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A GENERAL FIXED POINT THEOREM FOR A PAIR OF MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper a general fixed point theorem for a pair of mappings satisfying a new type of implicit relation, different than the type from [8], which generalize the results from Theorem 3.5 [4], Theorem 4.5 [4], Theorem 2.3 [2], Corollary 2.4 [2], Corollary 2.8 [2] and other results from [1] is proved.

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1. Introduction

In 1994, Matthews [5] introduced the concept of partial metric space as a part of the study of denotional semantics of dataflow networks and proved the Banach contraction principle in such spaces.

Many authors studied some contractive conditions in complete partial metric spaces.

Quite recently, in [1], [2], [4], some fixed point theorems under various contractive conditions in partial metric spaces are proved.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [6], [7] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, ultra - metric spaces, convex metric spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying contractive/extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces and G - metric spaces. Also, this method allows the study of local and global properties of fixed point structures.

The study of fixed point for self mappings in complete partial metric spaces satisfying an implicit relation is initiated in [8].

The purpose of this paper is to prove a general fixed point theorem for a pair of mappings satisfying a new type of implicit relation, different than the type from [8], which generalize Theorem 3.5 [4], Theorem 4.5 [4], Theorem 2.3 [2], Corollary 2.4 [2], Corollary 2.8 [2] and other results from [1].

2. Preliminaries

Definition 1 ([5]). *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

$(P_1) : p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$,

$(P_2) : p(x, x) \le p(x, y)$,

$(P_3) : p(x, y) = p(y, x)$,

$(P_4) : p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space.

If $p(x, y) = 0$, then $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as a base the family of open p - balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \le p(x, x) + \varepsilon$, for all $x \in X$ and $\varepsilon > 0\}$.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a limit $x \in X$, with respect to τ_p if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

If p is a partial metric on X, then the function $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ define a metric on X.

Further more, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x). \tag{1}$$

Definition 2 ([5]). *Let (X, p) be a partial metric space.*

a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ *exists and is finite.*

b) (X, p) is said to be complete if every Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Lemma 1 ([5], [2]). *Let (X, p) be a partial metric space. Then:*

a) A sequence in X is a Cauchy sequence in (X, p) if and only if is a Cauchy sequence in (X, p^s).

b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

The following results are obtained in [4].

Theorem 2 (Theorem 3.5 [4]). *Let (X, p) be a complete partial metric space and $f, g : X \to X$ be two self mappings. If there exist nonnegative constants A, B, C, D, E with $A + B + C + 2D + E < 1$ and $A + B + C + D + 2E < 1$ such that*

$$p(fx, gy) \leq Ap(x, y) + Bp(x, fx) + Cp(y, gy) + Dp(x, gy) + Ep(y, fx) \qquad (2)$$

for each $x \in X$, then f and g have an unique common fixed point z such that $p(z, z) = 0$.

Theorem 3 (Theorem 4.3 [4]). *Let (X, p) be a complete partial metric space and $f : X \to X$ be a self map. If there exists $k \in \left[0, \frac{1}{2}\right)$ such that*

$$p(fx, fy) \leq k \max\{p(x, y), p(x, fx), p(y, fy), p(x, fy), p(y, fx)\} \qquad (3)$$

for all $x, y \in X$, then f has an unique fixed point z such that $p(z, z) = 0$.

Definition 3 ([3]). *Let X be a nonempty set. Two mappings $f, g : X \to X$ are said to have property Q if $F(f) \cap F(g) = F(f^n) \cap F(g^n)$ for each $n \in \mathbb{N}$, where $F(f) = \{x \in X : x = fx\}$.*

The following theorem is proved in [4].

Theorem 4 (Theorem 3.9 [4]). *Let (X, p) be a complete metric space and let $f, g : X \to X$ be two self mappings. Suppose that*

$$p(fx, gy) \leq Ap(x, y) + B[p(x, fx) + p(y, gy)] + D[p(x, gy) + p(y, fx)] \qquad (4)$$

holds for all $x, y \in X$, where $A, B, D \geq 0$ are such that $A + 2B + 3D < 1$.

3. Implicit relations

Definition 4. *Let $\mathfrak{F}_p$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying*

(F_1) : *F is non - increasing in variable t_2, t_3, t_4, t_5, t_6,*

(F_2) : *There exists $h_1, h_2 \in [0, 1)$ such that for all $u, v \geq 0$ with*

(F_{2a}) : *$F(u, v, v, u, u + v, v) \leq 0$ implies $u \leq h_1 v$,*

(F_{2b}) : *$F(u, v, u, v, v, u + v) \leq 0$ implies $u \leq h_2 v$.*

In the following examples, property (F_1) is obviously.

Example 1. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a,b,c,d,e \geq 0$, $a+b+c+2d+e < 1$ and $a+b+c+d+2e < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - av - bv - cu - d(u+v) - ev \leq 0$. Then $u \leq h_1 v$, where $0 \leq h_1 = \frac{a+b+d+e}{1-(c+d)} < 1$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq h_2 v$, where $0 \leq h_2 = \frac{a+c+d+e}{1-(b+e)} < 1$.

Example 2. *$F(t_1,...,t_6) = t_1 - k\max\{t_2,...,t_6\}$, where $k \in \left[0,\frac{1}{2}\right)$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - k(u+v) \leq 0$. Then $u \leq h_1 v$, where $0 \leq h_1 = \frac{k}{1-k} < 1$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq h_2 v$, where $0 \leq h_2 = h_1 = \frac{k}{1-k} < 1$.

Example 3. *$F(t_1,...,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a,b,c,d \geq 0$ and $a+b+c+2d < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u^2 - u(av+bv+cu) - dv(u+v) \leq 0$. If $u > v$, then $u^2[1-(a+b+c+2d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \sqrt{a+b+c+2d} < 1$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq h_2 v$, where $0 \leq h_2 = h_1 = \sqrt{a+b+c+2d} < 1$.

Example 4. *$F(t_1,...,t_6) = t_1^2 - a\max\{t_2^2,t_3^2,t_4^2\} - bt_5t_6$, where $a,b \geq 0$ and $a+2b < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u^2 - a\max\{u^2,v^2\} - bv(u+v) \leq 0$. If $u > v$, then $u^2[1-(a+2b)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \sqrt{a+2b} < 1$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq h_2 v$, where $0 \leq h_2 = h_1 = \sqrt{a+2b} < 1$.

Example 5. *$F(t_1,...,t_6) = t_1^3 - at_1^2t_2 - ct_2t_3t_4 - dt_1t_5t_6$, where $a,b,c,d \geq 0$ and $a+b+c+2d < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u^3 - auv^2 - buv^2 - cv^2u - dv(u+v)v \leq 0$. If $u > v$, then $u^3[1-(a+b+c+2d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \sqrt[3]{a+b+c+2d} < 1$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq h_2 v$, where $0 \leq h_2 = h_1 = \sqrt[3]{a+b+c+2d} < 1$.

Example 6. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5+t_6\}$, where $a,b,c \geq 0$ and $a+b+3c < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,u+v,v) = u - av - bv - c\max\{2u, u+2v\} \leq 0$. If $u > v$, then $u[1-(a+b+3c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = a+b+3c < 1$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq h_2 v$, where $0 \leq h_2 = \frac{a+2c}{1-(b+c)} < 1$.

4. Fixed point theorems

Theorem 5. *Let (X,p) be a complete partial metric space and let $f,g : X \to X$ be two self mappings. If*

$$F(p(fx,gy),p(x,y),p(x,fx),p(y,gy),p(x,gy),p(y,fx)) \leq 0 \tag{5}$$

for all $x,y \in X$, where $F \in \mathfrak{F}_p$. Then, f and g have an unique common fixed point z such that $p(z,z)=0$.

Proof. Let $x_0 \in X$ be an arbitrary point in X. If $fx_0 = x_0$ and $gx_0 \neq x_0$, by (5) we have successively

$$F(p(fx_0,gx_0),p(x_0,x_0),p(x_0,fx_0),p(x_0,gx_0),p(x_0,gx_0),p(x_0,fx_0)) \leq 0$$

$$F(p(x_0,gx_0),p(x_0,x_0),p(x_0,x_0),p(x_0,gx_0),p(x_0,gx_0),p(x_0,x_0)) \leq 0.$$

By (F_1) and (P_2) we obtain

$$F(p(x_0,gx_0),p(x_0,gx_0),p(x_0,gx_0),p(x_0,gx_0),p(x_0,gx_0),p(x_0,gx_0)) \leq 0.$$

By (F_{2a}) we obtain

$$p(x_0,gx_0) \leq h_1 p(x_0,gx_0) < p(x_0,gx_0),$$

a contradiction. Hence, $p(x_0,gx_0)=0$ which implies $x_0 = gx_0$. Thus x_0 is a common fixed point of f and g. So we assume that $x_0 \neq fx_0$ and $x_0 \neq gx_0$. Then, we define the sequence $\{x_n\}$ in X such that

$$x_{2n+1} = fx_{2n} \text{ and } x_{2n+2} = gx_{2n+1}, \text{ for } n \in \mathbb{N} \tag{6}$$

By (5) for $x = x_{2n}$ and $y = x_{2n+1}$ we get that

$$\begin{gathered} F(p(fx_{2n},gx_{2n+1}),p(x_{2n},x_{2n+1}),p(x_{2n},fx_{2n}), \\ p(x_{2n+1},gx_{2n+1}),p(x_{2n},gx_{2n+1}),p(x_{2n+1},fx_{2n})) \leq 0. \end{gathered}$$

By (6) we obtain

$$\begin{gathered} F(p(x_{2n+1},x_{2n+2}),p(x_{2n},x_{2n+1}),p(x_{2n},x_{2n+1}), \\ p(x_{2n+1},x_{2n+2}),p(x_{2n},x_{2n+2}),p(x_{2n+1},x_{2n+1})) \leq 0. \end{gathered} \tag{7}$$

Since by (P_4)

$$\begin{aligned} p(x_{2n},x_{2n+2}) &\leq p(x_{2n},x_{2n+1}) + p(x_{2n+1},x_{2n+2}) - p(x_{2n+1},x_{2n+1}) \\ &\leq p(x_{2n},x_{2n+1}) + p(x_{2n+1},x_{2n+2}) \end{aligned}$$

and by (P_2)

$$p(x_{2n+1}, x_{2n+1}) \leq p(x_{2n}, x_{2n+1}),$$

by (7) and (F_1) we obtain

$$\begin{gathered}F(p(x_{2n+1}, x_{2n+2}), p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}),\\ p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}), p(x_{2n}, x_{2n+1})) \leq 0.\end{gathered}$$

By (F_{2a}) we obtain

$$p(x_{2n+1}, x_{2n+2}) \leq hp(x_{2n}, x_{2n+1}),$$

where $h = \max\{h_1, h_2\}$.

Similarly, by (5) we have

$$\begin{gathered}F(p(fx_{2n}, gx_{2n-1}), p(x_{2n}, x_{2n-1}), p(x_{2n}, fx_{2n}),\\ p(x_{2n-1}, gx_{2n-1}), p(x_{2n}, gx_{2n-1}), p(x_{2n-1}, fx_{2n})) \leq 0.\end{gathered}$$

By (6) we obtain

$$\begin{gathered}F(p(x_{2n+1}, x_{2n}), p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}),\\ p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n}), p(x_{2n-1}, x_{2n+1})) \leq 0.\end{gathered} \tag{8}$$

Since by (P_4)

$$\begin{aligned}p(x_{2n-1}, x_{2n+1}) &\leq p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n})\\ &\leq p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}),\end{aligned}$$

and by (P_2),

$$p(x_{2n}, x_{2n}) \leq p(x_{2n}, x_{2n-1}),$$

by (8) we obtain

$$\begin{gathered}F(p(x_{2n+1}, x_{2n}), p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}),\\ p(x_{2n}, x_{2n-1}), p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})) \leq 0.\end{gathered}$$

By (F_{2b}) we obtain

$$p(x_{2n}, x_{2n+1}) \leq hp(x_{2n}, x_{2n-1}).$$

Hence

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n) \leq ... \leq h^n p(x_0, x_1).$$

For $n > m$ we have

$$\begin{aligned} p(x_n, x_m) &\leq (h^m + h^{m+1} + ... + h^{n-1})p(x_0, x_1) \\ &\leq \frac{h^m}{1-h}p(x_0, x_1) \end{aligned}$$

and so $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. From the definition of p^s we get

$$p^s(x_n, x_m) \leq 2p(x_n, x_m) \to 0 \text{ as } n, m \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in metric space (X, p^s). Since (X, p) is complete, then by Lemma 1, (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $z \in X$ with respect to the metric p^s, that is, $\lim_{n\to\infty} p^s(x_n, z) = 0$. Again, by (1) we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{m,n\to\infty} p(x_n, x_m) = 0.$$

By (5) for $x = z$ and $y = x_{2n+1}$ we obtain

$$F(p(fz, gx_{2n+1}), p(z, x_{2n+1}), p(z, fz), p(x_{2n+1}, gx_{2n+1}), p(z, gx_{2n+1}), p(x_{2n+1}, fz)) \leq 0,$$

$$F(p(fz, x_{2n+2}), p(z, x_{2n+1}), p(z, fz), p(x_{2n+1}, x_{2n+2}), p(z, x_{2n+2}), p(x_{2n+1}, fz)) \leq 0.$$

Letting n tends to infinity we obtain

$$F(p(fz, z), 0, p(z, fz), 0, 0, p(z, fz)) \leq 0,$$

which implies by (F_{2b}) that $p(z, fz) = 0$, hence $z = fz$.

Similarly,

$$F(p(fx_{2n}, gz), p(x_{2n}, z), p(x_{2n}, fz), p(z, gz), p(x_{2n}, gz), p(z, fx_{2n})) \leq 0,$$

$$F(p(x_{2n+1}, gz), p(x_{2n}, z), p(x_{2n}, fz), p(z, gz), p(x_{2n}, gz), p(z, x_{2n+1})) \leq 0.$$

Letting n tends to infinity we obtain

$$F(p(z, gz), 0, 0, p(z, gz), p(z, gz), 0) \leq 0,$$

which implies by (F_{2b}) that $p(z, gz) = 0$. Hence $z = gz$ and z is the common fixed point of f and g. On the other hand by (P_2):

$$p(z, z) \leq p(z, fz) = 0,$$

hence $p(z, z) = 0$.

Suppose that there exists an other point u such that $fu = gu = u$. By (5) we obtain

$$F(p(fz, gu), p(z, u), p(z, fz), p(u, gu), p(z, gu), p(u, fz)) \leq 0,$$

$$F(p(z, u), p(z, u), p(z, z), p(u, u), p(z, u), p(u, z)) \leq 0,$$

By (P_2) and (F_1) we obtain

$$F(p(z, u), p(z, u), p(z, u), p(z, u), 2p(z, u), p(z, u)) \leq 0$$

which implies by (F_{2a}) that

$$p(z, u) \leq h_1 p(z, u) < p(z, u),$$

a contradiction. Hence $p(z, u) = 0$, i.e. $u = z$.

Remark 1. *1) By Theorem 1 and Example 1 we obtain Theorem 2.*
2) By Theorem 1 and Example 2 we obtain the following result

Theorem 6. *Let (X, p) be a complete partial metric space and let $f, g : X \to X$ be two self mappings. If there exists $k \in \left[0, \frac{1}{2}\right)$ such that*

$$p(fx, gy) \leq k \max\{p(x, y), p(x, fx), p(y, fy), p(x, gy), p(y, fx)\}, \tag{9}$$

for all $x, y \in X$, then, f and g have an unique common fixed point z such that $p(z, z) = 0$.

Example 7. *Let $X = [0, 1]$ and $p(x, y) = max\{x, y\}$. Then (X, p) is a complete partial metric space and p is not a metric. Then for mappings $f, g : X \to X$, where $f(x) = \dfrac{x^2}{1 + 3x}$ and $g(x) = 0$ we have*

$$p(fx, gy) = \max\{fx, 0\} = fx = \frac{x^2}{1 + 3x} \leq \frac{1}{4} \cdot x = \frac{1}{4} p(x, gy).$$

Hence

$$p(fx, gy) \leq h \max\{p(x, y), p(x, fx), p(y, gy), p(x, gy), p(y, , fx)\},$$

where $h = \dfrac{1}{4} < \dfrac{1}{2}$.

Then by Theorem 6, f and g have a unique common fixed point $x = 0$.

Remark 2. *By Theorem 5 and Examples 3 - 6 we obtain new particular results.*

If $f = g$, then by Theorem 1 we obtain

Theorem 7. *Let (X, p) be a complete partial metric space and let $f : X \to X$ be a self mapping. If*

$$F(p(fx, fy), p(x, y), p(x, fx), p(y, fy), p(x, fy), p(y, fx)) \leq 0, \tag{10}$$

for all $x, y \in X$ and F satisfies properties (F_1) and (F_{2a}), then, f has an unique fixed point z such that $p(z, z) = 0$.

Remark 3. *By Example 2 and Theorem 7 we obtain Theorem 3.*

Corollary 8 (Theorem 2.3 [2])**.** *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a self mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq \max\{ap(x, y), bp(x, Tx), cp(y, Ty), d[p(x, Ty)+p(y, Tx)], p(x, x), p(y, y)\},$$

where $a, b, c \in [0, 1)$ and $d \in \left[0, \frac{1}{2}\right)$. Then T has an unique fixed point z and $p(z, z) = 0$.

Proof. By (P_2),

$$\begin{aligned} p(x, x) &\leq p(x, Tx), \\ p(y, y) &\leq p(y, Ty) \end{aligned}$$

and

$$\begin{aligned} &\max\{ap(x, y), bp(x, Tx), cp(y, Ty), 2d\tfrac{[p(x,Ty)+p(y,Tx)]}{2}, p(x, x), p(y, y)\} \leq \\ &\leq k \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}, \end{aligned}$$

where $k = \max\{a, b, c, 2d\} < 1$ and the proof follows from Theorem 7 and Examples 2.

Remark 4. *A similar result with Corollary 8 we obtain by Example 1 and Theorem 7 (Corollary 2.8 [2]).*

5. Q - PROPERTY IN PARTIAL METRIC SPACES

Definition 5. *Let $\mathfrak{F}_q$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying*

(F_q) : *There exists $q \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, 0, u + v, u, v) \leq 0$ implies $u \leq qv$.*

Example 8. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d \geq 0$ and $a + 2c + d + e < 1$.*

(F_q) : *Let $u, v \geq 0$ such that $F(u, v, 0, u+v, u, v) = u-av-c(u+v)-du-ev \leq 0$, which implies $u \leq qv$, where $0 < q = \frac{a+c+e}{1-(c+d)} < 1$.*

Example 9. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, ..., t_6\}$, *where* $k \in \left(0, \frac{1}{2}\right)$.

(F_q) : *Let* $u, v \geq 0$ *such that* $F(u, v, 0, u + v, u, v) = u - k(u + v) \leq 0$, *which implies* $u \leq qv$, *where* $0 < q = \frac{k}{1-k} < 1$.

Example 10. $F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, *where* $a, b, c, d \geq 0$ *and* $a + 2c + d < 1$.

(F_q) : *Let* $u, v \geq 0$ *such that* $F(u, v, 0, u+v, u, v) = u^2 - u[av + c(u+v)] - duv \leq 0$. *If* $u > 0$, *then* $u - [av + c(u + v)] - dv \leq 0$ *which implies* $u \leq qv$, *where* $0 < q = \frac{a+c+d}{1-c} < 1$. *If* $u = 0$ *then* $u \leq qv$.

Example 11. $F(t_1, ..., t_6) = t_1^2 - a\max\{t_2^2, t_3^2, t_4^2\} - bt_5t_6$, *where* $a, b \geq 0$ *and* $0 < 4a + b < 1$.

(F_q) : *Let* $u, v \geq 0$ *such that* $F(u, v, 0, u+v, u, v) = u^2 - a\max\{v^2, (u+v)^2\} - buv = u^2 - a(u+v)^2 - buv \leq 0$. *If* $u > v$, *then* $u^2[1 - (4a + b)] < 0$, *a contradiction. Hence* $u \leq v$ *which implies* $u \leq qv$, *where* $0 < q = \sqrt{4a + b} < 1$.

Example 12. $F(t_1, ..., t_6) = t_1^3 - at_1^2t_2 - bt_1t_2^2 - ct_2t_3t_4 - dt_1t_5t_6$, *where* $a, b, c, d \geq 0$ *and* $0 < a + b + d < 1$.

(F_q) : *Let* $u, v \geq 0$ *such that* $F(u, v, 0, u+v, u, v) = u^3 - au^2v - buv^2 - du^2v \leq 0$. *Let* $u > 0$, *then* $u^2 - auv - bv^2 - duv \leq 0$. *If* $u > v$, *then* $u^2[1 - (a + b + d)] \leq 0$, *a contradiction. Hence* $u \leq v$ *which implies* $u \leq qv$, *where* $0 < q = \sqrt{a + b + d} < 1$. *If* $u = 0$, *then* $u \leq qv$.

Example 13. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5 + t_6\}$, *where* $a, b, c \geq 0$ *and* $0 < a + 4c < 1$.

(F_q) : *Let* $u, v \geq 0$ *such that* $F(u, v, 0, u + v, u, v) = u - av - 2c(u + v) \leq 0$, *which implies* $u \leq qv$, *where* $0 < q = \frac{a+2c}{1-2c} < 1$.

Theorem 9. *Let* (X, p) *be a complete partial metric space and let* $f, g : X \to X$ *be satisfying the conditions of Theorem 5. If* $F \in \mathfrak{F}_q$, *then* f *and* g *have property* Q.

Proof. By Theorem 5 we have that $F(f) \cap F(g) = \{z\}$, where z is the unique common fixed point of f and g and $p(z, z) = 0$. So $F(f^n) \cap F(g^n) \neq \emptyset$, for each $n \in \mathbb{N}$. Let $u \in F(f^n) \cap F(g^n)$, where $n > 1$ is arbitrary and suppose that $n \neq 2$. Then (5) we have successively

$$F(p(f^n z, g^n u), p(f^{n-1}z, g^{n-1}u), p(f^{n-1}z, f^n z),$$
$$p(g^{n-1}u, g^n u), p(f^{n-1}z, g^n u), p(g^{n-1}u, f^n z)) \leq 0,$$

$$F(p(z, g^n u), p(z, g^{n-1}u), p(z, z), p(g^{n-1}u, g^n u), p(z, g^n u), p(z, g^{n-1}u)) \leq 0.$$

By (P_2) and (F_1) we obtain

$$F(p(z, g^n u), p(z, g^{n-1}u), 0, p(g^{n-1}u, z) + p(z, g^n u), p(z, g^n u), p(z, g^{n-1}u)) \leq 0,$$

which implies by (F_q) that

$$p(z, g^n u) \leq qp(z, g^{n-1}u) \leq ... \leq q^n p(z, u).$$

Since $g^n u = gu$ then $p(z, u) \leq q^n p(z, u)$, so $p(z, u)(1 - q^n) \leq 0$ which implies $p(z, u) = 0$. Hence $u = z$, which implies that f and g have property Q.

Remark 5. *By Examples 1 and 8 and Theorem 9 we obtain Theorem 4.*

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