Proposition 2, we obtain

$$\hbar(\theta)-\lambda^{*}(\theta,\hbar(\theta))-(\hbar(0)-\lambda^{*}(0,\hbar(0)))=$$
$${}_{0}^{\mathcal{AB}}\mathcal{I}^{\gamma}\phi\Big(\theta,\hbar(\theta),\int_{0}^{x}\kappa_{1}(\theta,\zeta,\hbar(\zeta))d\zeta,\int_{0}^{\Phi}\kappa_{2}(\theta,\zeta,\hbar(\zeta))d\zeta\Big).$$

Since $\hbar(0)=\hbar_{0}$, then (4) is satisfied. Now, consider $\hbar(\theta)$ satisfies the (4), then by

$$\phi^{*}(0,\hbar(0),0,\int_{0}^{\Phi}\kappa_{2}(0,\zeta,\hbar(\zeta))d\zeta=\lambda^{*}(0,\hbar(0))=0,$$

it is visible that $\hbar(0)=\hbar_{0}$ using Applying $\mathcal{AB}-$derivative in R-L sense of (4) and by using $({}_{0}^{\mathcal{AB}}\mathcal{D}^{\gamma}({}_{0}^{\mathcal{AB}}\mathcal{D}^{\gamma})\hbar)(\theta)=\hbar(\theta)$, we get

$$({}_{0}^{\mathcal{ABR}}\mathcal{D}^{\gamma}\hbar)(\theta)=(\hbar_{0})({}_{0}^{\mathcal{ABR}}\mathcal{D}^{\gamma})I)(\theta)+({}_{0}^{\mathcal{ABR}}\mathcal{D}^{\gamma})\lambda^{*}(\theta,\hbar(\theta))$$
$$+({}_{0}^{\mathcal{ABR}}\mathcal{D}^{\gamma}{}_{0}^{\mathcal{AB}}\mathcal{I}^{\gamma})\phi\Big(\theta,\hbar(\theta),\int_{0}^{\theta}\kappa_{1}(\theta,\zeta,\hbar(\zeta))d\zeta,\int_{0}^{\Phi}\kappa_{2}(\theta,\zeta,\hbar(\zeta))d\zeta\Big)$$

Thus

$$({}_{0}^{\mathcal{ABR}}\mathcal{D}^{\gamma})(\hbar)(\theta)-\lambda^{*}(\theta,\hbar(\theta)))=(\hbar_{0})\mathbb{E}_{\gamma}\Big(\frac{-\gamma}{1-\gamma}\theta\gamma\Big)$$
$$+\phi\Big(\theta,\hbar(\theta),\int_{0}^{x}\kappa_{1}(\theta,\zeta,\hbar(\zeta))d\zeta,\int_{0}^{\Phi}\kappa_{2}(\theta,\zeta,\hbar(\zeta))d\zeta\Big)$$

Hence, the equation (1) can be obtained by the Theorem 1 in [9].

Now, let us define the operator $\mathcal{F}$ on $\mathfrak{B}_{\overline{r}}$ as follows

$$\begin{aligned}\mathcal{F}\hbar(\theta) &= \lambda^{*}(\theta,\hbar(\theta))+\hbar_{0}\\ &+ {}_{0}^{\mathcal{AB}}\mathcal{D}^{\gamma}\phi\Big(\theta,\hbar(\theta),\int_{0}^{\theta}\kappa_{1}(\theta,\zeta,\hbar(\zeta))d\zeta,\int_{0}^{\Phi}\kappa_{2}(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\end{aligned}$$

It is observed that $\hbar(\theta)$ is the solution of (1) iff the operator $\mathcal{F}$ has a fixed point.

Theorem 5. *Assume that* $\mathfrak{L}_1 - \mathfrak{L}_5$ *are satisfied and*

$$\begin{aligned}\rho(\theta_2-\theta_1) &= \mathcal{M}1[\|(\theta,\upsilon(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\upsilon(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\upsilon(\zeta))d\zeta)\|] \\ &+ x(\xi_1\|(\theta,\upsilon(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\upsilon(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\upsilon(\zeta))d\zeta))\|) \\ &+ \Phi(\Omega_1\|(\theta,\upsilon(\theta),\int_0^x \kappa_1(\theta,\zeta,\upsilon(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\upsilon(\zeta))d\zeta))\|).\end{aligned}$$

If $\mu_1 \le 1$ *Then problem* (1) *has a solution.*

Proof. Let us define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathfrak{B}_{\overline{r}}$ such that

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$$

$$\mathcal{F}_1\hbar(\theta) = \lambda^*(\theta,\hbar(\theta)) + \hbar_0$$

,

$$\mathcal{F}_2\hbar(\theta) = {}_0^{\mathcal{AB}}\mathcal{I}^\gamma\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big).$$

The following three steps are required to apply the Theorem 2.

1) $\|\mathcal{F}_1\hbar + \mathcal{F}_2\mathsf{z}\| \le \overline{r}$ where $\overline{r} \in \mathfrak{B}_{\overline{r}}$ For any $\hbar, \mathsf{z} \in \mathfrak{B}_{\overline{r}}$

$$\begin{aligned}&\|\mathcal{F}_1\hbar + \mathcal{F}_2\mathsf{z}\| = \sup_{\theta\in[0,1]}\Bigg\{\Big|\lambda^*(\theta,\hbar(\theta)) + \hbar_0 \\ &+\frac{1-\gamma}{\Psi(\gamma)}\phi\Big(\theta,\mathsf{z}(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\mathsf{z}(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\mathsf{z}(\zeta))d\zeta\Big) \\ &+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma\phi\Big(\theta,\mathsf{z}(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\mathsf{z}(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\mathsf{z}(\zeta))d\zeta\Big)\Big|\Bigg\} \\ &\le \sup_{\theta\in[0,1]}\Bigg\{|\lambda^*(\theta,\hbar(\theta))| + |\hbar_0| \\ &+\frac{1-\gamma}{\Psi(\gamma)}|\phi\Big(\theta,\mathsf{z}(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\mathsf{z}(\zeta))d\zeta,\mathsf{z}(\theta)\int_0^\Phi \kappa_2(\theta,\zeta,\mathsf{z}(\zeta))d\zeta\Big)| \\ &+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma|\phi\Big(\theta,\mathsf{z}(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\mathsf{z}(\zeta))d\zeta,\mathsf{z}(\theta)\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)|\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\theta\in[0,1]}\Big\{|\lambda^*(\theta,\hbar(\theta))-\lambda^*(\theta,\hbar(0))+\lambda^*(\theta,\hbar(0))|+|\hbar_0|\\
&+\frac{1-\gamma}{\Psi(\gamma)}|\phi\Big(\theta,\mathsf{z}(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\mathsf{z}(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\mathsf{z}(\zeta))d\zeta\Big)\\
&-\phi\Big(\theta,\mathsf{z}(0),\int_0^\theta\kappa_1(\theta,\zeta,\mathsf{z}(0))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\mathsf{z}(0))d\zeta\Big)\\
&+\phi\Big(\theta,\mathsf{z}(0),\int_0^\theta\kappa_1(\theta,\zeta,\mathsf{z}(0))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\mathsf{z}(0))d\zeta\Big)|\\
&+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma|\phi\Big(\theta,\mathsf{z}(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\mathsf{z}(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\mathsf{z}(\zeta))d\zeta\Big)\\
&-\phi\Big(\theta,\mathsf{z}(0),\int_0^\theta\kappa_1(\theta,\zeta,\mathsf{z}(0))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\mathsf{z}(0))d\zeta\Big)\\
&+\phi\Big(\theta,\mathsf{z}(0),\int_0^\theta\kappa_1(\theta,\zeta,\mathsf{z}(0))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\mathsf{z}(0))d\zeta\Big)|\Big\}\\
&\leq \mu_1\|\hbar\|+\frac{1-\gamma}{\Psi(\gamma)}\Bigg(\mathcal{M}_1[|\hbar\|+\theta(\xi_1\|\hbar\|+\xi_2)+\Phi(\Omega_1\|\mathsf{z}\|+\Omega_2)]\Bigg)\\
&+\frac{1}{\Psi(\gamma)\Gamma(\gamma)}\Bigg(\mathcal{M}_1[|\hbar\|+\theta(\xi_1\|\hbar\|+\xi_2)+\Phi(\Omega_1\|\mathsf{z}\|+\Omega_2)]\Bigg)\\
&+\frac{1-\gamma}{\Psi(\gamma)}\mathcal{M}_2+\left(\frac{\gamma}{\Psi(\gamma)}\mathcal{M}_2\right)\frac{(1)^\gamma}{\gamma\Gamma(\gamma)}\\
&\leq \mu_1\|\hbar\|+\left[\left(\frac{1-\gamma}{\Psi(\gamma)}+\frac{1}{\gamma\Gamma(\gamma)}\right)\mathcal{M}_1[|1+\xi_1+\Phi(\Omega 1]\right]\|\mathsf{z}\|\\
&\leq \mu_2+\left[\left(\frac{1-\gamma}{\Psi(\gamma)}+\frac{1}{\gamma\Gamma(\gamma)}\right)\mathcal{M}_2[|1+\xi_2+\Phi(\Omega_2]\right]\\
&=\|z\|\mathcal{P}+\mathcal{Q}\leq\mathcal{P}\overline{r}+\mathcal{Q}\leq\overline{r}.
\end{aligned}
$$

2) $\mathcal{F}_1$ is the contraction on $\mathfrak{B}_{\overline{r}}$ For any $\hbar,\mathsf{z}\in\mathfrak{B}_{\overline{r}}$ by using $\mathfrak{L}_2$ and $\mathfrak{L}_5$.

$$
\begin{aligned}
\|\mathcal{F}_1\hbar(\theta)-\mathcal{F}_1\mathsf{z}(\theta)\| &= \sup_{\theta\in[0,1]}\Big\{\Big|\lambda^*(\theta,\hbar(\theta))+\hbar_0-\lambda^*(\theta,\mathsf{z}(\theta))-\hbar_0\Big|\Big\}\\
&\leq \sup_{\theta\in[0,1]}\Big\{|\lambda^*(\theta,\hbar(\theta))-\lambda^*(\theta,\mathsf{z}(\theta))||\Big\}\\
&\leq \mu_1\|\hbar-\mathsf{z}\|.
\end{aligned}
$$

As $\mu_1 < 1$. Thus $\mathcal{F}_1$ is a contraction operator.

3) We prove that $\mathcal{F}_1$ is completely continuous operator.

For completeness of $\mathcal{F}_2$, firstly we prove that $\mathcal{F}_2$ is continuous.
With $\lim_{n\longrightarrow\infty}\|\hbar_n-\hbar\|=0$, for any $\hbar_n,\hbar\in\mathfrak{B}_{\overline{r}}, n=1,2,..$
Then $\lim_{n\longrightarrow\infty}\hbar_n(\theta)=\hbar(\theta) \quad ,\forall\theta\in[0,1]$.
Therefore

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\phi\Big(\theta,\hbar_n(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar_n(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar_n(\zeta))d\zeta\Big)\\&=\phi\Big(\theta,\hbar(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big).\end{aligned}$$

Now, for $\theta\in[0,1]$

$$\begin{aligned}&\|\mathcal{F}_2\hbar_n(\theta)-\mathcal{F}_2\hbar(\theta)\|\\
=&\sup_{\theta\in[0,1]}\Big\{\Big|\frac{(1-\gamma)}{\Psi(\gamma)}\phi\Big(\theta,\hbar_n(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar_n(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar_n(\zeta))d\zeta\Big)\\
-&\phi\Big(\theta,\hbar(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\\
+&\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma\phi\Big(\theta,\hbar_n(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar_n(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar_n(\zeta))d\zeta\Big)\\
-&\phi\Big(\theta,\hbar(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\Big\}\\
\leq&\Big(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Psi(\gamma)\Gamma(\gamma)}\Big)\\
\times&\sup_{\theta\in[0,1]}\|\phi\Big(\theta,\hbar_n(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar_n(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar_n(\zeta))d\zeta\Big)\\
-&\phi\Big(\theta,\hbar(\theta),\int_0^\theta\kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\|.\end{aligned}$$

Thus $\quad\|\mathcal{F}_2\hbar_n(\theta)-\mathcal{F}_2\hbar(\theta)\|\longrightarrow 0$ as $n\longrightarrow\infty$.

Now, we prove that $\mathcal{F}_2$ is compact.

$$\begin{aligned}
\|\mathcal{F}_2\hbar(\theta)\| &= \sup_{\theta\in[0,1]}\Big\{\Big|\frac{(1-\gamma)}{\Psi(\gamma)}\phi\Big(\theta,\hbar(\theta),\int_0^{\theta}\kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\\
&+ \frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^{\gamma}\phi\Big(\theta,\hbar_n(\theta),\int_0^{\theta}\kappa_1(\theta,\zeta,\hbar_n(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta,\zeta,\hbar_n(\zeta))d\zeta\Big)\Big|\Big\}\\
&\leq \Big(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Psi(\gamma)\Gamma(\gamma)}\Big)\\
&\times \sup_{\theta\in[0,1]}\|\phi\Big(\theta,\hbar(\theta),\int_0^{\theta}\kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\|\\
&\leq \Big(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Psi(\gamma)\Gamma(\gamma)}\Big)\\
&\times \Big(\mathcal{M}_1[\|\hbar\|+\theta(\xi_1\|\hbar\|+\xi_2)+\Phi(\Omega_1\|\hbar\|+\Omega_2)]+\mathcal{M}_2\Big)\\
&\leq \Big(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Psi(\gamma)\Gamma(\gamma)}\Big)\\
&\times \Big[\mathcal{M}_1[1+\xi_1+\Phi(\Omega_1]\|\hbar\|+\mathcal{M}_2[1+\xi_1+\Phi\Omega_2]\Big]\\
&\leq \Big[(\mathcal{P}-\mu_1)+(\mathcal{Q}-\mu_2)\Big]<\infty,
\end{aligned}$$

which shows that $\mathcal{F}_2$ is bounded on $\mathfrak{B}_{\overline{r}}$.

Next, we prove that $\mathcal{F}_2$ is equicontinuous. For any $0<\theta_1<\theta_2<\theta<1$, we have

$$\begin{aligned}
&\|\mathcal{F}_2\hbar(\theta_2)-\mathcal{F}_2\hbar(\theta_1)\|\\
&= \sup_{\theta\in[0,1]}\Big\{\Big|\frac{(1-\gamma)}{\Psi(\gamma)}\phi\Big(\theta_2,\hbar(\theta_2),\int_0^{\theta_2}\kappa_1(\theta_2,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta_2,\zeta,\hbar(\zeta))d\zeta\Big)\\
&+ \frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^{\gamma}\phi\Big(\theta_2,\hbar(\theta_2),\int_0^{\theta_2}\kappa_1(\theta_2,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta_2,\zeta,\hbar(\zeta))d\zeta\Big)\\
&- \frac{(1-\gamma)}{\Psi(\gamma)}\phi\Big(\theta_1,\hbar(\theta_1),\int_0^{\theta_1}\kappa_1(\theta_1,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta_1,\zeta,\hbar(\zeta))d\zeta\Big)\\
&- \frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^{\gamma}\phi\Big(\theta_1,\hbar(\theta_1),\int_0^{\theta_1}\kappa_1(\theta_1,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta_1,\zeta,\hbar(\zeta))d\zeta\Big)\Big|\Big\}\\
&\leq \frac{(1-\gamma)}{\Psi(\gamma)}\|\phi\Big(\theta_2,\hbar(\theta_2),\int_0^{\theta_2}\kappa_1(\theta_2,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta_2,\zeta,\hbar(\zeta))d\zeta\Big)\\
&- \phi\Big(\theta_1,\hbar(\theta_1),\int_0^{\theta_1}\kappa_1(\theta_1,\zeta,\hbar(\zeta))d\zeta,\int_0^{\Phi}\kappa_2(\theta_1,\zeta,\hbar(\zeta))d\zeta\Big)\|
\end{aligned}$$

$$
\begin{aligned}
+ & \quad \frac{\gamma}{\Psi(\gamma)} {}_0\mathcal{I}^{\gamma} \|\phi\Big(\theta_2, \hbar(\theta_2), \int_0^{\theta_2} \kappa_1(\theta_2, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta_2, \zeta, \hbar(\zeta))d\zeta\Big) \\
- & \quad \phi\Big(\theta_1, \hbar(\theta_1), \int_0^{\theta_1} \kappa_1(\theta_1, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta_1, \zeta, \hbar(\zeta))d\zeta\Big)\| \\
\leq & \quad \frac{(1-\gamma)}{\Psi(\gamma)} \Bigg(\mathcal{M}_1 \Bigg[\Bigg\|\Big(\theta, \hbar(\theta), \int_0^{\theta} \kappa_1(\theta, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta, \hbar(\zeta))d\zeta\Big)\Bigg\| \\
+ & \quad \theta\Bigg(\xi_1 \Bigg\|\Big(\theta, \hbar(\theta), \int_0^{\theta} \kappa_1(\theta, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta, \zeta, \hbar(\zeta))d\zeta\Big)\Bigg\|\Bigg) \\
+ & \quad \Phi\Bigg(\Omega_1 \Bigg\|\Big(\theta, \hbar(\theta), \int_0^{\theta} \kappa_1(\theta, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta, \zeta, \hbar(\zeta))d\zeta\Big)\Bigg\|\Bigg)\Bigg]\Bigg) \\
+ & \quad \frac{(\gamma)}{\Psi(\gamma)} \Bigg(\mathcal{M}_1 \Bigg[\Bigg\|\Big(\theta, \hbar(\theta), \int_0^{\theta} \kappa_1(\theta, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta, \zeta, \hbar(\zeta))d\zeta\Big)\Bigg\| \\
+ & \quad \theta\Bigg(\xi_1 \Bigg\|\Big(\theta, \hbar(\theta), \int_0^{\theta} \kappa_1(\theta, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta, \zeta, \mathsf{z}(\zeta))d\zeta\Big)\Bigg\|\Bigg) \\
+ & \quad \Phi\Bigg(\Omega_1 \Bigg\|\Big(\theta, \hbar(\theta), \int_0^{\theta} \kappa_1(\theta, \zeta, \hbar(\zeta))d\zeta, \int_0^{\Phi} \kappa_2(\theta, \zeta, \hbar(\zeta))d\zeta\Big)\Bigg\|\Bigg)\Bigg]\Bigg) \frac{(\theta_2-\theta_1)^{\gamma}}{\gamma\Gamma(\gamma)} \\
\leq & \quad \rho(\theta_2-\theta_1) + \frac{\gamma}{\Psi(\gamma)}\rho(\theta_2-\theta_1)\frac{(\theta_2-\theta_1)^{\gamma}}{\gamma\Gamma(\gamma)} \\
\leq & \quad \rho\Bigg(\frac{(1-\gamma)}{\Psi(\gamma)} + \frac{(\theta_2-\theta_1)^{\gamma}}{\gamma\Gamma(\gamma)}\Bigg)(\theta_2-\theta_1)
\end{aligned}
$$

$\|\mathcal{F}_2\hbar(\theta_2) - \mathcal{F}_2\hbar(\theta_1)\| \longrightarrow 0 \quad \text{as} \quad \theta_2 \longrightarrow \theta_1$

Consequently, $\mathcal{F}_2$ is equicontinuous operator on$\mathfrak{B}_{\bar{r}}$. Therefore by the Arzela-Ascoli theorem $\mathcal{F}_2$ is relatively compact on $\mathfrak{B}_{\bar{r}}$. Hence by the Theorem 3 $\mathcal{F}$ has at least one fixed point. Thus $\hbar$ is that fixed point of $\mathcal{F}$. Consequently, $\hbar$ is solution of the system (1).

4. Uniqueness Result

Theorem 6. *Assume that $\mathfrak{L}_1 - \mathfrak{L}_5$ are satisfied. If*

$$\phi^*\Big(0, \hbar(0).0. \int_0^{\Phi} \varphi(0, \zeta, \hbar(\zeta))d\zeta\Big) = \lambda^*(0, \hbar(0)) = 0$$

and

$$\mu_1 + \mathcal{M}_1(1+\xi_1+\Phi\Omega_1)\left(\frac{(1-\gamma)}{\Psi(\gamma)} + \frac{1}{\Gamma(\gamma)\Psi(\gamma)}\right) \leq 1$$

Then problem (1) *has unique solution on* $[0,1]$.

Proof. For any $\hbar \in \mathfrak{B}_{\overline{r}}$

$$\begin{aligned}
\|\mathcal{F}\hbar\| = \sup_{\theta\in[0,1]}\Bigg\{\Big|\lambda^*(\theta,\hbar(\theta)) + \hbar_0 \\
+\frac{1-\gamma}{\Psi(\gamma)}\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big) \\
+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)\Big|\Bigg\} \\
\leq \sup_{\theta\in[0,1]}\Bigg\{|\lambda^*(\theta,\hbar(\theta))| + |\hbar_0| \\
+\frac{1-\gamma}{\Psi(\gamma)}|\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)| \\
+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma|\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big)|\Bigg\} \\
\leq \sup_{\theta\in[0,1]}\Bigg\{|\lambda^*(\theta,\hbar(\theta)) - \lambda^*(\theta,\hbar(0)) + \lambda^*(\theta,\hbar(0))| + |\hbar_0| \\
+\frac{1-\gamma}{\Psi(\gamma)}|\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar z(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big) \\
-\phi\Big(\theta,\hbar(0),\int_0^\theta \kappa_1(\theta,\zeta,(\theta,\zeta,\hbar(0))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,(\theta,\zeta,\hbar(0))d\zeta\Big) \\
+\phi\Big(\theta,\hbar(0),\int_0^\theta \kappa_1(\theta,\zeta,\hbar((\theta,\zeta,\hbar(0))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,(\theta,\zeta,\hbar(0))d\zeta\Big)| \\
+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma|\phi\Big(\theta,\hbar(\theta),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(\zeta))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(\zeta))d\zeta\Big) \\
-\phi\Big(\theta,\hbar(0),\int_0^\theta \kappa_1(\theta,\zeta,(\theta,\zeta,\hbar(0))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(0))d\zeta\Big) \\
+\phi\Big(\theta,\hbar(0),\int_0^\theta \kappa_1(\theta,\zeta,\hbar(0))d\zeta,\int_0^\Phi \kappa_2(\theta,\zeta,\hbar(0))d\zeta\Big)|\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu_1\|\hbar\| + \frac{1-\gamma}{\Psi(\gamma)}\bigg(\mathcal{M}_1[|\hbar\| + \theta(\xi_1\|\hbar\| + \xi_2) + \Phi(\Omega_1\|\hbar\| + \Omega_2)]\bigg)\\
&+\frac{1}{\Psi(\gamma)\Gamma(\gamma)}\bigg(\mathcal{M}_1[|\hbar\| + \theta(\xi_1\|\hbar\| + \xi_2) + \Phi(\Omega_1\|\hbar\| + \Omega_2)]\bigg)\frac{1}{\gamma\Gamma(\gamma)}\\
&+\frac{1-\gamma}{\Psi(\gamma)}\mathcal{M}_2 + \bigg(\frac{\gamma}{\Psi(\gamma)}\mathcal{M}_2\bigg)\frac{1}{\gamma\Gamma(\gamma)}\\
&\leq \mu_1\|\hbar\| + \bigg[\bigg(\frac{1-\gamma}{\Psi(\gamma)} + \frac{1}{\gamma\Gamma(\gamma)}\bigg)\mathcal{M}_1[|1 + \xi_1 + \Phi(\Omega_1]\bigg]\|\mathsf{y}\|\\
&+\mu_2 + \bigg[\bigg(\frac{1-\gamma}{\Psi(\gamma)} + \frac{1}{\gamma\Gamma(\gamma)}\bigg)\mathcal{M}_2[|1 + \xi_2 + \Phi(\Omega_2]\bigg]\\
&= \|\hbar\|\mathcal{P} + \mathcal{Q} \leq \mathcal{P}\overline{r} + \mathcal{Q} \leq \overline{r}
\end{aligned}
$$

which shows that $\mathcal{F}$ is bounded on $\mathfrak{B}_{\overline{r}}$: Now to prove uniqueness

$$
\begin{aligned}
&\|\mathcal{F}\hbar_1(\theta) - \mathcal{F}\hbar_2(\theta)\| = \sup_{\theta\in[0,1]}\bigg\{\bigg|\lambda^*(\theta, \hbar_1(\theta)) + \hbar_0\\
&+\frac{1-\gamma}{\Psi(\gamma)}\phi\Big(\theta, \hbar_1(\theta), \int_0^\theta \kappa_1(\theta,\zeta,\hbar_1(\zeta))d\zeta, \int_0^\Phi \kappa_2(\theta,\zeta,\hbar_1(\zeta))d\zeta\Big)\\
&+\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma\phi\Big(\theta, \hbar_1(\theta), \int_0^\theta \kappa_1(\theta,\zeta,\hbar_1(\zeta))d\zeta, \int_0^\Phi \kappa_2(\theta,\zeta,\hbar_1(\zeta))d\zeta\Big)\bigg|\bigg\}\\
&-\lambda^*(\theta, \hbar_2(\theta)) - \hbar_0 - \frac{1-\gamma}{\Psi(\gamma)}\phi\Big(\theta, \hbar_2(\theta), \int_0^\theta \kappa_1(\theta,\zeta,\hbar_2(\zeta))d\zeta, \int_0^\Phi \kappa_2(\theta,\zeta,\hbar_2(\zeta))d\zeta\Big)\\
&-\frac{\gamma}{\Psi(\gamma)}{}_0\mathcal{I}^\gamma\phi\Big(\theta, \hbar_2(\theta), \int_0^\theta \kappa_1(\theta,\zeta,\hbar_2(\zeta))d\zeta, \int_0^\Phi \kappa_2(\theta,\zeta,\hbar_2(\zeta))d\zeta\Big)\bigg|\bigg\}\\
&\leq \mu_1\|\hbar_1 - \hbar_2\|\\
&+\frac{1-\gamma}{\Psi(\gamma)}\bigg(\mathcal{M}_1[\|\hbar_1(\theta) - \hbar_2(\theta)\| + \theta(\xi_1\|\hbar_1(\theta) - \hbar_2(\theta)\|) + \Phi(\Omega_1[\|\hbar_1(\theta) - \hbar_2(\theta)\|)]\bigg)\\
&+\frac{\gamma}{\Psi(\gamma)}\bigg(\mathcal{M}_1[\|\hbar_1(\theta) - \hbar_2(\theta)\| + \theta(\xi_1\|\hbar_1(\theta) - \hbar_2(\theta)\|) + \Phi(\Omega_1[\|\hbar_1(\theta) - \hbar_2(\theta)\|)]\bigg)\frac{(1)^\gamma}{\gamma\Gamma(\gamma)}\\
&\leq \Big[\mu_1 + \bigg(\frac{1-\gamma}{\Psi(\gamma)} + \frac{1}{\gamma\Gamma(\gamma)}\bigg)\mathcal{M}_1[1 + \xi_1 + \Phi\Omega_1]\Big]\|\hbar_1 - \hbar_2\|
\end{aligned}
$$

Since

$$\mu_1 + \left(\frac{1-\gamma}{\Psi(\gamma)} + \frac{1}{\gamma\Gamma(\gamma)}\right)\mathcal{M}_1[1+\xi_1+\Phi\Omega_1] \leq 1$$

Consequently, $\mathcal{F}$ is a contraction mapping. Therefore by the Banach contraction principle, the operator $\mathcal{F}$ has a unique fixed point. Hence the system (1) has a unique solution.

5. Examples

This section of the article produce examples related to EU of solutions of the discussed problem.

Example 1. *Consider the following ABC-fractional initial value problem:*

$$\begin{aligned} {}^{\mathcal{ABC}}_{0}\mathcal{D}^{\frac{3}{4}}\left[\hbar(\theta) + \frac{1}{100}e^{2\Phi}\hbar(\theta)\right] &= \frac{\zeta^4+2}{35}\Big(\frac{1}{45}\int_0^\theta (\theta^4+2)\frac{|\hbar(\zeta)|}{|1+\hbar(\zeta)|}d\zeta \\ &+ \frac{1}{65}\int_0^{\frac{3}{4}} e^{\theta^2+\zeta}\sin(\hbar(\zeta)))d\zeta\Big), \qquad \theta\in[0,1], \end{aligned} \tag{5}$$

$$\hbar(0) = \hbar_0$$

where $\gamma = \frac{3}{4}$ *and*

$$\lambda^*(\theta,\hbar(\theta)) = \frac{1}{100}e^{2\Phi}\hbar(\theta)$$

$$\kappa_1(\theta,\zeta,\hbar(\zeta)) = \frac{1}{45}(\theta^4+2)\frac{|\hbar(\zeta)|}{|1+\hbar(\zeta)|}$$

$$\kappa_2(\theta,\zeta,\hbar(\zeta)) = \frac{1}{65}e^{\theta^2+\zeta}\sin(\hbar(\zeta)))$$

Now

$$\begin{aligned} |\lambda^*(\theta,\hbar(\theta)) - \lambda^*(\theta,\hbar_1(\theta))| &= |\frac{1}{100}e^{2\Phi}\hbar(\theta) - \frac{1}{100}e^{2\Phi}\hbar_1(\theta)| \\ &\leq \frac{e^{2\Phi}}{100}\|\hbar-\hbar_1\| \end{aligned}$$

$$\begin{aligned} |\kappa_1(\theta,\zeta,\hbar(\zeta)) - \kappa_1(\theta,\zeta,\hbar_1(\zeta))| &= |\frac{1}{45}(\theta^4+2)\frac{|\hbar(\zeta)|}{|1+\hbar(\zeta)|} - \frac{1}{45}(\theta^4+2)\frac{|\hbar_1(\zeta)|}{|1+\hbar_1(\zeta)|} \\ &\leq \frac{1}{15}\|\hbar-\hbar_1\| \end{aligned}$$

$$
\begin{aligned}
|\kappa_2(\theta,\zeta,\hbar(\zeta))-\kappa_2(\theta,\zeta,\hbar_1(\zeta))| &= |\frac{1}{65}e^{\theta^2+\zeta}\sin(\hbar(\zeta)))-\frac{1}{65}e^{\theta^2+\zeta}\sin(\hbar_1(\zeta)))| \\
&\le \frac{e^2}{65}\|\hbar-\hbar_1\|
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg|\frac{\zeta^4+2}{35}\Big(\frac{1}{45}\int_0^\theta(\theta^4+2)\frac{|\hbar(\zeta)|}{|1+\hbar(\zeta)|}d\zeta+\frac{1}{65}\int_0^\Phi e^{\theta^2+\zeta}\sin(\hbar(\zeta)))d\zeta\Big) \\
&-\frac{\zeta^4+2}{35}\Big(\frac{1}{45}\int_0^\theta(\theta^4+2)\frac{|\hbar_1(\zeta)|}{|1+\hbar_1(\zeta)|}d\zeta+\frac{1}{65}\int_0^\Phi e^{\theta^2+\zeta}\sin(\hbar_1(\zeta)))d\zeta\Big)\Bigg| \\
&\le \frac{3}{35}\left(\frac{1}{15}+\frac{e(e^\Phi-1)}{65}\right)\|\hbar-\hbar_1\|,
\end{aligned}
$$

thus assumptions $(\mathfrak{L}_1);(\mathfrak{L}_2);(\mathfrak{L}_3)$ *and* $(\mathfrak{L}_4)$ *are hold true. Hence, Consequently,*

$$\mu_1=\frac{e^{2\Phi}}{100},\quad \Omega_1=\frac{e^2}{65},\quad \xi_1=\frac{1}{15},\quad \mathcal{M}_1=0.00972,\quad \Psi(\frac{3}{4})=1$$

the Theorem 5 implies that the system (5) *has a solution. In addition,*

$$\mu_1+\mathcal{M}_1(1+\xi_1+\Phi\Omega_1)\left(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Gamma(\gamma)\Psi(\gamma)}\right)=0.05676<1$$

hence using the Theorem 6, then the system (5) *has a unique solution.*

Example 2. *Consider the following ABC-fractional initial value problem:*

$$
\begin{aligned}
{}_0^{\mathcal{ABC}}\mathcal{D}^{\frac{3}{4}}\Big[\hbar(\theta)+\frac{1}{150+\theta^4}\hbar(\theta)\Big] &= \frac{1}{\theta^4+30}\Big(\frac{1}{60}\int_0^\theta(\theta^4+\zeta^4)\hbar(\zeta)d\zeta \qquad (6)\\
&+\ \frac{1}{20}\int_0^{\frac{\pi}{4}}(\theta^4\cos(\zeta)\hbar(\zeta)d\zeta\Big), \qquad \theta\in[0,1],\\
&\hbar(0)=\hbar_0, \qquad (7)
\end{aligned}
$$

where $\gamma=\frac{3}{4}$, *and*

$$\lambda^*(\theta,\hbar(\theta))=\frac{1}{150+\theta^4}\hbar(\theta)$$

$$\kappa_1(\theta,\zeta,\hbar(\zeta))=\frac{1}{60}(\theta^4+\zeta^4)\hbar(\zeta)$$

$$\kappa_2(\theta,\zeta,\hbar(\zeta))=\frac{1}{20}(\theta^4\cos(\zeta))\hbar(\zeta)$$

Now

$$\begin{aligned}|\lambda^*(\theta,\hbar(\theta))-\lambda^*(\theta,\hbar_1(\theta))| &= |\frac{1}{150+\theta^4}\hbar(\theta)-\frac{1}{150+\theta^4}\hbar_1(\theta)| \\ &\leq \frac{1}{150}\|\hbar-\hbar_1\|\end{aligned}$$

$$\begin{aligned}|\kappa_1(\theta,\zeta,\hbar(\zeta))-\kappa_1(\theta,\zeta,\hbar_1(\zeta))| &= |\frac{1}{60}(\theta^4+\zeta^4)\hbar(\zeta)-\frac{1}{60}(\theta^4+\zeta^4)\hbar_1(\zeta)| \\ &\leq \frac{1}{30}\|\hbar-\hbar_1\|\end{aligned}$$

$$\begin{aligned}|\kappa_2(\theta,\zeta,\hbar(\zeta))-\kappa_2(\theta,\zeta,\hbar_1(\zeta))| &= |\frac{1}{20}(\theta^4\cos(\zeta))\hbar(\zeta)-\frac{1}{20}(\theta^4\cos(\zeta))\hbar_1(\zeta)| \\ &\leq \frac{1}{20}\|\hbar-\hbar_1\|\end{aligned}$$

$$\begin{aligned}&\Bigg|\frac{1}{\theta^4+30}\Big(\frac{1}{60}\int_0^{\theta}(\theta^4+\zeta^4)\hbar(\zeta)d\zeta+\frac{1}{20}\int_0^{\Phi}\theta^4\cos(\zeta)\hbar(\zeta)d\zeta\Big) \\ &-\frac{1}{\theta^4+30}\Big(\frac{1}{60}\int_0^{\theta}(\theta^4+\zeta^4)\hbar_1(\zeta)d\zeta+\frac{1}{20}\int_0^{\Phi}\theta^4\cos(\zeta)\hbar_1(\zeta)d\zeta\Big)\Bigg| \\ &\leq \frac{1}{10}\left(\frac{1}{50}+\frac{\sin(\Phi)}{20}\right)\|\hbar-\hbar_1\|,\end{aligned}$$

thus assumptions $(\mathfrak{L}_1);(\mathfrak{L}_2);(\mathfrak{L}_3) and (\mathfrak{L}_4)$ *are hold true. Hence, Consequently,*

$$\mu_1=\frac{1}{150},\quad \Omega_1=\frac{1}{20},\quad \xi_1=\frac{1}{30},\quad \mathcal{M}_1=0.002615,\quad \Psi(\frac{3}{4})=1$$

the Theorem 5 implies that the system (6) *has a solution. In addition,*

$$\mu_1+\mathcal{M}_1(1+\xi_1+\Phi\Omega_1)\left(\frac{(1-\gamma)}{\Psi(\gamma)}+\frac{1}{\Gamma(\gamma)\Psi(\gamma)}\right)=0.00904<1,$$

hence using the Theorem 6, the system (6) *has a unique solution.*

6. Conclusion

In this research, we investigate the existence and uniqueness of fractional order integro-differential equations in Banach spaces with initial conditions and CAB-derivative of fractional order. By using fractional calculus, Banach's contraction principle, and other techniques, existence and uniqueness results of solutions are established, the theorem of Krasnoselskii fixed points. A few applications are provided to highlight the key findings.

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