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SOME PROPERTIES FOR SUBCLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS ASSOCIATED WITH NEW OPERATOR

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Abstract. The purpose of this paper is to introduce two subclasses of meromorphic multivalent functions by using new operator and investigate various properties for these subclasses.

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1. Introduction

Let Σ_p be the class of meromorphic $p-$valent functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \ (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$.

Let $\mathcal{P}_k(\rho)$ be the class of functions $p(z)$ analytic in $\mathbb{U}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{Rep(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \tag{2}$$

where $k \geq 2$ and $0 \leq \rho < 1$. This class was introduced by Padmanabhan and Parvatham [10]. For $\rho = 0$, the class $\mathcal{P}_k(0) = \mathcal{P}_k$ introduced by Pinchuk [12]. Also, $\mathcal{P}_2(\rho) = \mathcal{P}(\rho)$, where $\mathcal{P}(\rho)$ is the class of functions with positive real part greater than ρ and $\mathcal{P}_2(0) = \mathcal{P},$ is the class of functions with positive real part. From (2), we have $p(z) \in \mathcal{P}_k(\rho)$ if and only if there exist $p_1, p_2 \in \mathcal{P}(\rho)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \ (z \in \mathbb{U}). \tag{3}$$

It is known that the class $\mathcal{P}_k(\rho)$ is a convex set (see [7]).

The Hadamard product (or convolution) $f * g$ of $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p}, \tag{4}$$

is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} = (g * f)(z). \tag{5}$$

Define the functions

$$f_\alpha(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{n+\lambda}{\lambda}\right)^{\alpha} z^{n-p} \ (\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \lambda > 0),$$

and

$$\widetilde{\varphi}(a, b; z) = \frac{1}{z^p} F(1, a; b; z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \left| \frac{(a)_n}{(b)_n} \right| z^{n-p},$$

where $a \in \mathbb{C}^* = \mathbb{C} \backslash \{0\},\ b \in \mathbb{R} \backslash \mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \{0, -1, -2, ...\},$ $F(a, b; c; z)$ is the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \left| \frac{(a)_n (b)_n}{(c)_n (1)_n} \right| z^n \ (c \in \mathbb{R} \backslash \mathbb{Z}_0^-),$$

and

$$(\eta)_n = \begin{cases} 1 & \text{if } n = 0, \\ \eta(\eta+1)(\eta+2)...(\eta+n-1) & \text{if } n \in \mathbb{N}. \end{cases}$$

Also, let the associated function $f^*_{\alpha,\mu}(z)$ be defined by the Hadamard product (or convolution):

$$f_\alpha(z) * f^*_{p,\alpha,\mu}(z) = \frac{1}{z^p(1-z)^\mu} \ (\mu > 0;\ z \in \mathbb{U}^*).$$

Then, we have

$$\begin{aligned} \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z) &= f_{p,\alpha,\mu}^{*}(z) * \widetilde{\varphi}(a,b;z) * f(z) \\ &= \frac{1}{z^p} + \sum_{n=1}^{\infty} \left| \frac{(a)_n}{(b)_n} \right| \left(\frac{\lambda}{n+\lambda} \right)^{\alpha} \frac{(\mu)_n}{(1)_n} a_{n-p} z^{n-p}. \end{aligned} \tag{6}$$

It is easily verified from (6) that

$$z\left(\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)' = \mu \mathcal{I}_{a,b,\mu+1}^{p,\alpha,\lambda} f(z) - (\mu+p)\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z), \tag{7}$$

and

$$z\left(\mathcal{I}_{a,b,\mu}^{p,\alpha+1,\lambda} f(z)\right)' = \lambda \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z) - (\lambda+p)\mathcal{I}_{a,b,\mu}^{p,\alpha+1,\lambda} f(z). \tag{8}$$

We note that:

(i) $\mathcal{I}_{1,1,\mu}^{p,\alpha,\ell} f(z) = I_{p,\mu}^{\alpha}(\ell) f(z)$ (see El-Ashwah and Aouf [4, with $\lambda = 1$]);
(ii) $\mathcal{I}_{1,1,\mu}^{1,\alpha,\lambda} f(z) = \mathcal{I}_{\lambda,\mu}^{\alpha} f(z)$ (see Cho et al. [3] and Piejko and Sokol [11]) (see also Aouf et al. [2]).

Next, by using the operator $\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda}$, we introduce two subclasses of meromorphic multivalent functions of Σ_p as follows:
Definition 1. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{M}_{a,b,\mu}^{p,\alpha,\lambda}(\beta,\gamma,\rho,k)$ if it satisfies the condition:

$$\left[(1-\gamma)\left(z^p \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)^{\beta} + \gamma \left(\frac{\mathcal{I}_{a,b,\mu+1}^{p,\alpha,\lambda} f(z)}{\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)} \right) \left(z^p \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)^{\beta}\right] \in \mathcal{P}_k(\rho),$$

$$\left(k \geq 2;\ a \in \mathbb{C}^*;\ b \in \mathbb{R}\backslash\mathbb{Z}_0^-;\ \alpha \in \mathbb{N}_0;\ \beta,\gamma,\mu,\lambda > 0;\ 0 \leq \rho < 1\right). \tag{9}$$

We note that:

$\mathcal{M}_{1,1,\mu}^{1,\alpha,\lambda}(\beta,\gamma,\rho,k) = \mathcal{M}_{\lambda,\mu}^{\alpha}(\beta,\gamma,\rho,k)$ (see Aouf et al. [2]).

Also, we note that:

$$\mathcal{M}_{1,1,\mu}^{p,\alpha,\ell}(\beta,\gamma,\rho,k) = \mathcal{M}_{\mu}^{p,\alpha,\ell}(\beta,\gamma,\rho,k) = \Big\{f(z) \in \Sigma_p : \left[(1-\gamma)\left(z^p I_{p,\mu}^{\alpha}(\ell) f(z)\right)^{\beta} + \gamma \left(\tfrac{I_{p,\mu+1}^{\alpha}(\ell) f(z)}{I_{p,\mu}^{\alpha}(\ell) f(z)}\right)\left(z^p I_{p,\mu}^{\alpha}(\ell) f(z)\right)^{\beta}\right] \in \mathcal{P}_k(\rho)\Big\}.$$

Definition 2. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{N}_{a,b,\mu}^{p,\alpha,\lambda}(\beta,\gamma,\rho,k)$ if it satisfies the condition:

$$\left[(1-\gamma)\left(z^p\mathcal{I}_{a,b,\mu}^{p,\alpha+1,\lambda}f(z)\right)^{\beta}+\gamma\left(\frac{\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda}f(z)}{\mathcal{I}_{a,b,\mu}^{p,\alpha+1,\lambda}f(z)}\right)\left(z^p\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda}f(z)\right)^{\beta}\right]\in\mathcal{P}_k(\rho),$$

$$\left(k\geq 2;\ a\in\mathbb{C}^*;\ b\in\mathbb{R}\backslash\mathbb{Z}_0^-;\ \alpha\in\mathbb{N}_0;\ \beta,\gamma,\mu,\lambda>0;\ 0\leq\rho<1\right). \tag{10}$$

We note that:

$\mathcal{N}_{1,1,\mu}^{1,\alpha,\lambda}(\beta,\gamma,\rho,k)=\mathcal{N}_{\lambda,\mu}^{\alpha}(\beta,\gamma,\rho,k)$ (see Aouf et al. [2]).

Also, we note that:

$$\mathcal{N}_{1,1,\mu}^{p,\alpha,\ell}(\beta,\gamma,\rho,k)=\mathcal{N}_{\mu}^{p,\alpha,\ell}(\beta,\gamma,\rho,k)=\Big\{f(z)\in\Sigma_p: \left[(1-\gamma)\left(z^pI_{p,\mu}^{\alpha+1}(\ell)f(z)\right)^{\beta}+\gamma\left(\frac{I_{p,\mu+1}^{\alpha}(\ell)f(z)}{I_{p,\mu}^{\alpha+1}(\ell)f(z)}\right)\left(z^pI_{p,\mu}^{\alpha+1}(\ell)f(z)\right)^{\beta}\right]\in\mathcal{P}_k(\rho)\Big\}.$$

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $k\geq 2,\ a\in\mathbb{C}^*,\ b\in\mathbb{R}\backslash\mathbb{Z}_0^-,\ \alpha\in\mathbb{N}_0,\ \beta,\gamma,\mu,\lambda>0$ and $0\leq\rho<1$.

To establish our results, we need the following lemma due to Miller and Mocanu [5].
Lemma 2.1 [5]. Let $\phi(u,v)$ be a complex valued function $\phi: D\to\mathbb{C},\ D\subset\mathbb{C}^2$ and let $u=u_1+iu_2,\ v=v_1+iv_2$. Suppose that the function $\phi(u,v)$ satisfies
(i) $\phi(u,v)$ is continuous in D;
(ii) $(1,0)\in D$ and $Re\{\phi(1,0)\}>0$;
(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{n}{2}(1+u_2^2),\ Re\{\phi(iu_2,v_1)\}\leq 0$.
Let $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+...$ be regular in $\mathbb{U}$ such that $(p(z),zp'(z))\in D$ for all $z\in\mathbb{U}$. If $Re\{\phi(p(z),zp'(z))\}>0$ for all $z\in\mathbb{U}$, then $Rep(z)>0$.

Employing the techniques used by Owa [9] for univalent functions, Noor and Muhammad [8] and Aouf and Seoudy [1] for multivalent functions and Mostafa et al. [6] for meromorphic multivalent functions, we prove the following theorems.

Theorem 2.1. If $f(z) \in \mathcal{M}_{a,b,\mu}^{p,\alpha,\lambda}(\beta, \gamma, \rho, k)$, then

$$\left(z^p \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)^\beta \in \mathcal{P}_k(\rho_1), \tag{11}$$

where ρ_1 is given by

$$\rho_1 = \frac{2\mu\beta\rho + n\gamma}{2\mu\beta + n\gamma} \ (0 \le \rho_1 < 1). \tag{12}$$

Proof. Let

$$\begin{aligned} \left(z^p \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)^\beta &= (1-\rho_1)p(z) + \rho_1 \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\left[(1-\rho_1)p_1(z) + \rho_1\right] - \left(\frac{k}{4} - \frac{1}{2}\right)\left[(1-\rho_1)p_2(z) + \rho_1\right], \end{aligned} \tag{13}$$

where $p_i(z)$ is analytic in $\mathbb{U}$ with $p_i(0) = 1$ for $i = 1, 2$. Differentiating (13) with respect to z, and using identity (7) in the resulting equation, we get

$$\begin{aligned} &\left[(1-\gamma)\left(z^p \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)^\beta + \gamma\left(\frac{\mathcal{I}_{a,b,\mu+1}^{p,\alpha,\lambda} f(z)}{\mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)}\right)\left(z^p \mathcal{I}_{a,b,\mu}^{p,\alpha,\lambda} f(z)\right)^\beta\right] \\ = \ &\left[(1-\rho_1)p(z) + \rho_1\right] + \frac{\gamma(1-\rho_1)zp'(z)}{\mu\beta} \in \mathcal{P}_k(\rho). \end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{\left[(1-\rho_1)p_i(z) + \rho_1\right] - \rho + \frac{\gamma(1-\rho_1)zp_i'(z)}{\mu\beta}\right\} \in \mathcal{P} \ (z \in \mathbb{U};\ i = 1, 2).$$

Defining the function

$$\phi(u, v) = \left[(1-\rho_1)u + \rho_1\right] - \rho + \frac{\gamma(1-\rho_1)v}{\mu\beta}$$

where $u = p_i(z) = u_1 + iu_2,\ v = zp_i'(z) = v_1 + iv_2$, we have
(i) $\phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
(ii) $(1, 0) \in D$ and $Re\{\phi(1, 0)\} = 1 - \rho > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{n}{2}(1 + u_2^2)$,

$$\begin{aligned} Re\{\phi(iu_2, v_1)\} &= \rho_1 - \rho + \frac{\gamma(1-\rho_1)v_1}{\mu\beta} \\ &\le \rho_1 - \rho - \frac{n\gamma(1-\rho_1)(1+u_2^2)}{2\mu\beta} \\ &= \frac{A + Bu_2^2}{2C}, \end{aligned}$$

where $A = 2(\rho_1 - \rho)\mu\beta - n\gamma(1-\rho_1),\ B = -n\gamma(1-\rho_1),\ C = \mu\beta > 0$. We note that $Re\{\phi(iu_2, v_1)\} < 0$ if and only if $A = 0,\ B < 0$, this is true from (12). Therefore, by applying Lemma 2.1, $p_i(z) \in \mathcal{P}\ (i = 1, 2)$ and consequently $p(z) \in \mathcal{P}_k(\rho_1)$ for $z \in \mathbb{U}$. This completes the proof of Theorem 2.1.

Using similar arguments to those in the proof of Theorem 2.1 and the identity (8) instead of (7), we obtain the following theorem for the class $\mathcal{N}^{p,\alpha,\lambda}_{a,b,\mu}(\beta,\gamma,\rho,k)$.

Theorem 2.2. If $f(z) \in \mathcal{N}^{p,\alpha,\lambda}_{a,b,\mu}(\beta,\gamma,\rho,k)$, then

$$\left(z^p \mathcal{I}^{p,\alpha+1,\lambda}_{a,b,\mu} f(z)\right)^{\beta} \in \mathcal{P}_k(\rho_2), \tag{14}$$

where ρ_2 is given by

$$\rho_2 = \frac{2\lambda\rho\beta + n\gamma}{2\lambda\beta + n\gamma}\ (0 \le \rho_2 < 1). \tag{15}$$

Theorem 2.3. If $f(z) \in \mathcal{M}^{p,\alpha,\lambda}_{a,b,\mu}(\beta,\gamma,\rho,k)$, then

$$\left(z^p \mathcal{I}^{p,\alpha,\lambda}_{a,b,\mu} f(z)\right)^{\beta/2} \in \mathcal{P}_k(\rho_3), \tag{16}$$

where ρ_3 is given by

$$\rho_3 = \frac{n\gamma + \sqrt{(n\gamma)^2 + 4(\mu\beta + n\gamma)\mu\beta\rho}}{2(\mu\beta + n\gamma)}\ (0 \le \rho_3 < 1). \tag{17}$$

Proof. Let

$$\begin{aligned} \left(z^p \mathcal{I}^{p,\alpha,\lambda}_{a,b,\mu} f(z)\right)^{\beta/2} &= (1-\rho_3)p(z) + \rho_3 \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)[(1-\rho_3)p_1(z) + \rho_3] - \left(\frac{k}{4} - \frac{1}{2}\right)[(1-\rho_3)p_2(z) + \rho_3], \end{aligned} \tag{18}$$

where $p_i(z)$ is analytic in $\mathbb{U}$ with $p_i(0) = 1$ for $i = 1, 2$. Differentiating (18) with respect to z, and using identity (7) in the resulting equation, we get

$$\begin{aligned} &\left[(1-\gamma)\left(z^p \mathcal{I}^{p,\alpha,\lambda}_{a,b,\mu} f(z)\right)^{\beta} + \gamma\left(\frac{\mathcal{I}^{p,\alpha,\lambda}_{a,b,\mu+1} f(z)}{\mathcal{I}^{p,\alpha,\lambda}_{a,b,\mu} f(z)}\right)\left(z^p \mathcal{I}^{p,\alpha,\lambda}_{a,b,\mu} f(z)\right)^{\beta}\right] \\ = \ &[(1-\rho_3)p(z) + \rho_3]^2 + \frac{2\gamma(1-\rho_3)[(1-\rho_3)p(z) + \rho_3]\,zp'(z)}{\mu\beta} \in \mathcal{P}_k(\rho). \end{aligned}$$

This implies that

$$\frac{1}{1-\rho}\left\{\left[(1-\rho_3)p_i(z)+\rho_3\right]^2-\rho+\frac{2\gamma(1-\rho_3)\left[(1-\rho_3)p_i(z)+\rho_3\right]zp_i'(z)}{\mu\beta}\right\}\in\mathcal{P}\ (z\in\mathbb{U};\ i=1,2).$$

Defining the function

$$\phi(u,v)=\left[(1-\rho_3)u+\rho_3\right]^2-\rho+\frac{2\gamma(1-\rho_3)\left[(1-\rho_3)u+\rho_3\right]v}{\mu\beta}$$

where $u=p_i(z)=u_1+iu_2,\ v=zp_i'(z)=v_1+iv_2$, we have
(i) $\phi(u,v)$ is continuous in $D=\mathbb{C}^2$;
(ii) $(1,0)\in D$ and $Re\{\phi(1,0)\}=1-\rho>0$;
(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leq-\frac{n}{2}(1+u_2^2)$,

$$\begin{aligned}Re\{\phi(iu_2,v_1)\} &= -(1-\rho_3)^2u_2^2+\rho_3^2-\rho+\frac{2\gamma\rho_3(1-\rho_3)v_1}{\mu\beta}\\ &\leq -(1-\rho_3)^2u_2^2+\rho_3^2-\rho-\frac{n\gamma\rho_3(1-\rho_3)(1+u_2^2)}{\mu\beta}\\ &= \frac{A+Bu_2^2}{C},\end{aligned}$$

where $A=\mu\beta\rho_3^2-\mu\beta\rho-n\gamma\rho_3(1-\rho_3),\ B=-(1-\rho_3)\left[\mu\beta(1-\rho_3)+\gamma n\rho_3\right],\ C=\mu\beta>0$. We note that $Re\{\phi(iu_2,v_1)\}<0$ if and only if $A=0,\ B<0$, this is true from (17) and $0\leq\rho_3<1$. Therefore, by applying Lemma 2.1, $p_i(z)\in\mathcal{P}\ (i=1,2)$ and consequently $p(z)\in\mathcal{P}_k(\rho_3)$ for $z\in\mathbb{U}$. This completes the proof of Theorem 2.3.

Using similar arguments to those in the proof of Theorem 2.3 and the identity (8) instead of (7), we obtain the following theorem for the class $\mathcal{N}_{a,b,\mu}^{p,\alpha,\lambda}(\beta,\gamma,\rho,k)$.

Theorem 2.4. If $f(z)\in\mathcal{N}_{a,b,\mu}^{p,\alpha,\lambda}(\beta,\gamma,\rho,k)$, then

$$\left(z^p\mathcal{I}_{a,b,\mu}^{p,\alpha+1,\lambda}f(z)\right)^\beta\in\mathcal{P}_k(\rho_4), \tag{19}$$

where ρ_4 is given by

$$\rho_4=\frac{n\gamma+\sqrt{(n\gamma)^2+4\left(\lambda\beta+n\gamma\right)\lambda\beta\rho}}{2\left(\lambda\beta+n\gamma\right)}\ (0\leq\rho_4<1). \tag{20}$$

Remark. (i) Putting $p=a=b=1$ in our results, we will obtain the result obtained by Aouf et al [2];
(ii) Putting $a=b=1$ in our results, we will obtain new results for the classes $\mathcal{M}_{\mu}^{p,\alpha,\ell}(\beta,\gamma,\rho,k)$ and $\mathcal{N}_{\mu}^{p,\alpha,\ell}(\beta,\gamma,\rho,k)$ mentioned in the introduction.

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