Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 72/2022
pp. 79-93
doi: 10.17114/j.aua.2022.72.06

SOME RESULTS ON A SET OF λ-PSEUDO-STARLIKE FUNCTIONS INVOLVING A GENERALIZED RUSCHEWEYH OPERATOR

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Abstract. The set of starlike functions and in particular, the set of λ-pseudo-starlike functions has gained the attention of researchers in recent times. In this paper, we introduce a generalized set of λ-pseudo-starlike denoted by $\mathcal{S}_\lambda^\star(n,\sigma,\beta)$ and investigate some of its properties. Some of these properties include some conditions for univalence, integral representation, and estimates of some functionals: Fekete-Szegö and Hankel determinants. Finally, some remarks on some subsets of set $\mathcal{S}_\lambda^\star(n,\sigma,\beta)$ are discussed.

2010 *Mathematics Subject Classification*: 30C45, 30C50.

Keywords: Analytic function, λ-pseudo-starlike function, Babalola operator, Bazilevič function, convolution, Fekete-Szegö functional and Hankel determinants.

1. Background to the study

In this paper, we let $\mathcal{A}$ represent the set of analytic functions having Taylor's series representation

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (|z| < 1) \tag{1}$$

normalized such that $g(0) = 0 = g'(0) - 1$. We also represent by $\mathcal{S}$, a subset of $\mathcal{A}$, the set of analytic-univalent functions in $|z| < 1$. In the sequel, we let $\mathcal{S}^\star(\beta)$, a subset of $\mathcal{S}$, represent the set of starlike functions of order $\beta \in [0,1)$ such that

$$\mathfrak{Re}\frac{zg'(z)}{g(z)} > \beta \quad (|z| < 1).$$

If $\beta = 0$, then $\mathcal{S}^\star(0) \equiv \mathcal{S}^\star$ is simply called the set of starlike functions. The set of starlike functions is well-known and has been studied in various forms as evident in many available literature. In particular, the set of functions that satisfy the geometric condition

$$\mathfrak{Re}\frac{z(g'(z))^{\lambda}}{g(z)} > \beta \quad (\lambda \geqq 1,\ \beta \in [0,1),\ |z| < 1) \tag{2}$$

is called the set of λ-*pseudo-starlike functions* which was introduced by Babalola [4] and has been studied in different forms by a number of authors. An analytic function of the form

$$p_{\beta}(z) = 1 + \sum_{k=1}^{\infty}(1-\beta)p_k z^k \quad (\beta \in [0,1),\ |z| < 1)$$

normalized such that $p_{\beta}(0) = 1$ and $p_{\beta}(z) > \beta$ is said to be a function in the set $\mathcal{P}(\beta)$ of Caratheódory functions of order β. If $\beta = 0$, then set $\mathcal{P}(0) \equiv \mathcal{P}$ is simply called the set of Caratheódory functions whose form is

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (|z| < 1). \tag{3}$$

Another set of functions of interest in this work is the set $\mathcal{B}(\alpha, \eta, g, p)$ of Bazilevič functions introduced in [6] and having the integral form

$$b(z) = \frac{\alpha}{1+\eta^2}\int_0^z (p(z) - i\eta)\zeta^{-\left(1+\frac{i\alpha\eta}{1+\eta^2}\right)} g(\zeta)^{\frac{\alpha}{1+\eta^2}}\, d\zeta$$

for real numbers $\alpha > 0$ and η; $g(z) \in \mathcal{S}^{\star}$ and $p(z)$ in (3). Much is unknown of the set $\mathcal{B}(\alpha, \eta, g, p)$ except that it was proved to be the largest known subset of the set $\mathcal{S}$. However, the subset $\mathcal{B}(\alpha, 0, z, p) \equiv \mathcal{B}(\alpha)$ such that

$$\mathfrak{Re}\frac{g'(z)g(z)^{\alpha-1}}{z^{\alpha-1}} > 0 \quad (\alpha > 0,\ |z| < 1)$$

was studied by Singh [16]. Going by the declaration in [4], let $\mathcal{B}(\alpha, \beta)$ be the set of functions that satisfy the condition

$$\mathfrak{Re}\frac{g'(z)g(z)^{\alpha-1}}{z^{\alpha-1}} > \beta \quad (\alpha > 0,\ \beta \in [0,1),\ |z| < 1). \tag{4}$$

This set is known as the set of Bazilevič functions of type α and order β.

Let "$\star$" represent the *Hadamard product* or *convolution*. The convolution of two analytic functions $g(z)$ in (1) and $G(z) = z + \sum_{k=2}^{\infty} A_k z^k$ is defined by

$$(g \star G)(z) = z + \sum_{k=2}^{\infty} (a_k \times A_k) z^k.$$

Using the concept of convolution, Babalola [2, 3] defined two convolution operators $\varDelta_\sigma^n : \mathcal{A} \longrightarrow \mathcal{A}$ and $\nabla_\sigma^n : \mathcal{A} \longrightarrow \mathcal{A}$ as follows.

Definition 1. *For a fixed real parameter $\sigma \geqq n+1$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for $g(z) \in \mathcal{A}$, define the operator $\varDelta_\sigma^n : \mathcal{A} \longrightarrow \mathcal{A}$ by*

$$\varDelta_\sigma^n g(z) = (\tau_{\sigma,0} \star \tau_{\sigma,n}^{(-1)} \star g)(z) \tag{5}$$

and as a right inverse operator, the author gave

$$\nabla_\sigma^n g(z) = (\tau_{\sigma,0}^{(-1)} \star \tau_{\sigma,n} \star g)(z) \tag{6}$$

where

$$\tau_{\sigma,n}(z) = \frac{z}{(1-z)^{\sigma-n+1}} \quad (|z| < 1)$$

and $\tau_{\sigma,n}^{(-1)}(z)$ is such that

$$(\tau_{\sigma,n} \star \tau_{\sigma,n}^{(-1)})(z) = z + \sum_{k=2}^{\infty} z^k = \frac{z}{1-z} \quad (|z| < 1).$$

(5) *can be simplified as*

$$\varDelta_\sigma^n g(z) = z + \sum_{k=2}^{\infty} \left(\frac{(\sigma+k-1)!}{\sigma!} \times \frac{(\sigma-n)!}{(\sigma+k-n-1)!} \right) a_k z^k \tag{7}$$

or condensed as

$$\varDelta_\sigma^n g(z) = z + \sum_{k=2}^{\infty} \chi_k(n,\sigma) a_k z^k \quad (|z| < 1)$$

where

$$\chi_k \equiv \chi_k(n,\sigma) = \frac{(\sigma+k-1)!}{\sigma!} \times \frac{(\sigma-n)!}{(\sigma+k-n-1)!} \tag{8}$$

so that (6) *becomes*

$$\nabla_\sigma^n g(z) = z + \sum_{k=2}^{\infty} \chi_k^{-1} a_k z^k. \tag{9}$$

Remark 1 ([2, 3]). *The following relations are valid from* (5) *and* (6) *(or* (7) *and* (9)*).*

1. $\Delta_\sigma^0 g(z) = \Delta_0^0 g(z) = \nabla_\sigma^0 g(z) = \nabla_0^0 g(z) = g(z)$ *in* (1).
2. $\Delta_1^1 g(z) = \nabla_1^1 g(z) = zg'(z) = \mathcal{R}^1 g(z)$, *the Ruscheweyh operator of order 1 in [15].*
3. $\Delta_n^n g(z) = \mathcal{R}^n g(z)$, *the Ruscheweyh operator of order* n *in [15].*
4. $\Delta_0^{-n} g(z) = \mathcal{N}^n g(z)$, *the Noor operator in [13].*
5. $\nabla_n^n g(z) = \mathcal{N}^n g(z)$, *the Noor operator in [13].*
6. $\nabla_0^{-n} g(z) = \mathcal{R}^n g(z)$, *the Ruscheweyh operator in [13].*
7. *And note that*
$$\Delta_\sigma^n(\nabla_\sigma^n g(z)) = \nabla_\sigma^n(\Delta_\sigma^n g(z)) = g(z). \tag{10}$$

2. A new set of analytic functions

Henceforth, it shall means that $\sigma \geqq n+1$ is fixed, $n \in \mathbb{N}_0$, $\lambda \geqq 1$, $\beta \in [0,1)$ and $g(z) \in \mathcal{A}$. The new set of analytic functions studied in this paper is defined as follows. A function $g \in \mathcal{A}$ is said to be in the set $\mathcal{S}_\lambda^\star(n,\sigma,\beta)$ if, and only if,

$$\mathfrak{Re}\frac{z((\Delta_\sigma^n g(z))')^\lambda}{\Delta_\sigma^n g(z)} > \beta \quad (|z|<1) \tag{11}$$

where all powers are regarded as principal determinations only.

Remark 2. *We note the following subsets of* $\mathcal{S}_\lambda^\star(n,\sigma,\beta)$.

1. $\mathcal{S}_1^\star(0,\sigma,\beta) = \mathcal{S}_1^\star(0,0,\beta) = \mathcal{S}^\star(\beta)$ *is the set of starlike functions of order* β*, see [9].*
2. $\mathcal{S}_1^\star(0,\sigma,0) = \mathcal{S}_1^\star(0,0,0) = \mathcal{S}^\star$ *is the set of starlike functions, see [9].*
3. $\mathcal{S}_\lambda^\star(0,\sigma,\beta) = \mathcal{S}_\lambda^\star(0,0,\beta) = \mathcal{S}_\lambda^\star(\beta)$ *is the set of* λ*-pseudo-starlike functions, see [4].*
4. $\mathcal{S}_2^\star(0,\sigma,\beta) = \mathcal{S}_2^\star(0,0,\beta) = \mathcal{S}_2^\star(\beta)$ *is the set of functions satisfying the condition*
$$\mathfrak{Re}\left(g'(z)\frac{zg'(z)}{g(z)}\right) > \beta \quad (\beta \in [0,1),\ |z|<1). \tag{12}$$
Note that the expression in brackets is the product combination of geometric expressions for bounded-turning functions and starlike functions.

5. *Suppose $\lambda = n = \beta = 0$, we note that* (11) *will reduce to the reciprocal of the geometric expression $\mathfrak{Re}\,(z/g(z)) > 0$ of a set of functions studied by Yamaguchi [17].*

3. Initial lemmas

The following lemmas are necessary to proof our results.

Lemma 1 ([7, 9])**.** *If $p(z) \in \mathcal{P}$, then $|p_k| \leqq 2 \quad (k \in \mathbb{N})$.*
The result is sharp for the Möbius function $p_0(z) = (1+z)/(1-z)$.

Lemma 2 ([5])**.** *Let $p(z) \in \mathcal{P}$ and $u \in \mathbb{R}$, then*

$$\left|p_2 - u\frac{p_1^2}{2}\right| \leqq \begin{cases} 2(1-u) & for \quad u \leqq 0, \\ 2 & for \quad 0 \leqq u \leqq 2, \\ 2(u-1) & for \quad u \geqq 2. \end{cases}$$

Lemma 3 ([5])**.** *If $p(z) \in \mathcal{P}$ and $\mu \in \mathbb{C}$, then*

$$\left|p_2 - \mu\frac{p_1^2}{2}\right| \leqq 2\max\{1, |1-\mu|\}.$$

Lemma 4 ([4])**.** *Let $p_\beta(z) \in \mathcal{P}(\beta)$, then for $m \in [0,1]$, $h(z) = (p_\beta(z))^m$ implies that $h(0) = 1$ and $\mathfrak{Re}\, h(z) > \beta^m$.*

Lemma 5 ([4])**.** *Let $p(z)$ be analytic such that $p(0) = 1$. If*

$$\mathfrak{Re}\left(z\frac{p'(z)}{p(z)} + 1\right) > \frac{3\beta - 1}{2\beta} \quad (|z| < 1),$$

then for $m = (\beta - 1)/\beta$ ($\beta \in [1/2, 1)$), $\mathfrak{Re}\, p(z) > 2^m$. The constant 2^m is the best possible.

Lemma 6 ([10])**.** *If $p(z) \in \mathcal{P}$ and $i, j \in \mathbb{N}$, then*

$$|p_{i+j} - up_ip_j| \leqq \begin{cases} 2 & for \quad 0 \leqq u \leqq 1 \\ 2|2u-1| & \text{elsewhere.} \end{cases}$$

4. Main results

4.1. Basic properties

Theorem 7. *For $p_\beta(z) \in \mathcal{P}(\beta)$,*

$$\mathcal{S}^\star_\lambda(n, \sigma, \beta) \subset \mathcal{B}\left(1 - \frac{1}{\lambda},\ \beta^{1/\lambda}\right).$$

This implies that functions in set $\mathcal{S}_\lambda^\star(n,\sigma,\beta)$ are Bazilevič functions of type $1-\frac{1}{\lambda}$ and order $\beta^{1/\lambda}$.

Proof. Let $g(z)\in\mathcal{S}_\lambda^\star(n,\sigma,\beta)$, then for $p_\beta(z)\in\mathcal{P}(\beta)$, (11) can be written as

$$\frac{z((\varDelta_\sigma^n g(z))')^\lambda}{\varDelta_\sigma^n g(z)} = \left(\frac{z^{1/\lambda}(\varDelta_\sigma^n g(z))'}{(\varDelta_\sigma^n g(z))^{1/\lambda}}\right)^\lambda = p_\beta(z) \tag{13}$$

which means that

$$\frac{z^{1/\lambda}(\varDelta_\sigma^n g(z))'}{(\varDelta_\sigma^n g(z))^{1/\lambda}} = (p_\beta(z))^{1/\lambda}.$$

Now applying Lemma 4 means that

$$\mathfrak{Re}\frac{z^{1/\lambda}(\varDelta_\sigma^n g(z))'}{(\varDelta_\sigma^n g(z))^{1/\lambda}} > \beta^{1/\lambda}. \tag{14}$$

Letting $1-\alpha=\frac{1}{\lambda}$ and comparing (14) with (4) means that $g(z)\in\mathcal{B}\left(1-\frac{1}{\lambda},\ \beta^{1/\lambda}\right)$ as required.

Theorem 8. *Let $g(z)\in\mathcal{A}$, then if*

$$\mathfrak{Re}\left(\frac{\lambda z(\varDelta_\sigma^n g(z))''}{(\varDelta_\sigma^n g(z))'} - \frac{z(\varDelta_\sigma^n g(z))'}{\varDelta_\sigma^n g(z)}\right) > \frac{-(1+\beta)}{2\beta} \quad (|z|<1)$$

holds true, then $\varDelta_\sigma^n g(z)\in\mathcal{S}_\lambda^\star(n,\sigma,\beta)$.

Proof. Letting $p(z)=p_\beta(z)$ and taking the logarithmic derivative of (13), then we can write

$$z\frac{p'(z)}{p(z)} = 1 + \frac{\lambda z(\varDelta_\sigma^n g(z))''}{(\varDelta_\sigma^n g(z))'} - \frac{z(\varDelta_\sigma^n g(z))'}{\varDelta_\sigma^n g(z)}$$

so that

$$\mathfrak{Re}\left(z\frac{p'(z)}{p(z)}+1\right) = \mathfrak{Re}\left(2 + \frac{\lambda z(\varDelta_\sigma^n g(z))''}{(\varDelta_\sigma^n g(z))'} - \frac{z(\varDelta_\sigma^n g(z))'}{\varDelta_\sigma^n g(z)}\right)$$

and by conditions of Lemma 5 we have that

$$\mathfrak{Re}\left(z\frac{p'(z)}{p(z)}+1\right) = \mathfrak{Re}\left(\frac{\lambda z(\varDelta_\sigma^n g(z))''}{(\varDelta_\sigma^n g(z))'} - \frac{z(\varDelta_\sigma^n g(z))'}{\varDelta_\sigma^n g(z)}\right) > \frac{-(1+\beta)}{2\beta}$$

which implies that

$$\mathfrak{Re}\left(\frac{z((\varDelta_\sigma^n g(z))')^\lambda}{\varDelta_\sigma^n g(z)}\right) > 2^m$$

where $m=1-\frac{1}{\beta},\ \frac{1}{2}\leqq\beta<1$ and $|z|<1$.

Theorem 9. *Let $g(z) \in \mathcal{S}^{\star}_{\lambda}(n,\sigma,\beta)$, then $g(z)$ can be represented in the integral form*

$$g(z) = \nabla^n_\sigma \left(\int_0^z \alpha \eta^{\alpha-1} (p_\beta(\eta))^{1-\alpha} d\eta \right)^{1/\alpha}$$

for $\alpha = 1 - \frac{1}{\lambda}$ and $\lambda > 1$.

Proof. Let $g(z) \in \mathcal{S}^{\star}_{\lambda}(n,\sigma,\beta)$, then for $p_\beta \in \mathcal{P}(\beta)$ we have from (13) that

$$\frac{z^{1/\lambda}((\varDelta^n_\sigma g(z))')}{(\varDelta^n_\sigma g(z))^{1/\lambda}} = (p_\beta(z))^{1/\lambda}. \tag{15}$$

Now if we let $\frac{1}{\lambda} = 1 - \alpha$ $(\lambda > 1)$, then (15) can be expressed as

$$\frac{z^{1-\alpha}((\varDelta^n_\sigma g(z))')}{(\varDelta^n_\sigma g(z))^{1-\alpha}} = (p_\beta(z))^{1-\alpha}$$

or

$$\frac{(\varDelta^n_\sigma g(z))^{\alpha-1}(\varDelta^n_\sigma g(z))'}{z^{\alpha-1}} = (p_\beta(z))^{1-\alpha}$$

so that we can write

$$((\varDelta^n_\sigma g(z))^\alpha)' = \alpha z^{\alpha-1}(p_\beta(z))^{1-\alpha}$$

and

$$\varDelta^n_\sigma g(z) = \left(\int_0^z \alpha \eta^{\alpha-1} (p_\beta(\eta))^{1-\alpha} d\eta \right)^{\frac{1}{\alpha}} \quad (\lambda > 1).$$

Applying (10) now completes the proof.

Remark 3. *Setting $n(=\sigma) = 0$ gives the results of Theorems 1, 2 and 3 in [4].*

4.2. Coefficient estimates

Theorem 10. *Let $g(z) \in \mathcal{S}^{\star}_{\lambda}(n,\sigma,\beta)$, then*

$$|a_2| \leqq \frac{2(1-\beta)}{\chi_2(2\lambda-1)}, \tag{16}$$

$$|a_3| \leqq \frac{2(1-\beta)}{\chi_3(3\lambda-1)}\left|\frac{2(1-\beta)(2\lambda^2-4\lambda+1)}{(2\lambda-1)^2}-1\right|, \tag{17}$$

$$|a_4| \leqq \frac{2(1-\beta)}{\chi_4(4\lambda-1)} + \frac{4(1-\beta)^2}{\chi_4(4\lambda-1)}\left\{\frac{|6\lambda^2-11\lambda+2|}{(2\lambda-1)(3\lambda-1)}\right\} + \frac{8(1-\beta)^3}{\chi_4(4\lambda-1)}\left\{\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}\right\}, \tag{18}$$

$$|a_5| \leqq 2A\left|2\frac{B}{A}-1\right| + 4|C|\left|2\frac{D}{C}-1\right| + 16|E|, \tag{19}$$

where

$$\left.\begin{array}{l}
A = \frac{(1-\beta)}{\chi_5(5\lambda-1)}\\
B = \frac{(1-\beta)^2}{\chi_5(5\lambda-1)}\left\{\frac{2(4\lambda^2-7\lambda+1)}{(2\lambda-1)(4\lambda-1)}\right\}\\
C = \frac{(1-\beta)^2}{\chi_5(5\lambda-1)}\left\{\frac{9\lambda^2-15\lambda+2}{2(3\lambda-1)^2}\right\}\\
D = \frac{(1-\beta)^3}{\chi_5(5\lambda-1)}\left\{\frac{9\lambda(\lambda-1)(2\lambda^2-4\lambda+1)}{(2\lambda-1)^2(3\lambda-1)^2} + \frac{(8\lambda^2-10\lambda+1)(6\lambda^2-11\lambda+2)}{(2\lambda-1)^2(3\lambda-1)(4\lambda-1)} - \frac{6\lambda^3-16\lambda^2+8\lambda+1}{(2\lambda-1)^2(3\lambda-1)}\right\}\\
E = \\
\frac{(1-\beta)^4}{\chi_5(5\lambda-1)}\left\{\frac{\lambda(\lambda-1)(45\lambda^4-67\lambda^3+69\lambda^2+35\lambda-17)}{2(2\lambda-1)^4(3\lambda-1)^2} + \frac{(8\lambda^2-10\lambda+1)(24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3)}{3(2\lambda-1)^4(3\lambda-1)(4\lambda-1)}\right\}
\end{array}\right\} \tag{20}$$

Proof. Let $g(z) \in \mathcal{S}^{\star}_{\lambda}(n,\sigma,\beta)$ and for $p(z) \in \mathcal{P}$, (11) can be written as

$$\frac{z((\Delta^n_\sigma g(z))')^\lambda}{\Delta^n_\sigma g(z)} = (1-\beta)p(z) + \beta$$

or

$$z((\Delta^n_\sigma g(z))')^\lambda = \{(1-\beta)p(z)+\beta\}\Delta^n_\sigma g(z). \tag{21}$$

Simplifying the LHS of (21) by using (8) we get

$$\begin{aligned} z((\mathit{\Delta}_\sigma^n g(z))')^\lambda =& z + 2\lambda\chi_2 a_2 z^2 + \Big\{3\lambda\chi_3 a_3 + 2\lambda(\lambda-1)\chi_2^2 a_2^2\Big\} z^3 \\ &+ \Big\{4\lambda\chi_4 a_4 + 6\lambda(\lambda-1)\chi_2\chi_3 a_2 a_3 + \frac{4}{3}\lambda(\lambda-1)(\lambda-2)\chi_2^3 a_2^3\Big\} z^4 \\ &+ \Big\{5\lambda\chi_5 a_5 + \frac{1}{2}\lambda(\lambda-1)[16\chi_2\chi_4 a_2 a_4 + 9\chi_3^2 a_3^2] \\ &+ 6\lambda(\lambda-1)(\lambda-2)\chi_2^2\chi_3 a_2^2 a_3 + \frac{2}{3}\lambda(\lambda-1)(\lambda-2)(\lambda-3)\chi_2^4 a_2^4\Big\} z^5 \\ &+ \cdots \end{aligned} \tag{22}$$

and simplifying the RHS of (21) by using (8) we get

$$\begin{aligned} &\{(1-\beta)p(z)+\beta\}\mathit{\Delta}_\sigma^n g(z) = \\ &\qquad z + \{(1-\beta)p_1 + \chi_2 a_2\} z^2 \\ &\qquad + \{(1-\beta)p_2 + (1-\beta)p_1\chi_2 a_2 + \chi_3 a_3\} z^3 \\ &\qquad + \{(1-\beta)p_3 + (1-\beta)p_2\chi_2 a_2 + (1-\beta)p_1\chi_3 a_3 + \chi_4 a_4\} z^4 \\ &\qquad + \{(1-\beta)p_4 + (1-\beta)p_3\chi_2 a_2 + (1-\beta)p_2\chi_3 a_3 \\ &\qquad + (1-\beta)p_1\chi_4 a_4 + \chi_5 a_5\} z^5 + \cdots . \end{aligned} \tag{23}$$

A careful comparison of the coefficients in (22) and (23) shows that

$$2\lambda\chi_2 a_2 = (1-\beta)p_1 + \chi_2 a_2 \tag{24}$$

$$3\lambda\chi_3 a_3 + 2\lambda(\lambda-1)\chi_2^2 a_2^2 = (1-\beta)p_2 + (1-\beta)p_1\chi_2 a_2 + \chi_3 a_3 \tag{25}$$

$$\begin{aligned} &4\lambda\chi_4 a_4 + 6\lambda(\lambda-1)\chi_2\chi_3 a_2 a_3 + \frac{4}{3}\lambda(\lambda-1)(\lambda-2)\chi_2^3 a_2^3 \\ &\qquad = (1-\beta)p_3 + (1-\beta)p_2\chi_2 a_2 + (1-\beta)p_1\chi_3 a_3 + \chi_4 a_4 \end{aligned} \tag{26}$$

and

$$\begin{aligned} &5\lambda\chi_5 a_5 + \frac{1}{2}\lambda(\lambda-1)[16\chi_2\chi_4 a_2 a_4 + 9\chi_3^2 a_3^2] + 6\lambda(\lambda-1)(\lambda-2)\chi_2^2\chi_3 a_2^2 a_3 \\ &\qquad + \frac{2}{3}\lambda(\lambda-1)(\lambda-2)(\lambda-3)\chi_2^4 a_2^4 = (1-\beta)p_4 + (1-\beta)p_3\chi_2 a_2 \\ &\qquad\qquad + (1-\beta)p_2\chi_3 a_3 + (1-\beta)p_1\chi_4 a_4 + \chi_5 a_5. \end{aligned} \tag{27}$$

Now from (24) we get

$$a_2 = \frac{(1-\beta)p_1}{\chi_2(2\lambda-1)} \tag{28}$$

so that by applying triangle inequality and Lemma 1 we get (16). By putting (28) into (25) we get

$$a_3 = \frac{(1-\beta)}{\chi_3(3\lambda-1)}p_2 - \frac{(1-\beta)^2(2\lambda^2-4\lambda+1)}{\chi_3(2\lambda-1)^2(3\lambda-1)}p_1^2 \tag{29}$$

or

$$a_3 = \frac{(1-\beta)}{\chi_3(3\lambda-1)}\left\{p_2 - \left(\frac{(1-\beta)(2\lambda^2-4\lambda+1)}{(2\lambda-1)^2}\right)p_1^2\right\}$$

so that by applying triangle inequality leads to

$$|a_3| = \frac{(1-\beta)}{\chi_3(3\lambda-1)}\left|p_2 - \left(\frac{(1-\beta)(2\lambda^2-4\lambda+1)}{(2\lambda-1)^2}\right)p_1^2\right| \equiv \frac{(1-\beta)}{\chi_3(3\lambda-1)}\left|p_2 - up_1^2\right|$$

where

$$u = \frac{(1-\beta)(2\lambda^2-4\lambda+1)}{(2\lambda-1)^2}$$

and by applying Lemma 2 for $\lambda \geqq 1$ leads to (17). By putting (28) and (29) into (26) we get

$$\begin{aligned} a_4 = \frac{(1-\beta)}{\chi_4(4\lambda-1)}p_3 - \frac{(1-\beta)^2}{\chi_4(4\lambda-1)}\left\{\frac{6\lambda^2-11\lambda+2}{(2\lambda-1)(3\lambda-1)}\right\}p_1p_2 \\ + \frac{(1-\beta)^3}{\chi_4(4\lambda-1)}\left\{\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}\right\}p_1^3 \end{aligned} \tag{30}$$

so that applying triangle inequality in (30) leads to

$$\begin{aligned} |a_4| \leqq \frac{(1-\beta)}{\chi_4(4\lambda-1)}|p_3| + \frac{(1-\beta)^2}{\chi_4(4\lambda-1)}\left\{\frac{|6\lambda^2-11\lambda+2|}{(2\lambda-1)(3\lambda-1)}\right\}|p_1p_2| \\ + \frac{(1-\beta)^3}{\chi_4(4\lambda-1)}\left\{\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}\right\}|p_1|^3 \end{aligned}$$

and by applying Lemma 1 gives (18). Lastly, putting (28), (29) and (30) into (27) and simplifying completely leads to a summarized equation:

$$\begin{aligned} a_5 &= Ap_4 - Bp_1p_3 - Cp_2^2 + Dp_1^2p_2 - Ep_1^4 \\ &= A\left(p_4 - \frac{B}{A}p_1p_3\right) - Cp_2\left(p_2 - \frac{D}{C}p_1^2\right) - Ep_1^4 \end{aligned} \tag{31}$$

for A, B, C, D, and E in (20). Applying triangle inequality leads to

$$|a_5| \leqq A\left|p_4 - \frac{B}{A}p_1p_3\right| + C|p_2|\left|p_2 - \frac{D}{C}p_1^2\right| + E|p_1^4|$$

and applying Lemmas 1 and 6 gives (19).

Remark 4. *Setting $n(=\sigma) = 0$ and $\lambda = 1$ gives the results of the coefficient estimates of starlike functions in [7, 9] and setting $n(=\sigma) = 0$ gives the results in [4].*

4.3. Fekete-Szegö estimates

A frequently studied property of the coefficient problems of $g \in \mathcal{A}$ is the Fekete-Szegö functional introduced and defined in [8] by

$$\mathcal{FS}(\delta, g) = \left|a_3 - \delta a_2^2\right| \quad (\delta \in \mathbb{R}). \tag{32}$$

See [1, 11] for more details.

Theorem 11. *Let $g(z) \in \mathcal{S}_\lambda^\star(n, \sigma, \beta)$, then for $x \in \mathbb{R}$,*

$$\left|a_3 - xa_2^2\right| \leqq \begin{cases} \frac{2(1-\beta)(1-u)}{\chi_3(3\lambda-1)} & for \quad x \leqq \frac{\chi_2^2(4\lambda-2\lambda^2-1)}{\chi_3(3\lambda-1)} \\ \frac{2(1-\beta)}{\chi_3(3\lambda-1)} & for \quad \frac{\chi_2^2(4\lambda-2\lambda^2-1)}{\chi_3(3\lambda-1)} \leqq x \leqq \frac{\chi_2^2(4\lambda-2\lambda^2-1)}{\chi_3(3\lambda-1)} + \frac{\chi_2^2(2\lambda-1)^2}{\chi_3(1-\beta)(3\lambda-1)} \\ \frac{2(1-\beta)(1-u)}{\chi_3(3\lambda-1)} & for \quad x \geqq \frac{\chi_2^2(4\lambda-2\lambda^2-1)}{\chi_3(3\lambda-1)} + \frac{\chi_2^2(2\lambda-1)^2}{\chi_3(1-\beta)(3\lambda-1)} \end{cases} \tag{33}$$

and

$$u = \frac{2(1-\beta)[\chi_2^2(2\lambda^2 - 4\lambda + 1) + x\chi_3(3\lambda - 1)]}{\chi_2^2(2\lambda - 1)^2}. \tag{34}$$

Proof. Considering (28) and (29) in (32) for $x \in \mathbb{R}$ implies that

$$a_3 - xa_2^2 = \frac{(1-\beta)}{\chi_3(3\lambda - 1)} \left\{ p_2 - \frac{2(1-\beta)[\chi_2^2(2\lambda^2 - 4\lambda + 1) + x\chi_3(3\lambda - 1)]}{\chi_2^2(2\lambda - 1)^2} \times \frac{p_1^2}{2} \right\} \tag{35}$$

so that

$$|a_3 - xa_2^2| = \frac{(1-\beta)}{\chi_3(3\lambda - 1)} \left| p_2 - u\frac{p_1^2}{2} \right|$$

where u is given in (34). Now applying Lemma 2 in the ranges $u \leqq 0$, $0 \leqq u \leqq 2$ and $u \geqq 2$ leads to the result in (33).

Theorem 12. *Let $g(z) \in \mathcal{S}_\lambda^\star(n, \sigma, \beta)$, then for $y \in \mathbb{C}$,*

$$|a_3 - ya_2^2| \leqq \frac{2(1-\beta)}{\chi_3(3\lambda - 1)} \max\left\{1, |1-\mu|\right\} \tag{36}$$

where

$$\mu = \frac{2(1-\beta)[\chi_2^2(2\lambda^2 - 4\lambda + 1) + y\chi_3(3\lambda - 1)]}{\chi_2^2(2\lambda - 1)^2}. \tag{37}$$

Proof. Considering (35) for $y \in \mathbb{C}$ implies that

$$a_3 - ya_2^2 = \frac{(1-\beta)}{\chi_3(3\lambda-1)}\left\{p_2 - \frac{2(1-\beta)[\chi_2^2(2\lambda^2-4\lambda+1)+y\chi_3(3\lambda-1)]}{\chi_2^2(2\lambda-1)^2} \times \frac{p_1^2}{2}\right\}$$

so that

$$|a_3 - ya_2^2| = \frac{(1-\beta)}{\chi_3(3\lambda-1)}\left|p_2 - \mu\frac{p_1^2}{2}\right|$$

where μ is given in (37). Now applying Lemma 3 leads to the result in (36).

4.4. Estimates on Hankel determinants

The Hankel determinants introduced in [14] is well-known. The jth-Hankel determinant whose elements are the coefficients of g in (1) was defined in [14] by

$$\mathcal{HD}_{j,k}(g) = \begin{vmatrix} 1 & a_{k+1} & a_{k+2} & \cdots & a_{k+j-1} \\ a_{k+1} & a_{k+2} & \cdots & \cdots & a_{k+j} \\ a_{k+2} & a_{k+3} & \cdots & \cdots & a_{k+j+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k+j-1} & a_{k+j} & \cdots & \cdots & a_{k+2(j-1)} \end{vmatrix} \quad (i, j \in \mathbb{N}). \tag{38}$$

Observe that from (38), we can demonstrate that

$$\left.\begin{array}{rcl} |\mathcal{HD}_{2,1}(g)| & = & |a_3 - a_2^2|, \\ |\mathcal{HD}_{2,2}(g)| & = & |a_2a_4 - a_3^2|, \\ |\mathcal{HD}_{3,1}(g)| & \leqq & |a_3||\mathcal{HD}_{2,2}(g)| + |a_4||a_2a_3 - a_4| + |a_5||\mathcal{HD}_{2,1}(g)|. \end{array}\right\} \tag{39}$$

For some applications of Hankel determinants see [12] and the citations therein.

Theorem 13. *Let $g(z) \in \mathcal{S}_\lambda^\star(n, \sigma, \beta)$, then*

$$|\mathcal{HD}_{2,2}(g)| = |a_2a_4 - a_3^2| \leqq 4|K|\left|2\frac{L}{K} - 1\right| + 16M + 4|N| \tag{40}$$

where

$$\left.\begin{array}{l} K = \frac{(1-\beta)^2}{\chi_2\chi_4(2\lambda-1)(4\lambda-1)} \\ L = \frac{(1-\beta)^3}{(2\lambda-1)^2(3\lambda-1)}\left\{\frac{6\lambda^2-11\lambda+2}{\chi_2\chi_4(4\lambda-1)} - \frac{2(2\lambda^2-4\lambda+1)}{\chi_3^2(3\lambda-1)}\right\} \\ M = \frac{(1-\beta)^4}{(2\lambda-1)^4(3\lambda-1)}\left\{\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3\chi_2\chi_4(4\lambda-1)} - \frac{(2\lambda^2-4\lambda+1)^2}{\chi_3^2(3\lambda-1)}\right\} \\ N = \frac{(1-\beta)^2}{\chi_3^2(3\lambda-1)^2} \end{array}\right\} \tag{41}$$

Proof. Considering (28), (29) and (30) in (39) shows that

$$\begin{aligned}|a_2a_4-a_3^2| &= |Kp_1p_3-Lp_1^2p_2+Mp_1^4-Np_2^2|\\ &= \left|Kp_1\left(p_3-\frac{L}{K}p_1p_2\right)+Mp_1^4-Np_2^2\right|\\ &\leqq |K||p_1|\left|p_3-\frac{L}{K}p_1p_2\right|+M|p_1|^4+|N||p_2|^2\end{aligned}$$

for K, L, M and N in (41). Applying Lemmas 1 and 6 leads to (40).

Theorem 14. *Let $g(z)\in\mathcal{S}_\lambda^\star(n,\sigma,\beta)$, then*

$$|a_2a_3-a_4|\leqq 2J\left|2\frac{H}{J}-1\right|+8I \tag{42}$$

where

$$\left.\begin{aligned}H&=\tfrac{(1-\beta)^2}{(2\lambda-1)(3\lambda-1)}\left\{\tfrac{1}{\chi_2\chi_3}+\tfrac{6\lambda^2-11\lambda+2}{\chi_4(4\lambda-1)}\right\}\\ I&=\tfrac{(1-\beta)^3}{(2\lambda-1)^3(3\lambda-1)}\left\{\tfrac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3\chi_4(4\lambda-1)}+\tfrac{2\lambda^2-4\lambda+1}{\chi_2\chi_3}\right\}\\ J&=\tfrac{(1-\beta)}{\chi_4(4\lambda-1)}.\end{aligned}\right\} \tag{43}$$

Proof. Considering (28), (29) and (30) in (39) shows that

$$\begin{aligned}|a_2a_3-a_4| &= |Hp_1p_2-Ip_1^3-Jp_3|\\ &= \left|-J\left(p_3-\frac{H}{J}p_1p_2\right)-Ip_1^3\right|\\ &\leqq J\left|p_3-\frac{H}{J}p_1p_2\right|+I|p_1|^3\end{aligned}$$

for H, I and J in (43). Applying Lemmas 1 and 6 leads to (42).

Theorem 15. *Let $g(z)\in\mathcal{S}_\lambda^\star(n,\sigma,\beta)$, then*

$$\begin{aligned}&|\mathcal{HD}_{3,1}(g)|\\ &\leqq\left[\frac{2(1-\beta)}{\chi_3(3\lambda-1)}\left|\frac{2(1-\beta)(2\lambda^2-4\lambda+1)}{(2\lambda-1)^2}-1\right|\right]\left[4|K|\left|2\frac{L}{K}-1\right|+16M+4|N|\right]\\ &+\left[\frac{2(1-\beta)}{\chi_4(4\lambda-1)}+\frac{4(1-\beta)^2}{\chi_4(4\lambda-1)}\left\{\frac{|6\lambda^2-11\lambda+2|}{(2\lambda-1)(3\lambda-1)}\right\}\right.\\ &\qquad\left.+\frac{8(1-\beta)^3}{\chi_4(4\lambda-1)}\left\{\frac{24\lambda^4-80\lambda^3+84\lambda^2-28\lambda+3}{3(2\lambda-1)^3(3\lambda-1)}\right\}\right]\left[2J\left|2\frac{H}{J}-1\right|+8I\right]\\ &+\left[2A\left|2\frac{B}{A}-1\right|+4|C|\left|2\frac{D}{C}-1\right|+16|E|\right]\left[\frac{2(1-\beta)}{\chi_3(3\lambda-1)}\right]\end{aligned} \tag{44}$$

where A, B, C, D, E, H, I, J, K, L, M and N are defined in (20), (41) *and* (43).

Proof. Considering (17), (18), (19), (40), (42) and (36) in (39) shows that by simple calculation, we get the result in (44).

Acknowledgements. The authors thank the referees for their individual and collective valuable suggestions that helped to produce this quality paper.

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