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FEKETE-SZEGO INEQUALITY FOR SUBCLASSES OF ANALYTIC FUNCTION OF COMPLEX ORDER

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Abstract. In this paper, we introduce certain new subclasses of analytic functions of complex order by using the convolution operator. For these classes several Fekete-Szego type coefficient inequalities are derived. Some special cases are also discussed.

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1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in E, \tag{1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C}\colon |z| < 1\}$. Let S denote the subclass of A consisting of univalent functions in E. Let f, $g \in A$, with

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, z \in E, \tag{2}$$

and $f(z)$ is given by (1). Then convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n g_n z^n, z \in E. \tag{3}$$

Also, if f and g are analytic in E, we say that f is subordinate to g written as $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)), \quad z \in E.$$

It is well known that for $f \in S$, given by (1), the inequality $\left|a_3 - a_2^2\right| \leq 1$ holds [3]. Fekete and Szego [5] obtained the sharp upper bound for the functional $\left|a_3 - \mu a_2^2\right|$ as

$$\left|a_3 - \mu a_2^2\right| \leq 1 + \exp\left(\frac{-2\mu}{1-\mu}\right),$$

for $f \in S, 0 \leq \mu \leq 1$. The problemma of finding sharp upper bound for functional $\left|a_3 - \mu a_2^2\right|$ for different classes of functions in A is known as Fekete-Szego problemma. Many authors considered this problemma for different classes of univalent functions (see [10]-[6]). For brief history of this problemma for the classes of starlike, convex and close to convex functions see [16]. In [13], Fekete-Szego problemma for the classes $k - UCV$, $k - SP$ and some other related classes defined by using fractional calculus is settled. We discuss the Fekete-Szego type inequalities for the classes $S^*(b, g(z), \phi(z))$ and $C(b, g(z), \phi(z))$ defined as follows:

Definition 1. *Let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3.., \tag{4}$$

be convex univalent function with $Re\left(\phi(z)\right) > 0$ *with real coefficients in* E *and let* $g(z)$ *given by (2) with real coefficients and let*

$$(f * g)(z) \neq 0, z \in E.$$

A function $f \in A$ *is in the class* $S^*(b, g(z), \phi(z))$ *if*

$$1 + \frac{1}{b}\left(\frac{z\left(f * g\right)'}{f * g} - 1\right) \prec \phi(z), z \in E.$$

Definition 2. *A function* $f \in A$ *is in the class* $C(b, g(z), \phi(z))$ *if*

$$1 + \frac{1}{b}\left(\frac{z\left(f * g\right)''}{(f * g)'}\right) \prec \phi(z), z \in E.$$

We have the following special cases.

1. $S^*(1, \frac{z}{1-z}, \frac{1+z}{1-z}) = S^*$, studied in [3].
2. $S^*(1, \frac{z}{1-z}, \frac{1+Az}{1+Bz}) = S^*[A, B]$, see [7].
3. $S^*(1, z_2F_1(a, b, c; z), p_k(z)) = k - SP_c^{a,b}$, introduced in [13].
4. $C(1, \frac{z}{1-z}, \frac{1+z}{1-z}) = C$, see [3].
5. $C(1, \frac{z}{1-z}, \frac{1+Az}{1+Bz}) = C[A, B]$ we refer to [7].

2. Preliminaries

Lemma 1. *[3] Let $p \in P$ with*

$$p\left(z\right) = 1 + c_1 z + c_2 z + ..,$$

then

$$|c_n| \leq 2, n \geq 2.$$

Lemma 2. *[3] Let $p \in P$ with*

$$p(z) = 1 + c_1 z + c_2 z + ..,$$

then for any complex number ν

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\left\{1, |2\nu - 1|\right\},$$

and result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2},\ p(z) = \frac{1 + z}{1 - z}.$$

Lemma 3. *[3] Let $p \in P$ with*

$$p\left(z\right) = 1 + c_1 z + c_2 z^2 + ..,$$

then

$$\left|c_2 - \frac{1}{2}\mu c_1\right| \leq 2 + \frac{1}{2}\left(|\mu - 1| - 1\right)|c_1|^2.$$

Lemma 4. *Let $p \in P$ with*

$$p\left(z\right) = 1 + c_1 z + c_2 z + ..,$$

then

$$|c_2 - \upsilon c_1| \leq \begin{cases} -4\nu + 2 & if \quad \nu \leq 0 \\ 2 & if \quad 0 \leq \nu \leq 1 \\ 4\nu - 2 & if \quad \nu \geq 1 \end{cases}$$

3. Main Results

Theorem 5. *Let $\phi(z)$ be given by (4) and $g(z)$ by (2) where both $\phi(z)$ and $g(z)$ have real coefficients and $b \in \mathbb{C}\backslash\{0\}$.If $f \in S^*(b, g(z), \phi(z))$, then*

$$|a_2| \leq |b| \frac{B_1}{g_2},$$

$$|a_3| \leq |b| \frac{B_1}{2g_3} \max\left\{1, \left|\frac{B_2}{B_1} + bB_1\right|\right\}$$

and

$$\left|a_3 - \frac{1}{2}\left(\frac{g_2^2}{g_3}\right)a_2^2\right| \leq |b| \frac{B_2}{2g_3}.$$

Proof. Let $f \in S^*(b, g(z), \phi(z))$ then

$$1 + \frac{1}{b}\left(\frac{z(f*g)'}{f*g} - 1\right) \prec \phi(z), \quad z \in E,$$

so that

$$1 + \frac{1}{b}\left(\frac{z(f*g)'}{f*g} - 1\right) = \phi(w(z)), \quad z \in E,$$

where $w(z)$ is Schawarz function with $w(0) = 0$and $|w(z)| \leq 1$. Let us denote

$$(f*g)(z) = z + A_2z^2 + A_3z^3 + ..,$$

then by (3), we can write

$$A_2 = a_2g_2 \text{ and } A_3 = a_3g_3, \tag{5}$$

so that

$$\begin{aligned}\frac{z(1 + 2A_2z^2 + 3A_3z^2 + ...)}{z + A_2z^2 + A_3z^3 + ...} &= 1 - b + b(\phi(w(z))) \\ &= b(\phi(w(z)) - b + 1 \\ &= b(\phi(\frac{p(z)-1}{p(z)+1})) - b + 1\end{aligned}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with $Re(p(z)) > 0$. This implies that

$$\frac{z(1+2A_2z^2+3A_3z^2+...)}{z+A_2z^2+A_3z^3+...} = b\left(1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+...\right)-b+1$$

$$\begin{aligned} z\left(1+2A_2z^2+3A_3z^2+...\right) &= \left(z+A_2z^2+A_3z^3+...\right) \\ &\left(1+\frac{b}{2}B_1c_1z+\left(\frac{b}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{b}{4}B_2c_1^2\right)z^2+...\right) \\ &= z+\left(A_2+\left(\frac{b}{2}B_1c_1\right)\right)z^2+ \\ &\left[\left(A_3+\left(\frac{b}{2}B_1c_1\right)A_2+\frac{b}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+b\left(\frac{1}{4}B_2c_1^2\right)\right)\right]z^3+... \end{aligned}$$

Equating the coefficients on both sides, we have

$$A_2 = \frac{b}{2}B_1c_1 \text{ and } A_3 = \frac{bB_1}{4}\left(c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}-bB_1\right)c_1^2\right) \tag{6}$$

Taking into account (5), (6) and lemmama 1, we have

$$|a_2| \le |b|\frac{B_1}{g_2},$$

and lemmama 2 leads us to

$$|a_3| \le |b|\frac{B_1}{2g_3}\max\left\{1, \left|\frac{B_2}{B_1}+bB_1\right|\right\}.$$

Moreover, by lemmama 1, we get

$$\left|a_3-\frac{1}{2}\left(\frac{g_2^2}{g_3}\right)a_2^2\right| \le \frac{|b|\,B_2}{2g_3}.$$

which is our required result.

For $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B \le A \le 1)$ in theorem 5,we obtain the following corollaryollary.

Corollary 6. *If* $f \in S^*(b, g(z), \frac{1+Az}{1+Bz})$,$(-1 \le B \le A \le 1)$ *then*

$$|a_2| \le |b|\frac{(A-B)}{g_2},$$

$$|a_3| \le |b|\frac{(A-B)}{2g_3}\max\{1, |-B+b(A-B)|\},$$

and

$$\left|a_3-\frac{1}{2}\left(\frac{g_2^2}{g_3}\right)a_2^2\right| \le |b|\frac{-B(A-B)}{2g_3}.$$

We take $g(z) = \frac{z}{1-z}$, $\phi(z) = \frac{1+z}{1-z}$, $z \in E$ and $b = 1$ in theorem 5, it follows the known result, given in [10]

Corollary 7. *If* $f \in S^*\left(1, \frac{z}{1-z}, \frac{1+z}{1-z}\right)$, *then*

$$\left|a_3 - \frac{1}{2}a_2^2\right| \leq 1.$$

For $g(z) = z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n$, $a > 0$, $\delta \geq 0$ and $\phi(z) = \frac{1+z}{1-z}$, $z \in E$, in theorem 5, we get the following known result, see [2].

Corollary 8. *If* $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n, \frac{1+z}{1-z}\right), (a > 0,\ \delta \geq 0,\ z \in E)$, *then*

$$\left|a_3 - \frac{1}{2}\left(\frac{a(a+2)}{(a+1)^2}\right)^{\delta} a_2^2\right| \leq |b| \left(\frac{a+2}{a}\right)^{\delta}.$$

Theorem 9. *Let* $\phi(z)$ *be given by (4),* $g(z)$ *by (2), both with real coefficients and* $b \in \mathbb{C}\backslash\{0\}$. *If* $f \in S^*(b, g(z), \phi(z))$ *then for any* $\mu \in \mathbb{C}$ *we have*

$$\left|a_3 - \mu a_2^2\right| \leq |b| \frac{B_1}{2g_3} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left[1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right] bB_1\right|\right\}.$$

Proof. Taking into account (6), we have

$$\left|a_3 - \mu a_2^2\right| = \left| \begin{array}{c} \frac{bB_1}{4g_3}\left(c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bB_1\right)c_1^2\right) - \\ \mu\left(\frac{g_2^2}{g_3}\right)\left(\frac{b}{2g_2}B_1 c_1\right)^2 \end{array} \right|$$
$$= \left|\frac{bB_1}{4g_3}\left(c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bB_1 + 2\mu\left(\frac{g_3}{g_2^2}\right)bB_1\right)c_1^2\right)\right|.$$

From lemmama 2 we obtain,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\, B_1}{2g_3} \max\left\{1, \left|\frac{B_2}{B_1} + \left[1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right] bB_1\right|\right\}$$

We put $\phi(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1)$ in theorem 9 and obtain the following result.

Corollary 10. *If* $f \in S^*\left(b, g(z), \frac{1+Az}{1+Bz}\right)$, $(-1 \leq B \leq A \leq 1)$, *then for any* $\mu \in \mathbb{C}$ *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,(A-B)}{2g_3} \max\left\{1, \left|-B + \left[1 - 2\mu\left(\frac{g_3}{g_2^2}\right)\right] b\,(A-B)\right|\right\}.$$

We take $g(z) = z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n$, $(a > 0,\ \delta \geq 0)$ and $\phi(z) = \frac{1+z}{1-z}$, $z \in E$, in theorem 5, to get the following result.

Corollary 11. *If* $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\delta} z^n, \frac{1+z}{1-z}\right)$, $(a > 0,\ \delta \geq 0,\ z \in E)$, *then we for any* $\mu \in \mathbb{C}$, *we obtain the inequality*

$$\left|a_3 - \mu a_2^2\right| \leq |b|\left(\frac{a+2}{a}\right)^{\delta} \times \max\left\{1, \left|1 + 2b - 4\mu b\left(\frac{(a+1)^2}{a(a+2)}\right)^{\delta}\right|\right\}.$$

This result has been proved in [2]

We take

$g(z) = z + \sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n$, $(\alpha, \beta, \lambda, \delta \geq 0, \beta > \alpha, \lambda > \delta, k \in \mathbb{N}_0)$, in theorem 9, this implies the result proved in [1].

Corollary 12. *If* $f \in S^*\left(1, z + \sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n, \phi(z)\right)$, $(\alpha, \beta, \lambda, \delta \geq 0, \beta > \alpha, \lambda > \delta, k \in \mathbb{N}_0)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,B_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left[1 - 2\mu\left(\frac{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2k}}\right)\right] bB_1\right|\right\}.$$

Now we consider the case when both μ and b are real.

Theorem 13. *Let* $\phi(z)$ *be given by (4),* $g(z)$ *by (2), both with real coefficients and* $b \in \mathbb{R}$ *with* $b > 0$. *If* $f \in S^*(b, g(z), \phi(z))$ *then for any* $\mu \in \mathbb{R}$, *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{bB_1}{2g_3}\left[\frac{B_2}{B_1} + bB_1 - 2\mu\frac{g_3}{g_2^2} bB_1\right] & \text{if } \mu \leq \delta_1 \\ \frac{bB_1}{2g_3} & \text{if } \delta_1 \leq \mu \leq \delta_2 \\ \frac{bB_1}{2g_3}\left[-\frac{B_2}{B_1} - bB_1 + 2\mu\left(\frac{g_3}{g_2^2}\right) bB_1\right] & \text{if } \mu \geq \delta_2 \end{cases}$$

where $\delta_1 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1^2} - \frac{1}{bB_1}\right)\left(\frac{g_2^2}{g_3}\right)$, $\delta_2 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1} + \frac{1}{bB_1^2}\right)\left(\frac{g_2^2}{g_3}\right)$.

Proof. Using (6), we have

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|\frac{bB_1}{4g_3}\left(c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}-bB_1\right)c_1^2\right)-\mu\left(\frac{g_2^2}{g_3}\right)\left(\frac{b}{2g_2}B_1c_1\right)^2\right| \\ &= \left|\frac{bB_1}{4g_3}\left\{c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}-bB_1+2\mu\left(\frac{g_3}{g_2^2}\right)bB_1\right)c_1^2\right\}\right|,\end{aligned}$$

which can be written as

$$\left|a_3-\mu a_2^2\right| = \left|\frac{bB_1}{4g_3}\right|\left|(c_2-\nu c_1^2)\right|,$$

where $\nu=\frac{1}{2}\left(1-\frac{B_2}{B_1}-\left(1-2\mu\left(\frac{g_3}{g_2^2}\right)\right)bB_1\right)$. From lemmama 4, it follows that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{bB_1}{2g_3}[\frac{B_2}{B_1}+\left(1-2\mu\left(\frac{g_3}{g_2^2}\right)\right)bB_1] & \text{if } \mu\leq\delta_1 \\ \frac{bB_1}{2g_3} & \text{if } \delta_1\leq\mu\leq\delta_2 \\ \frac{bB_1}{2g_3}\left[-\frac{B_2}{B_1}-\left(1-2\mu\left(\frac{g_3}{g_2^2}\right)\right)bB_1\right] & \text{if } \mu\geq\delta_2 \end{cases}$$

where $\delta_1=\frac{1}{2}\left(1+\frac{B_2}{bB_1^2}-\frac{1}{bB_1}\right)\left(\frac{g_2^2}{g_3}\right)$ and $\delta_2=\frac{1}{2}\left(1+\frac{B_2}{bB_1}+\frac{1}{bB_1^2}\right)\left(\frac{g_2^2}{g_3}\right)$.

We take $\phi(z)=\frac{1+Az}{1+Bz}\,(-1\leq B\leq A\leq 1)$ in theorem 5, to obtain the following result.

Corollary 14. *If $f\in S^*(b,g(z),\frac{1+Az}{1+Bz})$, $(-1\leq B\leq A\leq 1)$ then for any $\mu\in\mathbb{R}$ we have*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{b(A-B)}{2g_3}\left[-B+\left(1-2\mu\left(\frac{g_3}{g_2^2}\right)\right)b\left(A-B\right)\right] & \textit{if } \mu\leq\delta_1 \\ \frac{b(A-B)}{2g_3} & \textit{if } \delta_1\leq\mu\leq\delta_2 \\ \frac{b(A-B)}{2g_3}\left[B-\left(1-2\mu\left(\frac{g_3}{g_2^2}\right)\right)b\left(A-B\right)\right] & \textit{if } \mu\geq\delta_2 \end{cases}$$

where $\delta_1=\frac{1}{2}\left(1+\frac{-B}{b(A-B)}-\frac{1}{b(A-B)}\right)\left(\frac{g_2^2}{g_3}\right)$ *and* $\delta_1=\frac{1}{2}\left(1+\frac{-B}{b(A-B)}-\frac{1}{b(A-B)}\right)\left(\frac{g_2^2}{g_3}\right)$.

For $g(z)=z+\sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n$, $(\alpha,\beta,\lambda,\delta\geq 0,\ \beta>\alpha,\lambda>\delta,k\in\mathbb{N}_0)$, in theorem 3.3, we obtain the known result provided in [1].

Corollary 15. *If $f\in S^*\left(1,z+\sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k z^n,\phi(z)\right)$,*

$(\alpha,\ \beta,\ \lambda,\ \delta \geq 0,\ \beta > \alpha,\ \lambda > \delta, k \in \mathbb{N}_0)$, *then for any* $\mu \in \mathbb{R}$, *we have*

$$\left|a_3 - \mu a_2^2\right| \leq \left\{ \begin{cases} c\frac{b}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left\{B_2 - \left[2\mu\left(\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^k}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}\right)-1\right]bB_1^2\right\} & \text{if } \mu \leq \delta_1 \\ \frac{bB_1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k} & \text{if } \delta_1 \leq \mu \leq \delta_2 \\ \frac{b}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left\{-B_2 + \left[2\mu\left(\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^k}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}\right)-1\right]bB_1^2\right\} & \text{if } \mu \geq \delta_2 \end{cases} \right\},$$

where

$$\delta_1 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1^2} - \frac{1}{bB_1}\right)\left(\frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right),$$

and

$$\delta_2 = \frac{1}{2}\left(1 + \frac{B_2}{bB_1} + \frac{1}{bB_1^2}\right)\left(\frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}}{[2(\lambda-\delta)(\beta-\alpha)+1]^k}\right).$$

Remark 1. *By setting* $g(z) = \frac{z}{(1-z)^{n+1}}$, $n \in \mathbb{N}$, *and* $\phi(z) = \frac{1+z}{1-z}$ *in theorem 5, theorem 9 and theorem 13, we get the known results proved by Kanas and Darwish in [8].*

Using similar techniques as given in proof of theorem **??**, we obtain the following result.

Theorem 16. *Let* $\phi(z)$ *be given by (4),* $g(z)$ *by (2), both with real coefficients and* $b \in \mathbb{C}/\{0\}$. *If* $f \in C(b, g(z), \phi(z))$, *then*

$$|a_2| \leq \frac{|b|}{2}\frac{B_1}{g_2},$$

$$|a_3| \leq |b|\frac{B_1}{6g_3}\max\left\{1, \left|\frac{B_2}{B_1} + bB_1\right|\right\}$$

and

$$\left|a_3 - \frac{2}{3}\left(\frac{g_3}{g_2^2}\right)a_2^2\right| \leq \frac{|b|}{6g_3}B_2.$$

When $\phi(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1)$ in theorem 3.4, we obtain the following result.

Corollary 17. *If* $f \in C(b, g(z), \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1))$, *then*

$$|a_2| \leq \frac{|b|}{2}\frac{(A-B)}{g_2},$$

$$|a_3| \leq |b| \frac{(A-B)}{6g_3} \max\{1, |-B + b(A-B)|\},$$

and

$$\left| a_3 - \frac{2}{3} \left(\frac{g_3}{g_2^2} \right) a_2^2 \right| \leq \frac{|b|}{6g_3} [-B(A-B)].$$

Reasoning in the same lines as in the proof of theorem 3.2, we obtain the following theorm.

Theorem 18. *Let $\phi(z)$ be given by (4), $g(z)$ by (2), both with real coefficients and $b \in \mathbb{C}/\{0\}$. If $f \in C(b, g(z), \phi(z))$, then for any $\mu \in \mathbb{C}$, we have*

$$\left| a_3 - \mu a_2^2 \right| \leq |b| \frac{B_1}{2g_3} \max(1, \left| \frac{B_2}{B_1} + \left(1 - 2\mu \frac{g_3}{g_2^2} \right) bB_1 \right|.$$

We take $\phi(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B \leq A \leq 1)$ in 18, to obtain the following corollary-ollary.

Corollary 19. *If $f \in C(b, g(z), \frac{1+Az}{1+Bz})$, $(-1 \leq B \leq A \leq 1)$, then for any $\mu \in \mathbb{C}$ we have*

$$\left| a_3 - \mu a_2^2 \right| \leq |b| \frac{(A-B)}{2g_3} \max(1, \left| -B + \left(1 - 2\mu \frac{g_3}{g_2^2} \right) b(A-B) \right|.$$

When $g(z) = \frac{z}{1-z}$, $\phi(z) = \frac{1+z}{1-z}$, $z \in E$ and $b = 1$ in theorem 3.1, we obtain the known result, see [10].

Corollary 20. *If $f \in C\left(1, \frac{z}{1-z}, \frac{1+z}{1-z}\right)$, then for any $\mu \in \mathbb{C}$, we have*

$$\left| a_3 - \mu a_2^2 \right| \leq \max\left\{ \frac{1}{3}, |\mu - 1| \right\}.$$

If we take $g(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1} \right)^{\delta} z^n$, $(a > 0,\ \delta \geq 0)$, and $\phi(z) = \frac{1+z}{1-z}$, $z \in E$, in theorem 3.5, we obtain the following corollaryollary.

Corollary 21. *If $f \in C\left(1, z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1} \right)^{\delta} z^n, \frac{1+z}{1-z}\right)$, $(a > 0,\ \delta \geq 0, z \in E,)$, then for $\mu \in \mathbb{C}$, we have*

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{|b|}{3} \max\left\{ 1, \left| 1 + 2b - 3\mu b \left(\frac{(a+1)^2}{a(a+2)} \right)^{\delta} \right| \right\},$$

which has been proved in [2].

Remark 2. *By setting* $g(z) = \frac{z}{(1-z)^{n+1}}$*,* $n \in \mathbb{N}$*, and* $\phi(z) = \frac{1+z}{1-z}$ *in theorem 3.4 and theorem 3.5, we get the results given in [8].*

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