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NEARLY PARTIAL TERNARY CUBIC DERIVATIONS ON NON-ARCHIMEDEAN RANDOM BANACH TERNARY ALGEBRAS

A. Ebadian, S. Zolfaghari, S. Ostadbashi

Abstract. Let (A_i, μ, t) $(i = 1, 2, \cdots, n)$ be non-Archimedean random normed ternary algebras over a non-Archimedean field $\mathcal{K}$ and let (B, μ, T) be non-Archimedean random Banach ternary algebra over a non-Archimedean field $\mathcal{K}$. A mapping δ_k from $A_1 \times \cdots \times A_n$ into B is called a *k-th partial ternary cubic derivation* if there exists a cubic mapping $g_k : A_k \to B$ such that

$$\begin{aligned} \delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) &= [g_k(a_k) g_k(b_k) \delta_k(x_1, \cdots, c_k, \cdots, x_n)] \\ &+ [g_k(a_k) \delta_k(x_1, \cdots, b_k, \cdots, x_n) g_k(c_k)] \\ &+ [\delta_k(x_1, \cdots, a_k, \cdots, x_n) g_k(b_k) g_k(c_k)] \end{aligned}$$

and

$$\begin{aligned} &\delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n) \\ &\quad = 2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n) + 2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) \\ &\quad\quad +12\delta_k(x_1, \cdots, a_k, \cdots, x_n) \end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$. We prove the stability of the partial ternary cubic derivations on non-Archimedean random Banach ternary algebras.

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1. Introduction

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of non-Archimedean random normed spaces (non-Archimedean RN-spaces) as in ([6, 30, 43, 45]). Let Δ^+ is the space of all probability distribution functions, i.e., the space $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$, and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, i.e., $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

A triangular norm (t-norm) is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$, which is commutative, associative, and monotone, and has 1 as the unit element.
By a non-Archimedean field, we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\ .\ |$ from $\mathcal{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \le \max(|r|, |s|)$ for all $r, s \in \mathcal{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. The function $|\ .\ |$ is called the trivial valuation if $|r| = 1$, $\forall r \in \mathcal{K}$, $r \ne 0$, and $|0| = 0$. Let $\mathcal{X}$ be a vector space over a field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\ .\ |$. A function $\|\ .\ \| : \mathcal{X} \to [0, \infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) for all $r \in \mathcal{K}$ and $x \in \mathcal{X}$, $\|rx\| = |r|\|x\|$.

(iii) the strong triangle inequality (ultrametric)

$$\|x + y\| \le \max(\|x\|, \|y\|) \qquad (x, y \in \mathcal{X})$$

is satisfied.
Then, $(\mathcal{X}, \|\ .\ \|)$ is called a non-Archimedean normed space due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n - 1\}$$

in which $n > m$, the sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. In a complete non-Archimedean normed space, every Cauchy sequence is convergent.

In 1897, Hensel [22] discovered the p-adic numbers, which is a theoretical analogue to power series in complex analysis. Fix a prime number p. For any non-zero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers that can not be divided by p. Then, $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

Definition 1. *A non-Archimedean RN-space is a triple $(\mathcal{X}, \mu, T)$, where $\mathcal{X}$ is a linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from $\mathcal{X}$ into D^+ such that the following conditions hold:*

$(NA - RN1)$ $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$;

$(NA - RN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ *for all* $x \in \mathcal{X}$, $t > 0$, *and* $\alpha \neq 0$;

$(NA - RN3)$ $\mu_{x+y}(\max(t, s)) \geq T(\mu_x(t), \mu_y(s))$ *for all* $x, y, z \in \mathcal{X}$ *and* $t, s \geq 0$.

It is easy to see that if $(NA - RN3)$ holds, then

$(RN3)$ $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$.

Definition 2. *Let $(\mathcal{X}, \mu, T)$ be a non-Archimedean RN-space and $\{x_n\}$ be a sequence in $\mathcal{X}$. Then, $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$$

for all $t > 0$, where x is the limit of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in $\mathcal{X}$ is called Cauchy if for each $\epsilon > 0$ and $t > 0$, there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have

$$\mu_{x_{n+p} - x_n}(t) > 1 - \epsilon.$$

If each Cauchy sequence is convergent, then the random norm is said to be complete, and the non-Archimedean RN-space is called a non-Archimedean random Banach space.

Let $\mathcal{K}$ be a non-Archimedean field and (X, μ, T) be a non-Archimedean random normed algebra.
A non-Archimedean random ternary algebra (X, μ, T) is a non-Archimedean random vector space over $\mathcal{K}$, endowed with a linear mapping, the so-called a ternary product, $(x, y, z) \mapsto [xyz]$ of $X \times X \times X \to X$ such that $[[xyz]tu] = [x[yzt]u] = [xy[ztu]]$ for all

$x, y, z, t, u \in X$. If $(X, \odot)$ is a usual binary non-Archimedean random algebra, then an induced ternary multiplication can be defined by $[xyz] := (x \odot y) \odot z$. Hence, the non-Archimedean random algebra is a natural generalization of the binary case. A normed non-Archimedean random ternary algebra (X, μ, T) is a non-Archimedean random ternary algebra such that

$$\mu_{[xyz]}(ste) \geq \mu_x(s)\mu_y(t)\mu_z(e)$$

for all $x, y, z \in X$ and all $s, t, e > 0$. Banach non-Archimedean random ternary algebra is a complete non-Archimedean random normed algebra.

The ternary algebras have been studied in nineteenth century. Their structures appeared more or less naturally in various domains of mathematical physics and data processing. The discovery of the Nambu mechanics and the progress of quantum mechanics [31], as well as work of Okubo [32] on Yang-Baxter equation gave a significant development on ternary algebras (see also [1, 5, 27, 44, 46]).

We say that a functional equation (ζ) is stable if any function g satisfying the equation (ζ) approximately is near to true solution of (ζ). We say that a functional equation (ζ) is superstable if every approximately solution of (ζ) is an exact solution of it (see [42]).

The stability of functional equations was first introduced by Ulam [47] in 1940. In 1941, Hyers [23] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1978, Rassias [41] generalized the theorem of Hyers for linear mappings by considering the stability problem with unbounded Cauchy differences $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$, $(\epsilon > 0, p \in [0, 1))$.

Moreover, D. G. Bourgin [3] and Găvruta [19] have considered the stability problem with unbounded Cauchy differences (see also [2, 18, 26, 33, 34]). On the other hand, J. M. Rassias (see [35]-[40]) considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruta [20].

In 1949, Bourgin [4] proved the following result, which is sometimes called the superstability of ring homomorphisms. Suppose that A and B are Banach algebras with unit. If $f : A \to B$ is a surjective mapping such that

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \\ &\|f(xy) - f(x)f(y)\| \leq \delta \end{aligned} \tag{1}$$

for some $\epsilon \geq 0$, $\delta \geq 0$ and for all $x, y \in A$, then f is a ring homomorphism.
Badora [2] and Miura et al. [29] proved the Ulam-Hyers stability and the Isac and Rassias type stability of derivations [24].

The cubic function $f(x) = ax^3$ satisfies the functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{2}$$

We promise that by a cubic function we mean a solution of the cubic functional equation (2). The functional equation (2) was solved by Jun and Kim [25]. In fact, they proved that a function $f : X \to Y$ between real vector spaces is a solution of (2) if and only if there exists a function $F : X \times X \times X \to Y$ such that $f(x) = F(x, x, x)$ for all $x \in X$, and F is symmetric for each fixed one variable and is additive for fixed two variables. The function F is given by

$$F(x,y,z) = \frac{1}{24}[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)$$

for all $x, y, z \in X$. Moreover, they investigated the Hyers–Ulam–Rassias stability for Eq. (2). For more detailed definitions of such terminologies, we can refer [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21] and [28]–[34].

2. Main results

Let (A_i, μ, T) $(i = 1, 2, \cdots, n)$ be non-Archimedean random normed ternary algebras over a non-Archimedean field $\mathcal{K}$ and (B, μ, T) be a non-Archimedean random Banach ternary algebra over a non-Archimedean field $\mathcal{K}$.

A mapping δ_k from $A_1 \times \cdots \times A_n$ into B is called a *k-th partial ternary cubic derivation* if there exists a cubic mapping $g_k : A_k \to B$ such that

$$\begin{aligned}\delta_k(x_1, \cdots, [a_k b_k c_k], \cdots, x_n) &= [g_k(a_k) g_k(b_k) \delta_k(x_1, \cdots, c_k, \cdots, x_n)] \\ &+ [g_k(a_k) \delta_k(x_1, \cdots, b_k, \cdots, x_n) g_k(c_k)] \\ &+ [\delta_k(x_1, \cdots, a_k, \cdots, x_n) g_k(b_k) g_k(c_k)]\end{aligned}$$

and

$$\begin{aligned}&\delta_k(x_1, \cdots, 2a_k + b_k, \cdots, x_n) + \delta_k(x_1, \cdots, 2a_k - b_k, \cdots, x_n) \\ &\quad = 2\delta_k(x_1, \cdots, a_k + b_k, \cdots, x_n) + 2\delta_k(x_1, \cdots, a_k - b_k, \cdots, x_n) \\ &\qquad + 12\delta_k(x_1, \cdots, a_k, \cdots, x_n)\end{aligned}$$

for all $a_k, b_k, c_k \in A_k$ and all $x_i \in A_i$ $(i \neq k)$. We denote that 0_k is zero element of A_k.

Let φ_i $(i = 1, 2, \cdots, n)$ be distribution function on $X \times X \times X \times [0, \infty)$ such that $\varphi_i(x, y, z, .)$ is nondecreasing, and

$$\varphi_i(cx, cx, cx, t) \geq \varphi_i(x, x, x, \frac{t}{|c|}), \qquad \forall x \in X, c \neq 0, (i = 1, 2, \cdots, n).$$

Theorem 1. *Let $F_k : A_1 \times \cdots \times A_n \to B$ be a mapping with $F_k(x_1, \cdots, 0_k, \cdots, x_n) = 0_B$. Assume that there exist a distribution function φ_k on $A_k \times A_k \times A_k \times [0, \infty)$ such that for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer ℓ, $\ell \geq 2$ with $|2^\ell| < \alpha$,*

$$\varphi_k(2^{-\ell}x_k, 2^{-\ell}y_k, 2^{-\ell}z_k, t) \geq \varphi_k(x_k, y_k, z_k, \alpha t) \tag{3}$$

for all $x_k, y_k, z_k \in A_k$ and all $t > 0$,
and

$$\lim_{m\to\infty} T_{\jmath=m}^{\infty} M(x_k, \frac{\alpha^\jmath}{|2|^{\ell\jmath}} t) = 1 \quad \forall x_k \in A_k, t > 0, \tag{4}$$

and a cubic mapping $g_k : A_k \to B$ such that satisfying

$$\mu_{\left(\begin{array}{l} F_k(x_1,\cdots,2a_k+b_k,\cdots,x_n)+F_k(x_1,\cdots,2a_k-b_k,\cdots,x_n)-2F_k(x_1,\cdots,a_k+b_k,\cdots,x_n) \\ -2F_k(x_1,\cdots,a_k-b_k,\cdots,x_n)-12F_k(x_1,\cdots,a_k,\cdots,x_n)\end{array}\right)}(t) \geq \varphi_k(a_k, b_k, 0_k, t) \tag{5}$$

and

$$\mu_{\left(\begin{array}{l} F_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n)-[g_k(a_k)g_k(b_k)F_k(x_1,\cdots,c_k,\cdots,x_n)] \\ -[g_k(a_k)F_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)]-[F_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]\end{array}\right)}(t) \geq \varphi_k(a_k, b_k, c_k, t) \tag{6}$$

for all $a_k, b_k, c_k \in A_k$, $x_i \in A_i$ $(i \neq k)$ and $t > 0$. Then there exists a unique k-th partial cubic derivation $\delta_k : A_1 \times \cdots \times A_n \to B$ such that

$$\mu_{F_k(x_1,\cdots,x_n)-\delta_k(x_1,\cdots,x_n)}(t) \geq T_{\imath=1}^{\infty} M(x_k, \frac{\alpha^{\imath+1}}{|2|^{\ell\imath}} t) \tag{7}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$ where

$$M(x_k, t) := T\left(\varphi_k(x_k, 0_k, 0_k, t), \varphi_k(2x_k, 0_k, 0_k, t), \cdots, \varphi_k(2^{\ell-1}x_k, 0_k, 0_k, t)\right)$$

for all $x_k \in A_k$ and $t > 0$.

Proof. First, we show by induction on $\jmath$ that, for all $x_i \in A_i$, $t > 0$ and $\jmath \geq 1$,

$$\begin{aligned}\mu_{F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)-2^{3\jmath}F_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \geq M_\jmath(x_k, t) = T\Big(&\varphi_k(x_k, 0_k, 0_k, t), \\ &\varphi_k(2x_k, 0_k, 0_k, t), \cdots, \varphi_k(2^{\jmath-1}x_k, 0_k, 0_k, t)\Big).\end{aligned} \tag{8}$$

Replacing $a_k = x_k$ and $b_k = 0_k$ in (5) yield

$$\mu_{2F_k(x_1,\cdots,2x_k,\cdots,x_n)-16F_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \geq \varphi_k(x_k, 0_k, 0_k, t), \quad \forall x_i \in A_i, t > 0,$$

$$\mu_{F_k(x_1,\cdots,2x_k,\cdots,x_n)-8F_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \geq \varphi_k(x_k, 0_k, 0_k, 2t) \geq \varphi_k(x_k, 0_k, 0_k, t)$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$. This proves (8) for $\jmath = 1$. Let (8) hold for some $\jmath > 1$. Replacing a_k by $2^\jmath x_k$ and b_k by 0_k in (5), we get

$$\mu_{F_k(x_1,\cdots,2^{\jmath+1}x_k,\cdots,x_n)-8F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)}(t) \geq \varphi_k(2^\jmath x_k, 0_k, 0_k, t)$$

where $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$. Since $|8| \leq 1$,

$$\begin{aligned}
&\mu_{F_k(x_1,\cdots,2^{\jmath+1}x_k,\cdots,x_n)-2^{3(\jmath+1)}F_k(x_1,\cdots,x_k,\cdots,x_n)}(t)\\
&\quad \geq T\Big(\mu_{F_k(x_1,\cdots,2^{\jmath+1}x_k,\cdots,x_n)-2^3F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)}(t)\\
&\qquad\quad ,\mu_{2^3F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)-2^{3(\jmath+1)}F_k(x_1,\cdots,x_k,\cdots,x_n)}(t)\Big)\\
&\quad = T\Big(\mu_{F_k(x_1,\cdots,2^{\jmath+1}x_k,\cdots,x_n)-2^3F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)}(t)\\
&\qquad\quad ,\mu_{F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)-2^{3\jmath}F_k(x_1,\cdots,x_k,\cdots,x_n)}(\frac{t}{|8|})\Big)\\
&\quad \geq T\Big(\mu_{F_k(x_1,\cdots,2^{\jmath+1}x_k,\cdots,x_n)-2^3F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)}(t)\\
&\qquad\quad ,\mu_{F_k(x_1,\cdots,2^\jmath x_k,\cdots,x_n)-2^{3\jmath}F_k(x_1,\cdots,x_k,\cdots,x_n)}(t)\Big)\\
&\quad \geq T\left(\varphi_k(2^\jmath x_k, 0_k, 0_k, t), M_\jmath(x_k, t)\right) = M_{\jmath+1}(x_k, t)
\end{aligned}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and all $t > 0$. Thus (8) holds for all $\jmath \geq 1$. In particular,

$$\mu_{F_k(x_1,\cdots,2^\ell x_k,\cdots,x_n)-2^{3\ell}F_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \geq M(x_k, t), \quad \forall x_i \in A_i, t > 0. \tag{9}$$

Replacing x_k by $2^{-(\ell m+\ell)}x_k$ in (9) and using inequality (3), we obtain

$$\mu_{F_k(x_1,\cdots,\frac{x_k}{2^{\ell m}},\cdots,x_n)-8^\ell F_k(x_1,\cdots,\frac{x_k}{2^{\ell m+\ell}},\cdots,x_n)}(t) \geq M(\frac{x_k}{2^{\ell m+\ell}}, t) \geq M(x_k, \alpha^{m+1}t) \tag{10}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$, $t > 0$ and $m \geq 0$. Then it follows that

$$\begin{aligned}
\mu_{\left((2^{3\ell})^m F_k(x_1,\cdots,\frac{x_k}{2^{\ell m}},\cdots,x_n)-(2^{3\ell})^{m+1}F_k(x_1,\cdots,\frac{x_k}{2^{\ell m+\ell}},\cdots,x_n)\right)}(t) &\geq M(x_k, \frac{\alpha^{m+1}}{|(2^{3\ell})^m|}t)\\
&\geq M(x_k, \frac{\alpha^{m+1}}{|(2^\ell)^m|}t),
\end{aligned} \tag{11}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$, $t > 0$ and $m \geq 0$, and so

$$\mu_{\left((2^{3\ell})^m F_k(x_1,\cdots,\frac{x_k}{2^{\ell m}},\cdots,x_n)-(2^{3\ell})^{m+p}F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^{m+p}},\cdots,x_n)\right)}(t)$$
$$\geq T_{\jmath=m}^{m+p}\left(\mu_{\left((2^{3\ell})^\jmath F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^\jmath},\cdots,x_n)-(2^{3\ell})^{\jmath+p}F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^{\jmath+p}},\cdots,x_n)\right)}(t)\right)$$
$$\geq T_{\jmath=m}^{m+p}M(x_k,\frac{\alpha^{\jmath+1}}{|(2^\ell)^\jmath|}t)$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$, $t > 0$ and $m \geq 0$. Since $\lim_{m\to\infty} T_{\jmath=m}^{\infty}M(x_k, \frac{\alpha^{\jmath+1}}{|(2^\ell)^\jmath|}t) = 1$ $(x_k \in A_k, t > 0)$, $\{(2^{3\ell})^m F_k(x_1, \cdots, \frac{x_k}{(2^\ell)^m}, \cdots, x_n)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in B. By the completeness of B, this sequence is convergent and so we can define a mapping $\delta_k : A_1 \times \cdots \times A_n \to B$ given by

$$\lim_{m\to\infty}\mu_{2^{3\ell m}F_k(x_1,\cdots,\frac{x_k}{2^{\ell m}},\cdots,x_n)-\delta_k(x_1,\cdots,x_n)}(t) = 1 \tag{12}$$

for all $x_i \in A_i$ $(i = 1, \cdots, n)$ and all $t > 0$.
As T is continuous, from a well-known result in the probabilistic metric space (see Chapter 12 in [45]), it follows that

$$\lim_{m\to\infty}\mu_{\Big((2^{3\ell})^m F_k(x_1,\cdots,2^{-\ell m}(2a_k+b_k),\cdots,x_n)+(2^{3\ell})^m F_k(x_1,\cdots,2^{-\ell m}(2a_k-b_k),\cdots,x_n) -2(2^{3\ell})^m F_k(x_1,\cdots,2^{-\ell m}(a_k+b_k),\cdots,x_n)-2(2^{3\ell})^m F_k(x_1,\cdots,2^{-\ell m}(a_k-b_k),\cdots,x_n) -12(2^{3\ell})^m F_k(x_1,\cdots,2^{-\ell m}(a_k),\cdots,x_n)\Big)}(t)$$
$$= \mu_{\Big(\delta_k(x_1,\cdots,2a_k+b_k,\cdots,x_n)+\delta_k(x_1,\cdots,2a_k-b_k,\cdots,x_n)-2\delta_k(x_1,\cdots,a_k+b_k,\cdots,x_n) -2\delta_k(x_1,\cdots,a_k-b_k,\cdots,x_n)-12\delta_k(x_1,\cdots,a_k,\cdots,x_n)\Big)}(t)$$

for all $a_k, b_k \in A_k$, $x_i \in A_i$ $(i \neq k, i = 1, 2, \cdots, n)$ and $t > 0$.
On the other hand, replacing a_k, b_k by $2^{-\ell m}a_k, 2^{-\ell m}b_k$ in (3) and (5), we get

$$\mu_{\Big((8^\ell)^m F_k(x_1,\cdots,2^{-\ell m}(2a_k+b_k),\cdots,x_n)+(8^\ell)^m F_k(x_1,\cdots,2^{-\ell m}(2a_k-b_k),\cdots,x_n) -2(8^\ell)^m F_k(x_1,\cdots,2^{-\ell m}(a_k+b_k),\cdots,x_n)-2(8^\ell)^m F_k(x_1,\cdots,2^{-\ell m}(a_k-b_k),\cdots,x_n) -12(8^\ell)^m F_k(x_1,\cdots,2^{-\ell m}(a_k),\cdots,x_n)\Big)}(t)$$
$$\geq \varphi_k(2^{-\ell m}a_k, 2^{-\ell m}b_k, 0_k, \frac{t}{|2^\ell|^m}) \geq \varphi_k(a_k, b_k, 0_k, \frac{\alpha^m}{|2^\ell|^m}t)$$

for all $a_k, b_k \in A_k$, $x_i \in A_i$ $(i \neq k, i = 1, 2, \cdots, n)$ and $t > 0$.
Since $\lim_{m\to\infty} \varphi_k(a_k, b_k, 0_k, \frac{\alpha^m}{|2^\ell|^m} t) = 1$, we infer that δ_k is cubic with respect to the k-th variable.
Next, for each $m \geq 1$, $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$,

$$\begin{aligned}
&\mu_{\left(F_k(x_1,\cdots,x_k,\cdots,x_n)-(2^{3\ell})^m F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^m},\cdots,x_n)\right)}(t)\\
&= \mu_{\left(\sum_{\imath=0}^{m-1}(2^{3\ell})^\imath F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^\imath},\cdots,x_n)-(2^{3\ell})^{\imath+1}F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^{\imath+1}},\cdots,x_n)\right)}(t)\\
&\geq T_{\imath=0}^{m-1}\left(\mu_{\left((2^{3\ell})^\imath F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^\imath},\cdots,x_n)-(2^{3\ell})^{\imath+1}F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^{\imath+1}},\cdots,x_n)\right)}(t)\right)\\
&\geq T_{\imath=0}^{m-1} M(x_k, \frac{\alpha^{\imath+1}}{|2^\ell|^\imath} t)
\end{aligned}$$

and so

$$\begin{aligned}
&\mu_{F_k(x_1,\cdots,x_k,\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)}(t)\\
&\geq T\Big(\mu_{\left(F_k(x_1,\cdots,x_k,\cdots,x_n)-(2^{3\ell})^m F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^m},\cdots,x_n)\right)}(t)\\
&\qquad\qquad, \mu_{\left((2^{3\ell})^m F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^m},\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)\right)}(t)\Big) \qquad (13)\\
&\geq T\Big(T_{\imath=0}^{m-1} M(x_k, \frac{\alpha^{\imath+1}}{|2^\ell|^\imath} t), \mu_{\left((2^{3\ell})^m F_k(x_1,\cdots,\frac{x_k}{(2^\ell)^m},\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)\right)}(t)\Big).
\end{aligned}$$

Taking the limit as $m \to \infty$ in (13), we obtain

$$\mu_{F_k(x_1,\cdots,x_k,\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \geq T_{\imath=1}^{\infty} M(x_k, \frac{\alpha^{\imath+1}}{|2^\ell|^\imath} t)$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$. This proves (7).
Replacing a_k, b_k, c_k with $2^{-\ell m}a_k, 2^{-\ell m}b_k, 2^{-\ell m}c_k$, respectively, in (6) and using (3), we obtain

$$\begin{aligned}
&\mu_{\Big(F_k(x_1,\cdots,\frac{[a_k b_k c_k]}{2^{3\ell m}},\cdots,x_n)-\left[\frac{g_k(a_k)}{2^{3\ell m}}\frac{g_k(b_k)}{2^{3\ell m}}F_k(x_1,\cdots,\frac{c_k}{2^{\ell m}},\cdots,x_n)\right]}\\
&\qquad _{-\left[\frac{g_k(a_k)}{2^{3\ell m}}F_k(x_1,\cdots,\frac{b_k}{2^{\ell m}},\cdots,x_n)\frac{g_k(c_k)}{2^{3\ell m}}\right]-\left[F_k(x_1,\cdots,\frac{a_k}{2^{\ell m}},\cdots,x_n)\frac{g_k(b_k)}{2^{3\ell m}}\frac{g_k(c_k)}{2^{3\ell m}}\right]\Big)}(t)\\
&\geq \varphi_k(\frac{a_k}{2^{\ell m}}, \frac{b_k}{2^{\ell m}}, \frac{c_k}{2^{\ell m}}, t).
\end{aligned}$$

Then we have

$$\begin{aligned}&\mu_{\left(2^{9\ell m}F_k(x_1,\cdots,\frac{[a_kb_kc_k]}{2^{3\ell m}},\cdots,x_n)-2^{9\ell m}\left[\frac{g_k(a_k)}{2^{3\ell m}}\frac{g_k(b_k)}{2^{3\ell m}}F_k(x_1,\cdots,\frac{c_k}{2^{\ell m}},\cdots,x_n)\right]\right.}\\&\quad{}_{\left.-2^{9\ell m}\left[\frac{g_k(a_k)}{2^{3\ell m}}F_k(x_1,\cdots,\frac{b_k}{2^{\ell m}},\cdots,x_n)\frac{g_k(c_k)}{2^{3\ell m}}\right]-2^{9\ell m}\left[F_k(x_1,\cdots,\frac{a_k}{2^{\ell m}},\cdots,x_n)\frac{g_k(b_k)}{2^{3\ell m}}\frac{g_k(c_k)}{2^{3\ell m}}\right]\right)}(t)\\&\geq \varphi_k(\frac{a_k}{2^{\ell m}},\frac{b_k}{2^{\ell m}},\frac{c_k}{2^{\ell m}},\frac{t}{|2|^{9\ell m}})\\&\geq \varphi_k(a_k,b_k,c_k,\frac{\alpha^m}{|2|^{\ell m}}t).\end{aligned}$$

for all $a_k,b_k,c_k\in A_k$, $x_i\in A_i$ $(i\neq k, i=1,2,\cdots,n)$ and $t>0$.
Since $\lim_{m\to\infty}\varphi_k(a_k,b_k,c_k,\frac{\alpha^m}{|2^\ell|^m}t)=1$, we have

$$\begin{aligned}\delta_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n)&=[g_k(a_k)g_k(b_k)\delta_k(x_1,\cdots,c_k,\cdots,x_n)]\\&\quad+[g_k(a_k)\delta_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)]\\&\quad+[\delta_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]\end{aligned}$$

for all $a_k,b_k,c_k\in A_k$, and all $x_i\in A_i$ $(i\neq k, i=1,2,\cdots,n)$.
Finally, to prove the uniqueness of δ_k, let $\delta_k^{'}:A_1\times\cdots\times A_n\to B$ be another k-th partial ternary cubic derivation satisfying (7). Then, for each $m\in\mathbb{N}$, $x_i\in A_i$ and $t>0$,

$$\begin{aligned}\mu_{\delta_k(x_1,\cdots,x_n)-\delta_k^{'}(x_1,\cdots,x_n)}(t)\geq T\Big(&\mu_{\left(\delta_k(x_1,\cdots,x_n)-(2^{3\ell})^mF_k(x_1,\cdots,\frac{x_k}{(2^\ell)^m},\cdots,x_n)\right)}(t)\\&,\mu_{\left((2^{3\ell})^mF_k(x_1,\cdots,\frac{x_k}{(2^\ell)^m},\cdots,x_n)-\delta_k^{'}(x_1,\cdots,x_n)\right)}(t)\Big).\end{aligned}$$

Therefore, from (12), we conclude that $\delta_k(x_1,\cdots,x_n)=\delta_k^{'}(x_1,\cdots,x_n)$. This proves the uniqueness of δ_k.

Theorem 2. *Let $F_k:A_1\times\cdots\times A_n\to B$ be a mapping with $F_k(x_1,\cdots,0_k,\cdots,x_n)=0_B$. Assume that there exist a distribution function φ_k on $A_k\times A_k\times A_k\times[0,\infty)$ such that for some $\alpha\in\mathbb{R}$, $\alpha>1$, and some integer ℓ, $\ell\geq 1$ with $\frac{1}{|2|^{6\ell}}<\alpha$,*

$$\varphi_k(2^\ell x_k,2^\ell y_k,2^\ell z_k,t)\geq\varphi_k(x_k,y_k,z_k,\frac{\alpha}{|2|^{3\ell}}t) \tag{14}$$

for all $x_k,y_k,z_k\in A_k$ and all $t>0$,
and

$$\lim_{m\to\infty}T_{\jmath=m}^{\infty}M(x_k,\alpha^\jmath t)=1 \quad \forall x_k\in A_k, t>0, \tag{15}$$

and a cubic mapping $g_k : A_k \to B$ such that satisfying (5) *and* (6). *Then there exists a unique k-th partial cubic derivation $\delta_k : A_1 \times \cdots \times A_n \to B$ such that*

$$\mu_{F_k(x_1,\cdots,x_n)-\delta_k(x_1,\cdots,x_n)}(t) \geq T_{\imath=1}^{\infty} M(x_k, \alpha^{\imath+1}t) \tag{16}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$ where

$$M(x_k, t) := T\left(\varphi_k(\frac{x_k}{2}, 0_k, 0_k, t), \varphi_k(\frac{x_k}{4}, 0_k, 0_k, t), \cdots, \varphi_k(\frac{x_k}{2^\ell}, 0_k, 0_k, t)\right)$$

for all $x_k \in A_k$ and $t > 0$.

Proof. It follows from (9) that,

$$\begin{aligned}
&\mu_{F_k(x_1,\cdots,x_k,\cdots,x_n)-2^{3\ell}F_k(x_1,\cdots,\frac{x_k}{2^\ell},\cdots,x_n)}(t) \geq M_\ell(x_k, t)\\
&= T\left(\varphi_k(\frac{x_k}{2}, 0_k, 0_k, t), \varphi_k(\frac{x_k}{4}, 0_k, 0_k, t), \cdots, \varphi_k(\frac{x_k}{2^{\ell-1}}, 0_k, 0_k, t), \varphi_k(\frac{x_k}{2^\ell}, 0_k, 0_k, t)\right)\\
&= M(x_k, t)
\end{aligned} \tag{17}$$

and so

$$\mu_{\frac{1}{2^{3\ell}}F_k(x_1,\cdots,x_k,\cdots,x_n)-F_k(x_1,\cdots,\frac{x_k}{2^\ell},\cdots,x_n)}(t) \geq M(x_k, |2|^{3\ell}t). \tag{18}$$

Replacing a_k by $2^{\ell m+\ell}x_k$ and b_k by 0_k in (18) and using inequality (14), we get

$$\begin{aligned}
\mu_{\frac{1}{2^{3\ell}}F_k(x_1,\cdots,2^{\ell m+\ell}x_k,\cdots,x_n)-F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)}(t) &\geq M(2^{\ell m+\ell}x_k, |2|^{3\ell}t)\\
&\geq M(x_k, |2|^{3\ell}\frac{\alpha^{m+1}}{|2|^{3\ell(m+1)}}t)\\
&= M(x_k, \frac{\alpha^{m+1}}{|2|^{3\ell m}}t)
\end{aligned} \tag{19}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$, $t > 0$ and $m \geq 0$. Then,

$$\begin{aligned}
\mu_{\left(\frac{1}{2^{3\ell+3\ell m}}F_k(x_1,\cdots,2^{\ell m+\ell}x_k,\cdots,x_n)-\frac{1}{2^{3\ell m}}F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)\right)}(t) &\geq M(x_k, |2|^{3\ell m}\frac{\alpha^{m+1}}{|2|^{3\ell m}}t)\\
&= M(x_k, \alpha^{m+1}t)
\end{aligned} \tag{20}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$, $t > 0$ and $m \geq 0$. Hence

$$\begin{aligned}
&\mu_{\left(\frac{1}{2^{3\ell(m+p)}}F_k(x_1,\cdots,2^{\ell(m+1)}x_k,\cdots,x_n)-\frac{1}{2^{3\ell m}}F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)\right)}(t)\\
&\geq T_{\jmath=m}^{m+p}\left(\mu_{\left(\frac{1}{2^{3\ell(p+\jmath)}}F_k(x_1,\cdots,2^{\ell(p+\jmath)}x_k,\cdots,x_n)-\frac{1}{2^{3\ell\jmath}}F_k(x_1,\cdots,2^{\ell\jmath}x_k,\cdots,x_n)\right)}(t)\right)\\
&\geq T_{\jmath=m}^{m+p}M(x_k, \alpha^{\jmath+1}t)
\end{aligned}$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$, $t > 0$ and $m \geq 0$. Since

$$\lim_{m\to\infty} T_{\jmath=m}^{\infty} M(x_k, \alpha^{\jmath+1}t) = 1$$

in which $x_k \in A_k$ and $t > 0$, $\{\frac{1}{2^{3\ell m}} F_k(x_1, \cdots, 2^{\ell m}x_k, \cdots, x_n)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in B. By the completeness of B, this sequence is convergent. Hence, we can define a mapping $\delta_k : A_1 \times \cdots \times A_n \to B$ given by

$$\lim_{m\to\infty} \mu_{\frac{1}{2^{3\ell m}} F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n) - \delta_k(x_1,\cdots,x_n)}(t) = 1 \tag{21}$$

for all $x_i \in A_i$ $(i = 1, \cdots, n)$ and all $t > 0$.
As T is continuous, from a well-known result in the probabilistic metric space (see Chapter 12 in [45]), it follows that

$$\begin{aligned}
&\lim_{m\to\infty} \mu_{\Big(\frac{1}{2^{3\ell m}} F_k(x_1,\cdots,2^{\ell m}(2a_k+b_k),\cdots,x_n) + \frac{1}{2^{3\ell m}} F_k(x_1,\cdots,2^{\ell m}(2a_k-b_k),\cdots,x_n)} \\
&\quad {}_{-2(\frac{1}{2^{3\ell m}})F_k(x_1,\cdots,2^{\ell m}(a_k+b_k),\cdots,x_n) - 2(\frac{1}{2^{3\ell m}})F_k(x_1,\cdots,2^{\ell m}(a_k-b_k),\cdots,x_n)} \\
&\quad {}_{-12(\frac{1}{2^{3\ell m}})F_k(x_1,\cdots,2^{\ell m}(a_k),\cdots,x_n)\Big)}(t) \\
&= \mu_{\Big(\delta_k(x_1,\cdots,2a_k+b_k,\cdots,x_n) + \delta_k(x_1,\cdots,2a_k-b_k,\cdots,x_n) - 2\delta_k(x_1,\cdots,a_k+b_k,\cdots,x_n)} \\
&\quad {}_{-2\delta_k(x_1,\cdots,a_k-b_k,\cdots,x_n) - 12\delta_k(x_1,\cdots,a_k,\cdots,x_n)\Big)}(t)
\end{aligned}$$

for almost all $t > 0$.
Moreover, replacing a_k, b_k by $2^{\ell m}a_k, 2^{\ell m}b_k$ in (14) and (5), we get

$$\begin{aligned}
&\mu_{\Big(\frac{1}{(8^\ell)^m} F_k(x_1,\cdots,2^{\ell m}(2a_k+b_k),\cdots,x_n) + \frac{1}{(8^\ell)^m} F_k(x_1,\cdots,2^{\ell m}(2a_k-b_k),\cdots,x_n)} \\
&\quad {}_{-2(\frac{1}{(8^\ell)^m})F_k(x_1,\cdots,2^{\ell m}(a_k+b_k),\cdots,x_n) - 2(\frac{1}{(8^\ell)^m})F_k(x_1,\cdots,2^{\ell m}(a_k-b_k),\cdots,x_n)} \\
&\quad {}_{-12(\frac{1}{(8^\ell)^m})F_k(x_1,\cdots,2^{\ell m}(a_k),\cdots,x_n)\Big)}(t) \\
&\geq \varphi_k(2^{\ell m}a_k, 2^{\ell m}b_k, 0_k, |2|^{3\ell m}t) \\
&\geq \varphi_k(a_k, b_k, 0_k, |2|^{3\ell m}\frac{\alpha^m}{|2|^{3\ell m}}t) \\
&= \varphi_k(a_k, b_k, 0_k, \alpha^m t)
\end{aligned}$$

for all $a_k, b_k \in A_k$, $x_i \in A_i$ $(i \neq k, i = 1, 2, \cdots, n)$ and $t > 0$. Since

$$\lim_{m\to\infty} \varphi_k(a_k, b_k, 0_k, \alpha^m t) = 1,$$

we infer that δ_k is cubic with respect to the k-th variable.
Next, for each $m \geq 1$, $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$,

$$\begin{aligned}
&\mu_{\left(F_k(x_1,\cdots,x_k,\cdots,x_n)-\frac{1}{2^{3\ell m}}F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)\right)}(t) \\
&= \mu_{\left(\sum_{\imath=0}^{m-1}\frac{1}{2^{3\ell\imath}}F_k(x_1,\cdots,2^{\ell\imath}x_k,\cdots,x_n)-\frac{1}{(2^{3\ell})^{\imath+1}}F_k(x_1,\cdots,(2^{\ell})^{\imath+1}x_k,\cdots,x_n)\right)}(t) \\
&\geq T_{\imath=0}^{m-1}\left(\mu_{\left(\frac{1}{2^{3\ell\imath}}F_k(x_1,\cdots,2^{\ell\imath}x_k,\cdots,x_n)-\frac{1}{(2^{3\ell})^{\imath+1}}F_k(x_1,\cdots,(2^{\ell})^{\imath+1}x_k,\cdots,x_n)\right)}(t)\right) \\
&\geq T_{\imath=0}^{m-1}M(x_k, \alpha^{\imath+1}t)
\end{aligned}$$

and so

$$\begin{aligned}
&\mu_{F_k(x_1,\cdots,x_k,\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \\
&\geq T\Big(\mu_{\left(F_k(x_1,\cdots,x_k,\cdots,x_n)-\frac{1}{2^{3\ell m}}F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)\right)}(t) \\
&\qquad\qquad ,\mu_{\left(\frac{1}{2^{3\ell m}}F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)\right)}(t)\Big) \qquad (22)\\
&\geq T\left(T_{\imath=0}^{m-1}M(x_k, \alpha^{\imath+1}t), \mu_{\left(\frac{1}{2^{3\ell m}}F_k(x_1,\cdots,2^{\ell m}x_k,\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)\right)}(t)\right).
\end{aligned}$$

By letting $m \to \infty$ in (22), we obtain

$$\mu_{F_k(x_1,\cdots,x_k,\cdots,x_n)-\delta_k(x_1,\cdots,x_k,\cdots,x_n)}(t) \geq T_{\imath=1}^{\infty}M(x_k, \alpha^{\imath+1}t)$$

for all $x_i \in A_i$ $(i = 1, 2, \cdots, n)$ and $t > 0$. This proves (16).
Replacing a_k, b_k, c_k with $2^{\ell m}a_k, 2^{\ell m}b_k, 2^{\ell m}c_k$, respectively, in (6) and using (14), we obtain

$$\begin{aligned}
&\mu_{\Big(F_k(x_1,\cdots,2^{3\ell m}[a_kb_kc_k],\cdots,x_n)-\left[(2^{3\ell m})g_k(a_k)(2^{3\ell m})g_k(b_k)F_k(x_1,\cdots,2^{\ell m}c_k,\cdots,x_n)\right]} \\
&\qquad _{-\left[(2^{3\ell m})g_k(a_k)F_k(x_1,\cdots,2^{\ell m}b_k,\cdots,x_n)(2^{3\ell m})g_k(c_k)\right]} \\
&\qquad _{-\left[F_k(x_1,\cdots,2^{\ell m}a_k,\cdots,x_n)(2^{3\ell m})g_k(b_k)(2^{3\ell m})g_k(c_k)\right]\Big)}(t) \\
&\geq \varphi_k(2^{\ell m}a_k, 2^{\ell m}b_k, 2^{\ell m}c_k, t).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mu\Big(\tfrac{1}{2^{9\ell m}}F_k(x_1,\cdots,2^{3\ell m}[a_kb_kc_k],\cdots,x_n)-\tfrac{1}{2^{9\ell m}}\Big[(2^{3\ell m})g_k(a_k)(2^{3\ell m})g_k(b_k)F_k(x_1,\cdots,2^{\ell m}c_k,\cdots,x_n)\Big]\\
&\qquad-\tfrac{1}{2^{9\ell m}}\Big[(2^{3\ell m})g_k(a_k)F_k(x_1,\cdots,2^{\ell m}b_k,\cdots,x_n)(2^{3\ell m})g_k(c_k)\Big]\\
&\qquad-\tfrac{1}{2^{9\ell m}}\Big[F_k(x_1,\cdots,2^{\ell m}a_k,\cdots,x_n)(2^{3\ell m})g_k(b_k)(2^{3\ell m})g_k(c_k)\Big]\Big)(t)\\
&\geq \varphi_k(2^{\ell m}a_k,2^{\ell m}b_k,2^{\ell m}c_k,|2|^{9\ell m}t)\\
&\geq \varphi_k(a_k,b_k,c_k,|2|^{9\ell m}\frac{\alpha^m}{|2|^{3\ell m}}t)\\
&= \varphi_k(a_k,b_k,c_k,|2|^{6\ell m}\alpha^m t).
\end{aligned}$$

for all $a_k,b_k,c_k\in A_k$, $x_i\in A_i$ $(i\neq k, i=1,2,\cdots,n)$ and $t>0$. Since $\frac{1}{|2|^{6\ell}}<\alpha$, we have

$$\lim_{m\to\infty}\varphi_k(a_k,b_k,c_k,|2|^{6\ell m}\alpha^m t)=1.$$

Hence

$$\begin{aligned}
\delta_k(x_1,\cdots,[a_kb_kc_k],\cdots,x_n)&=[g_k(a_k)g_k(b_k)\delta_k(x_1,\cdots,c_k,\cdots,x_n)]\\
&\quad+[g_k(a_k)\delta_k(x_1,\cdots,b_k,\cdots,x_n)g_k(c_k)]\\
&\quad+[\delta_k(x_1,\cdots,a_k,\cdots,x_n)g_k(b_k)g_k(c_k)]
\end{aligned}$$

for all $a_k,b_k,c_k\in A_k$, and all $x_i\in A_i$ $(i\neq k, i=1,2,\cdots,n)$.
Finally, to prove the uniqueness of δ_k, let $\delta_k^{'}:A_1\times\cdots\times A_n\to B$ be another k-th partial ternary cubic derivation satisfying (16). Then, for each $m\in\mathbb{N}$, $x_i\in A_i$ and $t>0$,

$$\begin{aligned}
\mu_{\delta_k(x_1,\cdots,x_n)-\delta_k^{'}(x_1,\cdots,x_n)}(t)\geq T\Big(&\mu_{\left(\delta_k(x_1,\cdots,x_n)-\frac{1}{(2^{3\ell})^m}F_k(x_1,\cdots,(2^\ell)^m x_k,\cdots,x_n)\right)}(t)\\
&,\mu_{\left(\frac{1}{(2^{3\ell})^m}F_k(x_1,\cdots,(2^\ell)^m x_k,\cdots,x_n)-\delta_k^{'}(x_1,\cdots,x_n)\right)}(t)\Big).
\end{aligned}$$

Therefore, from (21), we conclude that $\delta_k(x_1,\cdots,x_n)=\delta_k^{'}(x_1,\cdots,x_n)$. This proves the uniqueness of δ_k.

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Ali Ebadian
Department of Mathematics,
Urmia University,

P. O. Box 165, Urmia, Iran.
email: *ebadian.ali@gmail.com*

Somaye Zolfaghari (corresponding author)
Department of Mathematics,
Urmia University,
P. O. Box 165, Urmia, Iran.
email: *somaye.zolfaghari@gmail.com*

Saed Ostadbashi
Department of Mathematics,
Urmia University,
P. O. Box 165, Urmia, Iran.
email: *saedostadbashi@yahoo.com.*