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COMMON FIXED POINT RESULTS FOR SINGLE AND MULTIVALUED MAPS IN FUZZY METRIC SPACES USING COUNTABLE EXTENSION OF T-NORMS

T. Došenović, S. Sedghi, I. Altun

Abstract. The aim of this paper is to prove some common fixed point theorem for four single-valued mappings in complete fuzzy metric spaces and a common fixed point theorem for single-valued mappings $I, J : X \to X$ and multi-valued mappings F, $G : X \to CB(X)$ using an implicit relation. In this paper it is shown that the theorems can be apply to a much spreading class of t-norms using the theory of the countable extension given in [7].

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1. Introduction and Preliminaries

Banach fixed point theorem is one of the most cited theorem in this field of mathematics. It was followed by a large number of generalizations Banach contraction principle in the metric spaces, as well as in the spaces that represent a generalization of metric spaces. One of the expansions represent the fuzzy metric spaces. There are different definitions of this spaces, and in this paper we will use the George and Veeramani [4, 5] definition of fuzzy metric spaces. The reason for this is obtained a Hausdorff topology for this kind of fuzzy metric spaces. Accordingly, many authors translated the various contraction mappings from metric to fuzzy metric spaces, using mainly minimum t-norm (T_M). In the book [7] they investigated the classes of t-norms T and sequences (x_n) from the interval [0, 1] such that $\lim_{n\to\infty} x_n = 1$ and

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{\infty} x_{n+i} = 1. \tag{1}$$

V. Popa [13] introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. After that, the idea of using the implicit relations appeared in many papers and we mention only a few ([1]-[3], [10], [14]-[17], [19]). B. Singh and S. Jain [17] extended the result of Popa in fuzzy metric spaces. In this paper using the notion of compatible and weakly f-compatible mapping, and the notion of countable extension of t-norm we prove a common fixed point theorem for four mappings in complete fuzzy metric space. This result is generalization of result given in [14].

S. Sedghi et al. [15] proved a common fixed point theorem for multi-valued mappings satisfying an implicit relation in fuzzy metric spaces, where they used a t-norm satisfying condition $T(t,t) = t$. In this paper, it will be proved a common fixed point theorem for two pairs of weakly compatible mappings in fuzzy metric spaces using implicit relations for more general condition using the results given in [7, 9] about the theory of countable extension of a t-norm.

We begin our paper with the basic definitions and theorems.

Definition 1. *A triangular norm is a function $T : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties: commutativity, monotonicity, associativity and the number 1 acts as identity element.*

The following are the elementary t-norms :

$$T_M(x,y) = \min(x,y),\ T_P(x,y) = x \cdot y,\ T_L(x,y) = \max(x+y-1,0)$$

and

$$T_D(x,\,y) = \begin{cases} \min(x,\,y), & \max(x,\,y) = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Very important family of t-norms are Dombi, Aczél-Alsina, Sugeno-Weber and Schweizer-Sklar. For this familly of t-norms very useful results are obtained (see [7]).

- The Dombi family of t-norms $(T^D_\lambda)_{\lambda\in[0,\,\infty]}$, which is defined by

$$T^D_\lambda(x,\,y) = \begin{cases} T_D(x,\,y), & \lambda = 0 \\ T_M(x,\,y), & \lambda = \infty \\ \frac{1}{1+\left(\left(\frac{1-x}{x}\right)^\lambda+\left(\frac{1-y}{y}\right)\right)^{1/\lambda}}, & \lambda \in (0,\,\infty). \end{cases}$$

- The Aczél-Alsina family of t-norms $(T^{AA}_\lambda)_{\lambda\in[0,\,\infty]}$, which is defined by

$$T^{AA}_\lambda(x,\,y) = \begin{cases} T_D(x,\,y), & \lambda = 0 \\ T_M(x,\,y), & \lambda = \infty \\ e^{-((-\log x)^\lambda+(-\log y)^\lambda)^{1/\lambda}}, & \lambda \in (0,\,\infty). \end{cases}$$

- Sugeno-Weber family of t-norms $(T_\lambda^{SW})_{\lambda\in[-1,\infty]}$, which is defined by

$$T_\lambda^{SW}(x,\, y) = \begin{cases} T_D(x,\, y), & \lambda = -1 \\ T_P(x,\, y), & \lambda = \infty \\ \max\left(0,\, \frac{x+y-1+\lambda xy}{1+\lambda}\right), & \lambda \in (-1,\, \infty). \end{cases}$$

- The Schweizer-Sklar family of t-norms $(T_\lambda^{SS})_{\lambda\in[-\infty,\infty]}$, is defined by

$$T_\lambda^{SS}(x,\, y) = \begin{cases} T_M(x,\, y), & \lambda = -\infty \\ T_P(x,\, y), & \lambda = 0 \\ (\max(0,\, (x^\lambda + y^\lambda - 1))^{\frac{1}{\lambda}}, & \lambda \in (-\infty,\, 0) \cup (0,\, \infty), \\ T_D(x,\, y), & \lambda = \infty \end{cases}$$

Very important class of t-norms ([6]) is t-norms of H-type.

Let T be a t-norm and $T_n : [0,\, 1] \to [0,\, 1]$ $(n \in \mathbb{N})$ is defined in the following way:

$$T_1(x) = T(x,\, x),\ T_{n+1}(x) = T(T_n(x),\, x)\ (n \in \mathbb{N},\, x \in [0,\, 1]).$$

We say that t-norm T is of the H-type if T is continuous and the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$.

A trivial example of t-norms of H-type is $T = T_M$. A nontrivial example is given in the paper [6].

An arbitrary t-norm T can be extended (by associativity) (see [9]) in a unique way to an n-ary operation taking for $(x_1, \ldots, x_n) \in [0,1]^n$ the values

$$\mathbf{T}_{i=1}^{0} x_i = 1,\ \mathbf{T}_{i=1}^{n} x_i = T(\mathbf{T}_{i=1}^{n-1} x_i, x_n) = T(x_1, \ldots, x_n).$$

For example

$$T_L(x_1,\, x_2, \ldots, x_n) = \max\{\sum_{i=1}^{n} x_i - (n-1),\, 0\}$$

$$T_M(x_1,\, x_2, \ldots, x_n) = \min\{x_1,\, x_2, \ldots, x_n\}$$

$$T_P(x_1,\, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n.$$

We can extend T to a countable infinity operation taking for any sequence $(x_n)_{n\in\,\mathbb{N}}$ from $[0,1]$ the values

$$\mathbf{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{n} x_i.$$

Since the sequence $(\mathbf{T}_{i=1}^{n} x_i)_{n\in\,\mathbb{N}}$ is non-increasing and bounded from below, the limit of right side exists. In [7] the following results are obtained:

- If $T = T_L$ or $T = T_P$, then

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty}(1-x_i) < \infty,$$

  and also that for $T \geq T_L$ following implication holds

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = 1 \Rightarrow \sum_{i=1}^{\infty}(1-x_i) < \infty.$$

  For example, the condition $T \geq T_L$ is fulfilled by the families $(T_{\lambda}^{SS})_{\lambda\in[-\infty,1]}$ and $(T_{\lambda}^{SW})_{\lambda\in[0,\infty]}$.

- If $(T_{\lambda}^{*})_{\lambda\in(0,\,\infty)}$ is the Dombi family of t-norms or the Aczél-Alsina family of t-norms and if $(x_n)_{n\in\,\mathbb{N}}$ is a sequence of elements from $(0,\,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$, then

$$\lim_{n\to\infty} (\mathbf{T}_{\lambda}^{*})_{i=n}^{\infty} x_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty}(1-x_i)^{\lambda} < \infty. \tag{2}$$

- If $(T_{\lambda}^{SW})_{\lambda\in(-1,\,\infty]}$ is the Sugeno-Weber family of t-norms and $(x_n)_{n\in\,\mathbb{N}}$ is a sequence of elements from $(0,\,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$, then

$$\lim_{n\to\infty} (\mathbf{T}_{\lambda}^{SW})_{i=n}^{\infty} x_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty}(1-x_i) < \infty. \tag{3}$$

Proposition 1. *Let $(x_n)_{n\in\,\mathbb{N}}$ be a sequence of numbers from $[0,\,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$ and t-norm T is of H-type. Then*

$$\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} x_{n+i} = 1.$$

Definition 2 ([4])**.** *The 3-tuple $(X,\,M,\,T)$ is said to be a fuzzy metric space if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\,\infty)$ satisfies the following conditions:*
$M(x,\,y,\,t) > 0$, for every $x,\,y \in X,\ t > 0$,
$M(x,\,y,\,t) = 1$ for every $t > 0 \Leftrightarrow x = y$,
$M(x,\,y,\,t) = M(y,\,x,\,t)$, for every $x,\,y \in X,\ t > 0$,
$T(M(x,\,y,\,t),\,M(y,\,z,\,s)) \leq M(x,\,z,\,t+s)$, for every $x,\,y,\,z \in X,\ t,\,s > 0$,
$M(x,\,y,\,\cdot) : (0,\,\infty) \to [0,\,1]$ is continuous for every $x,\,y \in X$.

It is well known that $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$. If (X, M, T) is a fuzzy metric space the family $\{B(x, r, t) : x \in X, r \in (0, 1), t > 0\}$, where $B(x, r, t) = \{y : y \in X, M(x, y, t) > 1 - r\}$ is a neighborhood's system for a Hausdorff topology on X.

Definition 3. *A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a fuzzy metric space (X, M, T) is a Cauchy sequence if and only if for every $\varepsilon \in (0, 1)$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for every $n, m \geq n_0$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a fuzzy metric space (X, M, T) is said to converge to x if and only if for every $\varepsilon \in (0, 1)$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for every $n \geq n_0$. A fuzzy metric space (X, M, T) is comlete if every Cauchy sequence is convergent in X.*

Lemma 1 ([11])**.** *If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

Definition 4 ([11])**.** *Let f and g be self maps on a fuzzy metric space. The pair f and g is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.*

Definition 5 ([8])**.** *Let f and g be self maps on a fuzzy metric space. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$.*

Definition 6 ([12])**.** *The pair (f, g) is said to be weakly f-compatible if either $\lim_{n\to\infty} gfx_n = fz$ or $\lim_{n\to\infty} ggx_n = fz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ and $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = fz$ for some $z \in X$.*

Similarly, it can be defined weak g-compatibility of the pair (f, g).

Definition 4 and Definition 6 imply that the pair (f, g) is coincidentally commuting or a weakly compatible pair and Definition 4 imply Definition 6. The weakly f-compatible pair (f, g) need not be compatible (see Example 1.11 [14]).

Definition 7. *[12] The pair (f, g) is said to be f-continuous if $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = fz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.*

2. Main result for single-valued case

At the beginning, we mention the theorem that has been proved in [18].

Theorem 2 ([18]). *Let T be a continuous t-norm. For every monotone non-increasing function $H : (0, \infty) \to [0, 1]$, the following implication holds: If for some $\sigma_0 \in (0, 1)$, $\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} H(\sigma_0^i) = 1$, then $\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} H(\sigma^i) = 1$, for every $\sigma \in (0, 1)$.*

Proof. If $\sigma < \sigma_0$ then $\sigma^i < \sigma_0^i$, $i \in \mathbb{N}$, and since H is non-increasing $H(\sigma^i) \geq H(\sigma_0^i)$, $i \in \mathbb{N}$, thence $\mathbf{T}_{i=n}^{\infty} H(\sigma^i) \geq \mathbf{T}_{i=n}^{\infty} H(\sigma_0^i)$, for every $n \in \mathbb{N}$. Obviously, previous implication hold. Let $\sigma \geq \sigma_0$. First, we shall suppose that $\sigma = \sqrt{\sigma_0}$. Then

$$\begin{aligned} \mathbf{T}_{i=2n}^{\infty} H(\sigma^i) &= T(\mathbf{T}_{i=n}^{\infty} H(\sigma^{2i}), \mathbf{T}_{i=n}^{\infty} H(\sigma^{2i+1})) \\ &\geq T(\mathbf{T}_{i=n}^{\infty} H(\sigma_0^i), \mathbf{T}_{i=n}^{\infty} H(\sigma_0^i)) \end{aligned}$$

an if $\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} H(\sigma_0^i) = 1$, since T is continuous, we obtain that

$$\lim_{n\to\infty} \mathbf{T}_{i=2n}^{\infty} H(\sigma^i) \geq T(1, 1) = 1.$$

Since

$$\lim_{n\to\infty} \mathbf{T}_{i=2n+1}^{\infty} H(\sigma^i) \geq \lim_{n\to\infty} \mathbf{T}_{i=2n}^{\infty} H(\sigma^i)$$

it follows that

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} H(\sigma^i) = 1$$

for $\sigma = \sqrt{\sigma_0}$. For an arbitrary $\sigma > \sigma_0$, there exist $n \in \mathbb{N}$, such that $\sigma_0^{2^{-n}} > \sigma$, and we can repeat the above process n-times.

Theorem 3 ([18]). *Let (X, M, T) be a fuzzy metric space, such that $\lim_{t\to\infty} M(x, y, t) = 1$. If for some $\sigma_0 \in (0, 1)$ and some $x, y \in X$ the following hold:*

$$\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} M(x_0, y_0, \frac{1}{\sigma_0}) = 1,$$

then

$$\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} M(x_0, y_0, \frac{1}{\sigma}) = 1$$

for every $\sigma \in (0, 1)$.

Proof. Define a function $H(t) = M(x_0, y_0, \frac{1}{t})$, $t > 0$. Since $M(x_0, y_0, \cdot)$ is non-decreasing, it is obvious that H is non-increasing mapping from $(0, \infty)$ into $[0, 1]$. So, from Theorem 2, the condition is satisfied.

Let Φ denote the set of all continuous functions $\phi : [0,1]^6 \to \mathbb{R}^+$ satisfies the conditions

(ϕ_1) : ϕ is decreasing in t_2, t_3, t_4, t_5 and t_6

(ϕ_2) : $\phi(u, v, v, v, v, T(v, v)) \geq 0$ or $\phi(u, v, v, v, T(v, v), v) \geq 0$ implies $u \geq v$ for all u, $v \in [0, 1]$.

Example 1. *(i) Let* $T(a, b) = \min(a,\ b)$ *and*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\},$$

then $\phi \in \Phi$.

(ii) Let $T(a, b) = \min(a,\ b)$ *or* $T(a, b) = a \cdot b$ *or* $T(a, b) = \max(a + b - 1, 0)$ *and*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4, t_5, t_6\},$$

then $\phi \in \Phi$.

Theorem 4. *Let* (X, M, T) *be a complete fuzzy metric space such that* $\lim_{t\to\infty} M(x, y, t) = 1$ *and let* f, g, S *and* P *be self maps of* X *such that*

a) $f(X) \subseteq P(X)$ *and* $g(X) \subseteq S(X)$,

b1)

$$\begin{array}{r} \phi(M(fx, gy, kt), M(Sx, Py, t), M(fx, Sx, t), M(gy, Py, t), \\ M(fx, Py, t), M(gy, Sx, 2t)) \end{array} \geq 0$$

for all $x, y \in X$, *for all* $t > 0$, *where* $k \in (0,\ 1)$ *and* $\phi \in \Phi$.

b2)

$$\begin{array}{c} \phi(M(fx, gy, kt), M(Sx, Py, t), M(fx, Sx, t), M(gy, Py, t), \\ M(fx, Py, 2t), M(gy, Sx, t)) \end{array} \geq 0$$

for all $x, y \in X$, *for all* $t > 0$, *where* $k \in (0,\ 1)$ *and* $\phi \in \Phi$.

c) (f, S) *is weakly* S*-compatible,* (g, P) *is weakly* P*-compatible and either* (f, S) *is* S*-continuous or* (g, P) *is* P*-continuous*

or

d) (f, S) *is weakly* f*-compatible,* (g, P) *is weakly* g*-compatible and either* (f, S) *is* f*-continuous or* (g, P) *is* g*-continuous.*

Suppose there exist $x_0, x_1 \in X$, $f(x_0) = P(x_1)$, *and* $\mu \in (k, 1)$ *such that*

$$\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} M(f(x_0), g(x_1), \frac{1}{\mu^i}) = 1.$$

Then f, g, S *and* P *have a unique common fixed point* $z \in X$, *and* z *is the unique common fixed point of* f *and* S *and of* g *and* P.

Proof. Let $x_0 \in X$ be arbitrary point. By (a) there exist $x_1 \in X$ such that $fx_0 = Px_1$, and there exist $x_2 \in X$ such that $gx_1 = Sx_2$. Inductively we construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$fx_{2n} = Px_{2n+1} = y_{2n},\ gx_{2n+1} = Sx_{2n+2} = y_{2n+1}.$$

Let $d_n(t) = M(y_n, y_{n+1}, t)$ for all $t > 0$. Now using (b1) with $x = x_{2n}$ and $y = x_{2n+1}$ we get

$$\begin{aligned} 0 &\le \phi(M(fx_{2n}, gx_{2n+1}, kt), M(Sx_{2n}, Px_{2n+1}, t), M(fx_{2n}, Sx_{2n}, t), \\ &\quad M(gx_{2n+1}, Px_{2n+1}, t), M(fx_{2n}, Px_{2n+1}, t), M(gx_{2n+1}, Sx_{2n}, 2t)) \\ &= \phi(M(y_{2n}, y_{2n+1}, kt), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), \\ &\quad M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, 2t)) \\ &\le \phi(d_{2n}(kt), d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), 1, T(d_{2n}(t), d_{2n-1}(t))) \end{aligned}$$

since the function ϕ is decreasing in six variable. Now, suppose that $d_{2n-1}(t) > d_{2n}(t)$, then

$$\begin{aligned} 0 &\le \phi(d_{2n}(kt), d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), 1, T(d_{2n}(t), d_{2n-1}(t))) \\ &\le \phi(d_{2n}(kt), d_{2n}(t), d_{2n}(t), d_{2n}(t), d_{2n}(t), T(d_{2n}(t), d_{2n}(t))) \\ &= \phi(u, v, v, v, v, T(v, v)). \end{aligned}$$

Then from (ϕ_2) we have $d_{2n}(kt) \ge d_{2n}(t)$ a contradiction. Hence $d_{2n-1}(t) \le d_{2n}(t)$. Again, using (b1), (ϕ_1) and (ϕ_2) we have

$$0 \le \phi(d_{2n}(kt), d_{2n-1}(t), d_{2n-1}(t), d_{2n-1}(t), d_{2n-1}(t), T(d_{2n-1}(t), d_{2n-1}(t)))$$

and so,

i) $d_{2n}(kt) \ge d_{2n-1}(t)$.

Similarly, putting $x = x_{2n}, y = x_{2n-1}$ in (b2) we can show that

ii) $d_{2n-1}(kt) \ge d_{2n-2}(t)$.

From (i) and (ii) we have $d_n(kt) \ge d_{n-1}(t)$ and

$$M(y_n, y_{n+1}, t) \ge M(y_{n-1}, y_n, \frac{t}{k}) \ge \cdots \ge M(y_0, y_1, \frac{t}{k^n}) \to 1$$

as $n \to \infty$.

It remains to prove that the sequence $\{y_n\}$ is a Cauchy sequence. Let $\sigma = \frac{k}{\mu}$. Since $0 < \sigma < 1$ the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0}^{\infty} \sigma^i < 1$ Hence for $n > n_0$ and every $m \in \mathbb{N}$

$$t > t \sum_{i=n_0}^{\infty} \sigma^i > t \sum_{i=n}^{n+m-1} \sigma^i.$$

Then we have,

$$\begin{aligned}
M(y_{n+m}, y_n, t) &\geq M(y_{n+m}, y_n, t\sum_{i=n}^{n+m-1}\sigma^i) \\
&\geq \underbrace{T(T(\ldots T}_{m-times}(M(y_{n+m}, y_{n+m-1}, t\sigma^{n+m-1}), \ldots, M(y_{n+1}, y_n, t\sigma^n))) \\
&\geq \underbrace{T(T(\ldots T}_{m-times}(M(y_0, y_1\frac{t\sigma^{n+m-1}}{k^{n+m-1}}), \ldots, M(y_0, y_1, \frac{t\sigma^n}{k^n}))) \\
&= \underbrace{T(T(\ldots T}_{m-times}(M(y_0, y_1\frac{t}{\mu^{n+m-1}}), \ldots, M(y_0, y_1, \frac{t}{\mu^n}))) \\
&= \mathbf{T}_{i=n}^{n+m-1} M(y_0, y_1, \frac{t}{\mu^i}) \\
&\geq \mathbf{T}_{i=n}^{\infty} M(y_0, y_1, \frac{t}{\mu^i}).
\end{aligned}$$

Based on Theorem 3, $\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} M(y_0, y_1, \frac{1}{\mu^i}) = 1$ implies $\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} M(y_0, y_1, \frac{t}{\mu^i}) = 1$ for every $t > 0$.

Now, for every $t > 0$ and every $\lambda \in (0, 1)$ there exist $n_1(t, \lambda)$ such that $M(y_{n+m}, y_n, t) > 1 - \lambda$ for every $n \geq n_1(t, \lambda)$ and every $m \in \mathbb{N}$.

So, the sequence $\{y_n\}$ is a Cauchy sequence and from the completeness of X, there exist $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Now, suppose that (f, S) is S-continuous. Then $Sfx_{2n} \to Sz$ and $SSx_{2n} \to Sz$ as $n \to \infty$. Since (f, S) is weakly S-compatible we have $fSx_{2n} \to Sz$ or $ffx_{2n} \to Sz$ as $n \to \infty$. Suppose that $fSx_{2n} \to Sz$ as $n \to \infty$. Then putting $x = Sx_{2n}$, $y = x_{2n+1}$ in (b) we get

$$\begin{aligned}
0 \leq\ & \phi(M(fSx_{2n}, gx_{2n+1}, kt),\ M(SSx_{2n}, Px_{2n+1}, t), M(fSx_{2n}, SSx_{2n}, t), \\
& M(gx_{2n+1}, Px_{2n+1}, t), M(fSx_{2n}, Px_{2n+1}, t),\ M(gx_{2n+1}, SSx_{2n}, 2t))
\end{aligned}$$

Letting $n \to \infty$, and using condition (ϕ_1) we have

$$\begin{aligned}
0 &\leq \phi(M(Sz, z, kt), M(Sz, z, t), M(Sz, Sz, t), \\
&\quad M(z, z, t), M(Sz, z, t), M(z, Sz, 2t)) \\
&\leq \phi(M(Sz, z, kt), M(Sz, z, t), M(Sz, z, t), M(Sz, z, t), \\
&\quad M(Sz, z, t), T(M(Sz, z, t), M(Sz, z, t)))
\end{aligned}$$

From (ϕ_2), we have $M(Sz, z, kt) \geq M(Sz, z, t)$, which imply by Lemma 1 that $Sz = z$.

Now, putting $x = z$ and $y = x_{2n+1}$ in (b1), using condition (ϕ_1) and letting $n \to \infty$ we have

$$\begin{aligned} 0 &\leq \phi(M(fz,z,kt), M(z,z,t), M(fz,z,t), \\ &\quad M(z,z,t), M(fz,z,t), M(z,z,2t)) \\ &\leq \phi(M(fz,z,kt), M(fz,z,t), M(fz,z,t), M(fz,z,t), \\ &\quad M(fz,z,t), T(M(z,fz,t), M(fz,z,t))). \end{aligned}$$

From (ϕ_2) we have $M(fz,z,kt) \geq M(fz,z,t)$, which imply by Lemma 1 that $z = fz = Sz$.

Since $f(X) \subseteq P(X)$, there exists $w \in X$ such that $z = fz = Pw$. Putting $x = x_{2n}$, $y = w$ in (b1), using condition (ϕ_1) and letting $n \to \infty$ we have

$$\begin{aligned} 0 &\leq \phi(M(z,gw,kt), M(z,Pw,t), M(z,z,t), \\ &\quad M(gw,z,t), M(z,z,t), M(gw,z,2t)) \\ &\leq \phi(M(z,gw,kt), M(gw,z,t), M(gw,z,t), \\ &\quad M(gw,z,t), M(gw,z,t), T(M(gw,z,t), M(gw,z,t))) \end{aligned}$$

Again, from (ϕ_2) we have $M(z,gw,kt) \geq M(z,gw,t)$, which imply by Lemma 1 that $z = gw = Pw$.

Since (g,P) is weakly P-compatible it follows that (g,P) is weakly compatible pair. Hence $Pgw = gPw$, so that $Pz = gz$.

Putting $x = x_{2n}$, $y = z$ in (b1), using condition (ϕ_1) and letting $n \to \infty$ we have

$$\begin{aligned} 0 &\leq \phi(M(z,gz,kt), M(z,gz,t), M(z,z,t), \\ &\quad M(gz,gz,t), M(z,Pz,t), M(gz,z,2t)). \end{aligned}$$

From (ϕ_1), (ϕ_2) and Lemma 1 we have that $gz = Pz = z$.

Suppose now that z_0 is another common fixed point of f, g, S and P. Putting $x = z$, $y = z_0$ in (b1) we have

$$\begin{aligned} 0 &\leq \phi(M(z,z_0,kt), M(z,z_0,t), M(z,z,t), \\ &\quad M(z_0,z_0,t), M(z,z_0,t), M(z_0,z,2t) \end{aligned}$$

From (ϕ_1), (ϕ_2) we have $M(z_0,z,kt) \geq M(z_0,z,t)$ and from Lemma 1 we have $z = z_0$.

Suppose that z_1 is another common fixed point of f and S. Putting $x = z_1$ and $y = z$ in (b1) we have

$$\begin{aligned} 0 &\leq \phi(M(z_1,z,kt), M(z_1,z,t), M(z_1,z_1,t), \\ &\quad M(z,z,t), M(z_1,z,t), M(z,z_1,2t)). \end{aligned}$$

From (ϕ_1), (ϕ_2) we have $M(z_1, z, kt) \geq M(z_1, z, t)$ and from Lemma 1 we have $z = z_1$.

Similarly we can show that z is the unique common fixed point of g and P, and we can prove the theorem if (g, P) is P-continuous.

Corollary 5. *Let (X, M, T) be a complete fuzzy metric space, $\lim_{t\to\infty} M(x, y, t) = 1$ and f, g, S, $P : X \to X$ satisfy conditions (a), (b), (c) and (d) of previous theorem. If t-norm T is of H-type then there exists a unique common fixed point for the mappings f, g, S and P.*

Proof. By Proposition 1 all the conditions of the Theorem 4 are satisfied.

Corollary 6. *Let (X, M, T_λ^D) for some $\lambda > 0$ be a complete fuzzy metric space, $\lim_{t\to\infty} M(x, y, t) = 1$ and f, g, S, $P : X \to X$ satisfy conditions (a), (b), (c) and (d) of previous theorem. Suppose there exists $x_0, x_1 \in X$, $f(x_0) = P(x_1)$, and $\mu \in (k, 1)$ such that $\sum_{i=1}^{\infty}(1 - M(f(x_0), g(x_1), \frac{1}{\mu^i}))^\lambda < \infty$ then there exists a unique common fixed point for the mappings f, g, S and P.*

Proof. From equivalence (2) we have

$$\sum_{i=1}^{\infty}(1 - M(f(x_0), g(x_1), \frac{1}{\mu^i}))^\lambda < \infty \Leftrightarrow \lim_{n\to\infty}(T_\lambda^D)_{i=n}^{\infty} M(f(x_0), g(x_1), \frac{1}{\mu^i}) < \infty = 1.$$

Similarly, using equivalences (2) and (3) we have prove the following corollaries.

Corollary 7. *Let (X, M, T_λ^{AA}) for some $\lambda > 0$ be a complete fuzzy metric space, $\lim_{t\to\infty} M(x, y, t) = 1$ and f, g, S, $P : X \to X$ satisfy conditions (a), (b), (c) and (d) of previous theorem. Suppose there exists $x_0, x_1 \in X$, $f(x_0) = P(x_1)$ and $\mu \in (k, 1)$ such that $\sum_{i=1}^{\infty}(1 - M(y_0, y_1, \frac{1}{\mu^i}))^\lambda < \infty$ then there exists a unique common fixed point for the mappings f, g, S and P.*

Corollary 8. *Let (X, M, T_λ^{SW}) for some $\lambda > -1$ be a complete fuzzy metric space, $\lim_{t\to\infty} M(x, y, t) = 1$ and f, g, S, $P : X \to X$ satisfy conditions (a), (b), (c) and (d) of previous theorem. Suppose there exists $x_0, x_1 \in X$, $f(x_0) = P(x_1)$ and $\mu \in (k, 1)$ such that $\sum_{i=1}^{\infty}(1 - M(f(x_0), g(x_1), \frac{1}{\mu^i})) < \infty$ then there exists a unique common fixed point for the mappings f, g, S and P.*

3. Fixed point theorem for $T \geq T_L$

Theorem 9. *Let (X, M, T) be a complete fuzzy metric spaces such that $\lim_{t\to\infty} M(x, y, t) = 1$, $f : X \to X$ and t-norm $T \geq T_L$ such that for every $x, y \in X$, $k \in (0, 1)$ and $t > 0$ the following is satisfied*

$$\begin{aligned} M(fx, fy, kt) \quad &\geq \quad a(M(x, fx, t) + M(y, fy, kt)) + b(M(x, fy, (k+1)t) \\ &\quad\quad +M(fx, y, t)) + cM(x, y, t) \end{aligned} \tag{4}$$

where a, b, c are non-negative constants such that $2a+2b+c \geq 1$. If for some $x_0 \in X$ and $x_1 = fx_0$ and $\mu \in (q, 1)$ the following is satisfied

$$\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} M(x_0, x_1, \frac{1}{\mu^i}) = 1,$$

then there exist a unique fixed point of the mapping f.

Proof. Let $x_0 \in X$ and $x_{n+1} = fx_n$. From (4), for every $t > 0$, and $x = x_{n-1}$, $y = x_n$, we obtain

$$\begin{aligned} M(x_n, x_{n+1}, kt) \quad &\geq \quad a(M(x_{n-1}, x_n, t) + M(x_n, x_{n+1}, kt)) \\ &+ \quad b(M(x_{n-1}, x_{n+1}, (k+1)t) + M(x_n, x_n, t)) + cM(x_{n-1}, x_n, t) \\ &\geq \quad a(M(x_{n-1}, x_n, t) + M(x_n, x_{n+1}, kt)) \\ &+ \quad b(T_L(M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, kt)) + 1) + cM(x_{n-1}, x_n, t). \end{aligned}$$

Then we have

$$M(x_n, x_{n+1}, kt) \geq \frac{a+b+c}{1-a-b} M(x_{n-1}, x_n, t) \geq M(x_{n-1}, x_n, t).$$

As the same method as in previous theorems we conclude that the sequence $\{x_n\}$ is Cauchy sequence, so there exist $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Applying now (4) for $x = x_{n-1}$ and $y = z$ we have

$$\begin{aligned} M(fx_{n-1}, fz, kt) \quad &\geq \quad a(M(x_{n-1}, x_n, t) + M(z, fz, kt)) \\ &\quad\quad +b(M(x_{n-1}, fz, (k+1)t) + M(x_n, z, t)) + cM(x_{n-1}, z, t). \end{aligned}$$

Letting $n \to \infty$ we have

$$M(z, fz, kt) \geq \frac{a+b+c}{1-a-b} \geq 1,$$

so we conclude that $z = fz$. Suppose now that there exist another fixed point $p \in X$, $z \neq p$. Then applying (4) for $x = z$ and $y = p$ we have

$$M(z, p, kt) \geq \frac{2a}{1-2b-c} \geq 1.$$

So, $z = p$.

Remark 1. *Since the condition $T \geq T_L$ is fulfilled by the families $(T_\lambda^{SS})_{\lambda \in [-\infty, 1]}$, $(T_\lambda^{SW})_{\lambda \in [0, \infty]}$, we can apply previous theorem for this two family of t-norms.*

4. Main result for multi-valued case

Throw this section $CB(X)$ is the set of all non-empty, closed and bounded subsets of X. For A, $B \in CB(X)$ and every $t > 0$

$$\mathcal{M}(A, B, t) = \sup\{M(a, b, t);\ a \in A,\ b \in B\}$$

and

$$\delta_M(A, B, t) = \inf\{M(a, b, t);\ a \in A,\ b \in B\}.$$

If $A = \{a\}$ then we have $\delta_M(A, B, t) = \delta_M(a, B, t)$. In the paper [8] the following definition is introduced.

Definition 8. *The mappings $F : X \to X$ and $G : X \to CB(X)$ are weakly compatible if they commute at coincidence points, i.e., for each point $u \in X$ such that $Gu = \{Fu\}$ we have $GFu = FGu$.*

Let Φ be the set of all continuous real functions $\phi : [0, 1]^5 \to [0, 1]$ such that
(ϕ_1) $\phi(t_1, t_2, t_3, t_4, t_5)$ is non-decreasing in $t_1, \ldots, t_5$.
(ϕ_2) $\phi(t, t, t, T(t, t), t) > t$ and $\phi(t, t, t, t, T(t, t)) > t$.

Theorem 10. *Let (X, M, T) be a complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ and $F, G : X \to CB(X)$, $I, J : X \to X$ be mappings. Suppose that the following conditions hold*
(a) $Fx \subseteq J(X)$ and $Gx \subseteq I(X)$ for all $x \in X$
(b) the pairs (F, I) and (G, J) are weakly compatible
(c) for some $\phi \in \Phi$ there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ and every $t > 0$
(c1)

$$\begin{aligned} \delta_M(Fx, Gy, kt) \ \geq \ & \phi(M(Ix, Jy, t), \mathcal{M}, \mathcal{M}(Jy, Gy, kt), \\ & \mathcal{M}(Ix, Gy, (k+1)t), \mathcal{M}(Jy, Fx, t)) \end{aligned}$$

and
(c2)

$$\begin{aligned} \delta_M(Fx, Gy, kt) \ \geq \ & \phi(M(Ix, Jy, t), \mathcal{M}(Ix, Fx, kt), \mathcal{M}(Jy, Gy, t), \\ & \mathcal{M}(Ix, Gy, t), \mathcal{M}(Jy, Fx, (k+1)t)) \end{aligned}$$

are satisfied

(d) one of $I(X)$ and $J(X)$ are closed subsets of X.

(e) There exists $x_0, x_1 \in X$ such that for $y_0 = Jx_1 \in Fx_0$, $y_1 \in Gx_1$ and $\mu \in (k,1)$ the following condition holds:

$$\lim_{n\to\infty} \boldsymbol{T}_{i=n}^{\infty} M(y_0, y_1, \frac{1}{\mu^i}) = 1.$$

Then there exists a unique $p \in X$ such that $\{p\} = \{Jp\} = \{Ip\} = Fp = Gp$.

Proof. As $Gx \subseteq I(X)$ there exists $x_2 \in X$ such that $y_1 = Ix_2 \in Gx_1$. Continuing in this way we can construct a sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$y_{2n} = Jx_{2n+1} \in Fx_{2n} \text{ and } y_{2n+1} = Ix_{2n+2} \in Gx_{2n+1}$$

for $n = 0, 1, 2, \dots$

STEP 1. Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (c1) we have

$$\begin{aligned}\delta_M(Fx_{2n}, Gx_{2n+1}, kt) \quad \geq \quad & \phi(M(Ix_{2n}, Jx_{2n+1}, t), \mathcal{M}(Ix_{2n}, Fx_{2n}, t), \\ & \mathcal{M}(Jx_{2n+1}, Gx_{2n+1}, kt), \mathcal{M}(Ix_{2n}, Gx_{2n+1}, (k+1)t), \\ & \mathcal{M}(Jx_{2n+1}, Fx_{2n}, t)).\end{aligned}$$

So we have

$$\begin{aligned}\delta_M(y_{2n}, y_{2n+1}, kt) \quad \geq \quad & \phi(M(y_{2n-1}, y_{2n}, t), \mathcal{M}(y_{2n-1}, y_{2n}, t), \mathcal{M}(y_{2n}, y_{2n+1}, kt), \\ & \mathcal{M}(y_{2n-1}, y_{2n+1}, (k+1)t), \mathcal{M}(y_{2n}, y_{2n}, t))\end{aligned}$$

and from (ϕ_1) we have

$$\begin{aligned}M(y_{2n}, y_{2n+1}, kt) \quad \geq \quad & \phi(M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, kt), \\ & M(y_{2n-1}, y_{2n+1}, (k+1)t), 1).\end{aligned}$$

Since $M(y_{2n-1}, y_{2n+1}, (k+1)t) \geq T(M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, kt))$ and using (ϕ_1) we have

$$\begin{aligned}M(y_{2n}, y_{2n+1}, kt) \quad \geq \quad & \phi(M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, kt), \\ & T(M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, kt)), 1).\end{aligned}$$

If $M(y_{2n-1}, y_{2n}, t) > M(y_{2n}, y_{2n+1}, kt)$,then using (ϕ_1) we have

$$\begin{aligned}M(y_{2n}, y_{2n+1}, kt) \quad \geq \quad & \phi(M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n+1}, kt), \\ & T(M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n+1}, kt)), M(y_{2n}, y_{2n+1}, kt))\end{aligned}$$

and then using (ϕ_2) we have

$$v \geq \phi(v, v, v, T(v, v), v) > v.$$

This is a contradiction and so

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t).$$

STEP 2. Putting $x = x_{2n+2}$ and $y = x_{2n+1}$ in (c2) we have

$$\begin{aligned}\delta_M(Fx_{2n+2}, Gx_{2n+1}, kt) \geq \ & \phi(M(Ix_{2n+2}, Jx_{2n+1}, t), \mathcal{M}(Ix_{2n+2}, Fx_{2n+2}, kt),\\ & \mathcal{M}(Jx_{2n+1}, Gx_{2n+1}, t), \mathcal{M}(Ix_{2n+2}, Gx_{2n+1}, t),\\ & \mathcal{M}(Jx_{2n+1}, Fx_{2n+2}, (k+1)t))).\end{aligned}$$

So we have

$$\begin{aligned}\delta_M(y_{2n+2}, y_{2n+1}, kt) \geq \ & \phi(M(y_{2n+1}, y_{2n}, t), \mathcal{M}(y_{2n+1}, y_{2n+2}, kt),\\ & \mathcal{M}(y_{2n}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+1}, t),\\ & \mathcal{M}(y_{2n}, y_{2n+2}, (k+1)t))\end{aligned}$$

and from (ϕ_1) we have

$$\begin{aligned}M(y_{2n+2}, y_{2n+1}, kt) \geq \ & \phi(M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n+2}, kt),\\ & M(y_{2n}, y_{2n+1}, t), 1,\\ & T(M(y_{2n}, y_{2n+1}, t), M(y_{2n+2}, y_{2n+1}, kt))).\end{aligned}$$

If $M(y_{2n+1}, y_{2n}, t) > M(y_{2n+2}, y_{2n+1}, kt)$ using (ϕ_1) we have

$$\begin{aligned}M(y_{2n+2}, y_{2n+1}, kt) \geq \ & \phi(M(y_{2n+2}, y_{2n+1}, kt), M(y_{2n+2}, y_{2n+1}, kt),\\ & M(y_{2n+2}, y_{2n+1}, kt), M(y_{2n+2}, y_{2n+1}, kt),\\ & T(M(y_{2n+2}, y_{2n+1}, kt), M(y_{2n+2}, y_{2n+1}, kt))).\end{aligned}$$

and then using (ϕ_2) we have

$$v \geq \phi(v, v, v, v, T(v, v)) > v.$$

This is a contradiction and so

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t).$$

Thus for every $n \in \mathbb{N}$ we have

$$M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$$

i.e.,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, \frac{t}{k}) \geq \cdots \geq M(y_0, y_1, \frac{t}{k^n}) \to 1$$

when $n \to \infty$.

STEP 3. The proof that sequence $\{y_n\}$ is a Cauchy sequence is as in Theorem 4 and from the completeness of X, there exist $p \in X$ such that $\lim_{n\to\infty} y_n = p$.

Thus

$$p = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Jx_{2n+1} \in \lim_{n\to\infty} Fx_{2n}$$

and

$$p = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Ix_{2n+2} \in \lim_{n\to\infty} Gx_{2n+1}.$$

Without loss of generality we can assume that the set $J(X)$ is closed. Then for some $u \in X$ we have

$$p = Ju \in J(X).$$

STEP 4. Putting $x = x_{2n}$ and $y = u$ in (c.1) we have

$$\begin{aligned}\delta_M(Fx_{2n}, Gu, kt) \quad &\geq \quad \phi(M(Ix_{2n}, Ju, t), \mathcal{M}(Ix_{2n}, Fx_{2n}, t), \mathcal{M}(Ju, Gu, kt),\\ &\qquad \mathcal{M}(Ix_{2n}, Gu, (k+1)t), \mathcal{M}(Ju, Fx_{2n}, t)).\end{aligned}$$

So we have

$$\begin{aligned}M(y_{2n}, Gu, kt) \quad &\geq \quad \phi(M(y_{2n-1}, p, t), M(y_{2n-1}, y_{2n}, t), M(p, Gu, kt),\\ &\qquad M(y_{2n-1}, Gu, (k+1)t), M(p, y_{2n}, t)).\end{aligned}$$

On making $n \to \infty$ we have

$$\begin{aligned}M(p, Gu, kt) \quad &\geq \quad \phi(M(p, p, t), M(p, p, t), M(p, Gu, kt),\\ &\qquad M(p, Gu, (k+1)t), M(p, p, t))\\ &\geq \quad \phi(1, 1, M(p, Gu, kt), T(M(p, p, kt), M(p, Gu, t)), 1).\end{aligned}$$

From the condition (ϕ_1) we have

$$\begin{aligned}M(p, Gu, kt) \quad &\geq \quad \phi(1, 1, 1, T(1, M(p, Gu, t)), 1)\\ &\geq \quad \phi(M(p, Gu, t), M(p, Gu, t), M(p, Gu, t), T(M(p, Gu, t),\\ &\qquad M(p, Gu, t)), M(p, Gu, t))\\ &> \quad M(p, Gu, t).\end{aligned}$$

So

$$\{p\} = Gu = \{Ju\}.$$

Since the pair (G, J) is weakly compatible it follows $GJu = JGu$. Hence

$$Gp = \{Jp\}.$$

STEP 5. Putting $x = x_{2n}$ and $y = p$ in (c.1) we have

$$\begin{aligned}
\delta_M(Fx_{2n}, Gp, kt) \;&\geq\; \phi(M(Ix_{2n}, Jp, t), \mathcal{M}(Ix_{2n}, Fx_{2n}, t), \mathcal{M}(Jp, Gp, kt),\\
&\quad \mathcal{M}(Ix_{2n}, Gp, (k+1)t), \mathcal{M}(Jp, Fx_{2n}, t)).
\end{aligned}$$

So we have,

$$\begin{aligned}
M(y_{2n}, Gp, kt) \;&\geq\; \phi(M(y_{2n-1}, Gp, t), M(y_{2n-1}, y_{2n}, t), M(Gp, Gp, kt),\\
&\quad M(y_{2n-1}, Gp, (k+1)t), M(Gp, y_{2n}, t))\\
&\geq\; \phi(M(y_{2n-1}, Gp, t), M(y_{2n-1}, y_{2n}, t), M(Gp, Gp, kt),\\
&\quad T(M(y_{2n-1}, y_{2n}, kt), M(y_{2n}, Gp, t)), M(Gp, y_{2n}, t)).
\end{aligned}$$

On making $n \to \infty$ we have

$$\begin{aligned}
M(p, Gp, kt) \;&\geq\; \phi(M(p, Gp, t), 1, 1, T(1, M(p, Gp, t)), M(Gp, p, t))\\
&\geq\; \phi(M(p, Gp, t), M(p, Gp, t), M(p, Gp, t), T(M(p, Gp, t),\\
&\quad M(p, Gp, t)), M(Gp, p, t))\\
&>\; M(p, Gp, t).
\end{aligned}$$

So,

$$\{p\} = Gp.$$

STEP 6. Since $Gp \subseteq I(X)$, there exists $w \in X$ such that

$$\{Iw\} = Gp = \{Jp\} = \{p\}.$$

Putting $x = w$, $y = p$ in (c.1) we have

$$\begin{aligned}
\delta_M(Fw, Gp, kt) \;&\geq\; \phi(M(Iw, Jp, t), \mathcal{M}(Iw, Fw, t), \mathcal{M}(Jp, Gp, kt),\\
&\quad \mathcal{M}(Iw, Gp, (k+1)t), \mathcal{M}(Jp, Fw, t)).
\end{aligned}$$

and so

$$\begin{aligned}
M(Fw, p, kt) \;&\geq\; \phi(M(p, p, t), M(p, Fw, t), M(p, p, kt),\\
&\quad M(p, p, (k+1)t), M(p, Fw, t))\\
&=\; \phi(1, M(p, Fw, t), 1, 1, M(p, Fw, t))\\
&\geq\; \phi(M(p, Fw, t), M(p, Fw, t), M(p, Fw, t),\\
&\quad M(p, Fw, t), M(p, Fw, t)).
\end{aligned}$$

Therefore $Fw = \{p\}$. The pair (F, I) is weakly compatible i.e., from $Fw = \{Iw\}$ follows $FIw = IFw$, and we obtain that $Fp = \{Ip\}$. Well,

$$\{Iw\} = Fw = Gp = \{Jp\} = \{p\}.$$

STEP 7. Putting $x = p$, $y = x_{2n+1}$ in (c.1) we have

$$\begin{aligned}\delta_M(Fp, Gx_{2n+1}, kt) \;\geq\;& \phi(M(Ip, Jx_{2n+1}, t), \mathcal{M}(Ip, Fp, t), \mathcal{M}(Jx_{2n+1}, Gx_{2n+1}, kt),\\ & \mathcal{M}(Ip, Gx_{2n+1}, (k+1)t), \mathcal{M}(Jx_{2n+1}, Fp, t)).\end{aligned}$$

and making $n \to \infty$ we have

$$\begin{aligned}M(Fp, p, kt) &\geq \phi(M(Fp, p, t), M(Fp, Fp, t), M(p, p, kt),\\ &\quad M(Fp, p, (k+1)t), M(p, Fp, t))\\ &\geq \phi(M(Fp, p, t), 1, 1, T(1, M(Fp, p, t)), M(p, Fp, t))\\ &\geq \phi(M(Fp, p, t), M(Fp, p, t), M(Fp, p, t),\\ &\quad T(M(Fp, p, t), M(Fp, p, t)), M(p, Fp, t))\\ &> M(Fp, p, t).\end{aligned}$$

Now,

$$Fp = \{p\} = \{Ip\}.$$

Thus,

$$Fp = Gp = \{Ip\} = \{Jp\} = \{p\}.$$

STEP 8. Let z be another common fixed point i.e.,

$$Fz = Gz = \{Iz\} = \{Jz\} = \{z\}.$$

Putting $x = p$ and $y = z$ in (c.1) we have

$$\begin{aligned}\delta_M(Fp, Gz, kt) \;\geq\;& \phi(M(Ip, Jz, t), \mathcal{M}(Ip, Fp, t), \mathcal{M}M(Jz, Gz, kt),\\ & \mathcal{M}(Ip, Gz, (k+1)t), \mathcal{M}(Jz, Fp, t)).\end{aligned}$$

Accordingly,

$$\begin{aligned}M(p, z, kt) &\geq \phi(M(p, z, t), M(p, p, t), M(z, z, kt),\\ &\quad M(p, z, (k+1)t), M(p, z, t))\\ &= \phi(M(p, z, t), M(p, z, t), M(p, z, t),\\ &\quad T(M(p, z, t), M(p, z, t)), M(p, z, t))\\ &> M(p, z, t)\end{aligned}$$

and so $p = z$. Similarly the theorems follows when $I(X)$ is closed.

Remark 2. *Again the second condition can be substituted with assumption "t-norm T is of H-type". Also, analogously to the Corollary 6-Corollary 8, the previous theorem can be reformulated for a t-norm from Dombi, Aczél-Alsina or Sugeno-Weber class.*

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Tatjana Došenović
Faculty of Technology,
Bulevar Cara Lazara 1,
Novi Sad, Serbia
email: *tatjanad@tf.uns.ac.rs*

Shaban Sedghi
Department of Mathematics, Qaemshahr Branch,
Islamic Azad University,
Qaemshahr, Iran
email: *sedghi_gh@yahoo.com*

Ishak Altun
Department of Mathematics, Faculty of Science and Arts,
Kirikkale University,
Yahsihan, Kirikkale, Turkey
email: *ishakaltun@yahoo.com*