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Weierstrass Gaps and Curves on a Scroll

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Abstract. The aim of this paper is to study the Weierstrass semigroup of ramified points on non-singular models for curves on a rational normal scroll. We find properties of this semigroup and determine it in some special cases, finding also a geometrical interpretation for some of the Weierstrass gaps.

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Introduction

The Weierstrass gap sequences at ramification points of a (non-singular) trigonal curve have been determined by Coppens in [4] and [5]. These sequences also appeared in a work by Stöhr and Viana (cf. [12]), where they were both obtained by a method based on the fact that trigonal curves are canonically immersed on a rational normal scroll (Coppens had already used this fact in [5]). On the other hand, Weierstrass gap sequences at non-singular points of a singular plane curve (or, more precisely, at the inverse image of the non-singular point by the normalization morphism over the curve) have been studied in recent papers (e.g. [6], [7], [2]), specially when the non-singular point is ramified with respect to some morphism over the projective line. In the present work, we study the Weierstrass gap sequences at non-singular ramification points of possibly singular curves on a rational normal scroll, generalizing the results in [4] and [5] (the ramification being with respect to the morphism over the projective line defined by a ruling of the scroll). Also, we obtain a geometrical interpretation for some gaps, when the singularity locus of the curve is contained in the directrix of the scroll, and contains only simple cusps or simple nodes.

1. Divisors on curves on a scroll

A rational normal scroll $\mathcal{S}_{m\,n} \subset \mathbb{P}^{m+n+1}(k)$ defined over an algebraically closed field k is a surface which after a suitable choice of projective coordinates is given by

$$\mathcal{S}_{m\,n} := \{ (x_0 : \ldots : x_{m+n+1}) \in \mathbb{P}^{m+n+1}(k) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{n+m} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_{n+m+1} \end{pmatrix} < 2 \}$$

where the positive integers m and n are such that $m \leq n$.

$\mathcal{S}_{m\,n}$ has a ruling given by the union of the disjoint lines

$$L_{b/a} := \overline{(a^n : a^{n-1}b : \ldots : b^n : 0 : \ldots : 0)\,,\,(0 : \ldots : 0 : a^m : a^{m-1}b : \ldots : b^m)},$$

where $b/a \in \mathbb{P}^1(k) = k \cup \{\infty\}$, which join points of the non-singular rational curves

$$D := \{(a^n : a^{n-1}b : \ldots : b^n : 0 : \ldots : 0) \in \mathbb{P}^{m+n+1}(k) \mid (a : b) \in \mathbb{P}^1(k)\} \text{ and}$$

$$E := \{(0 : \ldots : 0 : a^m : a^{m-1}b : \ldots : b^m) \in \mathbb{P}^{m+n+1}(k) \mid (a : b) \in \mathbb{P}^1(k)\}.$$

Following [12] we cover $\mathcal{S}_{m\,n}$ with four affine open sets, all isomorphic to $\mathbb{A}^2(k)$ and defined by

$$U_0 := \mathcal{S}_{m\,n} \setminus (L_\infty \cup E) = \{(a^0 : \ldots : a^n : a^0 b : \ldots : a^m b) \in \mathbb{P}^{m+n+1}(k) \mid (a, b) \in \mathbb{A}^2(k)\},$$

$$U_n := \mathcal{S}_{m\,n} \setminus (L_0 \cup E) = \{(a^n : \ldots : a^0 : a^m b : \ldots : a^0 b) \in \mathbb{P}^{m+n+1}(k) \mid (a, b) \in \mathbb{A}^2(k)\},$$

$$U_{n+1} := \mathcal{S}_{m\,n} \setminus (L_\infty \cup D) = \{(a^0 b : \ldots : a^n b : a^0 : \ldots : a^m) \in \mathbb{P}^{m+n+1}(k) \mid (a, b) \in \mathbb{A}^2(k)\},$$

$$U_{n+m+1} := \mathcal{S}_{m\,n} \setminus (L_0 \cup D) = \{(a^n b : \ldots : a^0 b : a^m : \ldots : a^0) \in \mathbb{P}^{m+n+1}(k) \mid (a, b) \in \mathbb{A}^2(k)\}.$$

Associating to each affine curve in $\mathbb{A}^2(k)$ the Zariski closure of its image in U_0 under the isomorphism $(a : b) \mapsto (a^0 : \ldots : a^n : a^0 b : \ldots : a^m b)$ we get a bijection between affine plane curves and the projective curves on $\mathcal{S}_{m\,n}$ that do not have L_∞ or E as a component (we do not assume that a curve is irreducible, unless explictly stated).

We deal in this paper with (possibly) singular curves and divisors on them, following in this matter [11] (cf. also [9]). Thus let C be an integral curve defined over k and let $k(C)$ be its function field, a *divisor* $\mathcal{D}$ on C is a non-zero coherent fractional ideal sheaf of C, which we denote by the product of its stalks $\mathcal{D} = \prod_{P \in C} \mathcal{D}_P$. We denote by $\mathcal{O}$ the structure sheaf of C. The *local degree at* $P \in C$ *of* D is the integer $\deg_P(D)$ defined by requiring that $\deg_P(\mathcal{O}) = 0$ and that $\deg_P(\mathcal{D}) - \deg_P(\mathcal{E}) = \dim_k(\mathcal{D}_P/\mathcal{E}_P)$ whenever $\mathcal{D}_P \supseteq \mathcal{E}_P$. The *degree* of $\mathcal{D}$ is the integer $\deg(\mathcal{D}) := \sum_{P \in C} \deg_P(\mathcal{D})$. The *divisor of a rational function* $h \in k(C)^*$ is defined by $\operatorname{div} h := \prod_{P \in C}(1/h)\mathcal{O}_P$. If F is a (Cartier) divisor on $\mathcal{S}_{m\,n}$ and $C \subset \mathcal{S}_{m\,n}$ is not a component of F then we define the *intersection divisor* of C and F as $C \cdot F := \prod_{P \in C}(1/f_P)\mathcal{O}_P$, where F is locally defined by f_P on a open set containing P. We observe that the local degree at P of $C \cdot F$ coincides with the intersection number $i(C, F; P)$ of C and F at P as divisors on $\mathcal{S}_{m\,n}$.

We also note that the divisors on a singular curve are not necessarily locally principal, i.e. of the form $\mathcal{D} = \prod d_P \mathcal{O}_P$, where $d_P \in k(C)^*$ for all $P \in C$ (cf. [9, Ex. 1.6.1] or [3, Ex. 2.4]) and they do not form a group under the operation defined by $\mathcal{D} * \mathcal{E} := \prod \mathcal{D}_P \mathcal{E}_P$. Nevertheless, the locally principal divisors do form a commutative group under this operation and since the divisors on C appearing on this paper are all (intersection divisors and hence) locally principal we will denote this operation as a sum, thus $\prod d_P \mathcal{O}_P + \prod e_P \mathcal{O}_P = \prod (d_P e_P) \mathcal{O}_P$ and $\prod d_P \mathcal{O}_P - \prod e_P \mathcal{O}_P = \prod (d_p/e_P) \mathcal{O}_P$. Accordingly, instead of $\prod d_P \mathcal{O}_P \supseteq \prod \mathcal{O}_P$ we write $\prod d_P \mathcal{O}_P \geq 0$ and say that $\prod d_P \mathcal{O}_P$ is a *non-negative divisor*. Two divisors $\mathcal{D}$ and $\mathcal{E}$ on C are *linearly equivalent* if $\mathcal{D} - \mathcal{E} = \operatorname{div} h$ for some $h \in k(C)^*$ and the set $|\mathcal{K}|$ of all non-negative divisors linearly equivalent to a canonical divisor $\mathcal{K}$ on C is called the *canonical linear series of* C.

Now let C be a curve on $\mathcal{S}_{mn}$ that does not have E or L_∞ as a component and let $c_\ell(X)Y^\ell + c_{\ell-1}Y^{\ell-1} + \cdots + c_0(X) = 0$ be the equation of the affine curve that corresponds to $C \cap U_0$ under the isomorphism $\mathbb{A}^2(k) \simeq U_0$ described above. Then $\deg(C \cdot L_a) = \ell$ for all $a \in k \cup \{\infty\}$, $\deg(C \cdot E) = d_\ell$ and $\deg(C \cdot D) = d_\ell + \ell(n-m)$, where d_ℓ is the smallest integer such that $\deg c_i(X) \leq d_\ell + (\ell - i)(n-m)$ for all $i \in \{0, \ldots, \ell\}$ (and hence the equality holds for some i). The Picard group of $\mathcal{S}_{mn}$ is the free group generated by the classes of D and a line L, and the canonical divisor of $\mathcal{S}_{mn}$ is linearly equivalent to $-2D + (n-m-2)L$ (cf. [1, page 121]). From this we may deduce that $C \sim \ell D + d_\ell L$, where $\sim$ denotes the linear equivalence of divisors on $\mathcal{S}_{mn}$ and, if C is irreducible, from the adjunction formula $2g-2 = C \cdot (C + (n-m-2)L - 2D)$ (cf. [10, page 75]) we get $g = (\ell-1)(2d_\ell + \ell(n-m) - 2)/2$, where g is the arithmetic genus of C. In what follows L will always denote a line of the ruling on $\mathcal{S}_{mn}$. We recall that any two lines of the ruling on $\mathcal{S}_{mn}$ are linearly equivalent and we also have $E \sim D - (n-m)L$ (cf. [12]).

Theorem 1.1. *The divisors of the canonical linear series of an irreducible curve* $C \in \mathcal{S}_{mn}$ *are exactly the intersections of* C *with curves linearly equivalent to* $(\ell-2)E + (d_\ell + (\ell-1)(n-m) - 2)L$.

Proof. Let x and y be the rational functions defined on $C \cap U_0$ by $(a^0 : \ldots : a^n : a^0 b : \ldots : a^m b) \mapsto a$ and $(a^0 : \ldots : a^n : a^0 b : \ldots : a^m b) \mapsto b$, respectively and let $\mathcal{K} := (\ell-2) C \cdot E + (d_\ell + (\ell-1)(n-m) - 2) C \cdot L_\infty$. We have $\operatorname{div} x = C \cdot L_0 - C \cdot L_\infty$ and $\operatorname{div} y = C \cdot D - C \cdot E - (n-m) C \cdot L_\infty$, thus $\{x^i y^j \mid 0 \leq j \leq \ell-2,\ 0 \leq i \leq d_\ell + (\ell-1-j)(n-m) - 2\} \subset H^0(\mathcal{K})$. The degree of $\mathcal{K}$ is $(\ell-2)d_\ell + (\ell-1)\ell(n-m) + (d_\ell - 2)\ell = 2g-2$ hence the set of the g linearly independent elements $x^i y^j$ form a basis for $H^0(\mathcal{K})$ and $\mathcal{K}$ is canonical divisor of C. Now let $f := \sum_{j=0}^{\ell-2} \sum_{i=0}^{d_\ell + (\ell-1-j)(n-m)-2} a_{ij} x^i y^j$ be a non-zero element of $H^0(\mathcal{K})$, let r be the greatest integer such that $a_{ir} \neq 0$ for some i and let e_r be the least non-negative integer satifying $\max\{i \mid a_{ij} \neq 0; i = 0, \ldots, d_\ell + (\ell-1-j)(n-m) - 2\} \leq e_r + (r-j)(n-m)$ for all $j = 0, \ldots, r$ such that $a_{ij} \neq 0$ for some i. Then $0 \leq e_r \leq d_\ell + (\ell-1-r)(n-m) - 2$, $a_{ij} = 0$ if $j > r$ or $i > e_r + (r-j)(n-m)$ and let F be the curve on S_{mn} whose correspondent curve on $\mathbb{A}^2(k) \simeq U_0$ is $\sum_{j=0}^{r} \sum_{i=0}^{e_r + (r-j)(n-m)} a_{ij} X^i Y^j = 0$. We claim that $\operatorname{div}(\sum_{j=0}^{r} \sum_{i=0}^{e_r + (r-j)(n-m)} a_{ij} x^i y^j) + \mathcal{K}$ is the intersection divisor of C and $G := (\ell-2-r)E + (d_\ell + (\ell-1-r)(n-m) - 2 - e_r)L_\infty + F$. In fact, if $P \in C \cap U_0$ then $(C \cdot G)_P = (1/\sum_{j=0}^{r} \sum_{i=0}^{e_r + (r-j)(n-m)} a_{ij} x^i y^j)\mathcal{O}_P$ and the claim holds because $\mathcal{K}_P = \mathcal{O}_P$. Suppose now that $P \in C \cap U_{n+m+1}$ and let $\tilde{x}$ and $\tilde{y}$ be the rational functions defined on $C \cap U_{n+m+1}$ by $(a^n b : \ldots : a^0 b : a^m : \ldots : a^0) \mapsto a$ and $(a^n b : \ldots : a^0 b : a^m : \ldots : a^0) \mapsto b$

respectively, we have $x = 1/\tilde{x}$ and $y = 1/(\tilde{x}^{(n-m)}\tilde{y})$ on $C \cap U_0 \cap U_{n+m+1}$. Let $(a^n b : \dots : a^0 b : a^m : \dots : a^0) \mapsto (a, b)$ be an isomorphism between U_{n+m+1} and $\mathbb{A}^2(k)$ and let $\tilde{X}$ and $\tilde{Y}$ be the affine coordinates in $\mathbb{A}^2(k)$, then $F \cap U_{n+m+1}$, $E \cap U_{n+m+1}$ and $L_\infty \cap U_{n+m+1}$ correspond to the plane curves given by $\sum_{j=0}^{r} \sum_{i=0}^{e_r+(r-j)(n-m)} a_{ij} \tilde{X}^{e_r+(r-j)(n-m)-i} \tilde{Y}^{r-j} = 0$, $\tilde{Y} = 0$ and $\tilde{X} = 0$ respectively. Now it is easy to check that $(C \cdot G)_P = (\mathrm{div}(\sum_{j=0}^{r} \sum_{i=0}^{e_r+(r-j)(n-m)} a_{ij} x^i y^j) + \mathcal{K})_P = (1/\sum_{j=0}^{r} \sum_{i=0}^{e_r+(r-j)(n-m)} a_{ij}\, \tilde{x}^{d_\ell+(l-1-j)(n-m)-2-i}\, \tilde{y}^{\ell-2-j})\mathcal{O}_P$. The proof of the claim for $P \in U_n$ and $P \in U_{n+1}$ is similar. Thus any divisor in $|\mathcal{K}|$ is the intersection of C and a curve linearly equivalent to $(\ell-2)E + (d_\ell + (\ell-1)(n-m) - 2)L$.

Conversely, if H is a curve linearly equivalent to $(\ell-2)E + (d_\ell + (\ell-1)(n-m)-2)L$ then we may write $H = sE + tL_\infty + G$, with s and t non-negative integers, G a curve that does not have E or L_∞ as a component, and $G \sim (\ell-2-s)E + (d_\ell + (\ell-1)(n-m) - 2 - t)L \sim (\ell-2-s)D + (d_\ell + (s+1)(n-m) - 2 - t)L$. Thus $G \cap U_0$ is an affine curve given in $\mathbb{A}^2(k) \simeq U_0$ by an equation of the form $\sum_{j=0}^{\ell-2-s} \sum_{i=0}^{d_\ell+(s+1)(n-m)-2-t} a_{ij} X^i Y^j = 0$, and as above one may check that $\mathrm{div}\,(\sum_{j=0}^{\ell-2-s} \sum_{i=0}^{d_\ell+(s+1)(n-m)-2-t} a_{ij}\, x^i\, y^j) + \mathcal{K} = C \cdot H$. This completes the proof of the theorem. □

2. Weierstrass gaps at ramification points

From now on C will always denote an irreducible curve on $\mathcal{S}_{mn}$. Let $\eta : \tilde{C} \to C$ be the normalization of C, let $\tilde{P} \in \tilde{C}$ and let $\tilde{\mathcal{K}}$ be a canonical divisor on $\tilde{C}$. The set of positive integers $WG(\tilde{P}) := \{1 + \dim_k \mathcal{D}_{\tilde{P}}/\mathcal{O}_{\tilde{P}} \mid \mathcal{D} \in |\tilde{\mathcal{K}}|\}$ is called the Weierstrass gap sequence at $\tilde{P}$. The cardinality of this set is equal to the genus of $\tilde{C}$ and its complementary in the set of the non-negative integers is called the Weierstrass semigroup at $\tilde{P}$ (cf. [13]). Let $P \in C$ be a non-singular point and let $\tilde{P} = \eta^{-1}(P)$. In this case we will refer to the set $WG(\tilde{P})$ as the *Weierstrass gap sequence at* P and write $WG(P)$. Also, if T is the line of the ruling passing through P and $r := i(C, T; P)$ then we say that P is an *r-ramification point of* C. We want to determine $WG(P)$ at r-ramification points of $C \sim \ell D + d_\ell L$ for $r = \ell, \ell-1$ (observe that $r \leq \deg(C \cdot L) = \ell$). Let's begin with the case where C is non-singular.

Theorem 2.1. *Let C be a non-singular curve on a scroll $\mathcal{S}_{mn}$ such that $C \sim \ell D + d_\ell L$. Let $P \in C$ be an r-ramification point with $r \geq 2$ and let $WG(P)$ be the Weierstrass gap sequence at P.*

a) *If $P \notin E$ then $\{ir + j + 1 \mid j = 0, 1, \dots, \ell-2;\ i = 0, 1, \dots, d_\ell + (\ell-1-j)(n-m) - 2\} \subseteq WG(P)$ and equality holds when $r \in \{\ell, \ell-1\}$.*

b) *If $P \in E$ then $\{ir + \ell - 1 - j \mid j = 0, 1, \dots, \ell-2;\ i = 0, 1, \dots, d_\ell + (\ell-1-j)(n-m) - 2\} \subseteq WG(P)$ and equality holds when $r \in \{\ell, \ell-1\}$.*

Proof. Let T be the line of the ruling through P. After a suitable automorphism of $\mathcal{S}_{mn}$ we may assume that $P = T \cap D$, if $P \notin E$ (cf. [12, Prop. 1.2]) and of course $P = T \cap E$, if $P \in E$. Since $i(C, T; P) \geq 2$ we have $i(C, D; P) = 1$, if $P \notin E$ or $i(C, E; P) = 1$, if $P \in E$. Let $L \neq T$ be another line of the ruling and hence $P \notin L$. From Theorem 1.1 we get that

$$WG(P) \supseteq \{1 + i(C, jD + (\ell-2-j)E + (d_\ell + (\ell-1-j)(n-m) - 2 - i)L + iT; P) \mid 0 \leq j \leq \ell-2, 0 \leq i \leq d_\ell + (\ell-1-j)(n-m) - 2\}.$$

The right hand side set is equal to $\{ir+j+1 \mid 0 \le j \le \ell-2, 0 \le i \le d_\ell+(\ell-1-j)(n-m)-2\}$ if $P \notin E$, or is equal to $\{ir+\ell-1-j \mid 0 \le j \le \ell-2,\ 0 \le i \le d_\ell+(\ell-1-j)(n-m)-2\}$ if $P \in E$. Moreover, if $r \in \{\ell, \ell-1\}$ these sets have cardinality equal to $(\ell-1)(2d_\ell+\ell(n-m)-2)/2$ which is the genus of C and hence equality holds in either case. □

From now on we do not supppose that C is a smooth curve. Let $\mathcal{F}$ be the conductor divisor on C defined by $\mathcal{F}_P = (\mathcal{O}_P : \widetilde{\mathcal{O}_P})$ for all $P \in C$, where $\widetilde{\mathcal{O}_P}$ is the integral closure of $\mathcal{O}_P$ in $k(C)$. We call a divisor F on $\mathcal{S}_{mn}$ an *adjoint curve* if $F \sim (\ell-2)E+(d_\ell+(\ell-1)(n-m)-2)L$ and $\widetilde{\mathcal{O}_P} \subseteq (F \cdot C)_P \mathcal{F}_P$ for all $P \in C$. If Q is a singular point of C then $\mathcal{F}_Q \subset \mathcal{M}_Q$, where $\mathcal{M}_Q$ is the maximal ideal of $\mathcal{O}_Q$, and if f_Q defines an adjoint curve F locally in an open set of $\mathcal{S}_{mn}$ containing Q we get $\widetilde{\mathcal{O}_Q} f_Q \subset \mathcal{F}_Q \subset \mathcal{M}_Q$, thus F intersects C at Q. Exactly as in the case of plane curves one may show that the divisors of the canonical series of $\tilde{C}$ are the scheme theoretic inverse image under η of the divisors $\prod (F \cdot C)_P \mathcal{F}_P$, where F is an adjoint curve. At a non-singular point $P \in C$ we have $(F \cdot C)_P \mathcal{F}_P = (F \cdot C)_P$ since $\mathcal{F}_P = \mathcal{O}_P$, thus from the preceeding theorem we obtain the following result.

Lemma 2.2. *Let $P \in C \sim \ell D + d_\ell L$ be an r-ramification point, where $r \in \{\ell, \ell-1\}$.*

a) *If $P \notin E$ then $WG(P) \subset \{ir+j+1 \mid j = 0,1,\dots,\ell-2;\ i = 0,1,\dots,d_\ell+(\ell-1-j)(n-m)-2\}$.*

b) *If $P \in E$ then $WG(P) \subset \{ir+\ell-1-j \mid j = 0,1,\dots,\ell-2;\ i = 0,1,\dots,d_\ell+(\ell-1-j)(n-m)-2\}$.*

The next result shows that the so called Namba's Lemma holds for curves on $\mathcal{S}_{mn}$.

Lemma 2.3. *Let C, C_1 and C_2 be curves on a scroll $\mathcal{S}_{mn}$ and let $P \in \mathcal{S}_{mn}$ be a non-singular point of C. Then $i(C_1, C_2; P) \ge \min\{i(C, C_1; P), i(C, C_2; P)\}$.*

Proof. Let $F = 0$, $G_1 = 0$ and $G_2 = 0$ be local equations for C, C_1 and C_2 respectively, in an open affine subset of $\mathcal{S}_{mn}$ isomorphic to $\mathbb{A}^2(k)$. For $i \in \{1,2\}$ we get $i(C, C_i; P) = \dim_k \mathcal{O}_{\mathbb{A}^2(k),P}/(F, G_i) = \dim_k \mathcal{O}_{C,P}/(g_i) = \mathrm{ord}_P(g_i)$ where $g_i \in k(C)$ is the rational function determined by the polynomial G_i. Then $i(C_1, C_2; P) = \dim_k \mathcal{O}_{\mathbb{A}^2(k),P}/(G_1, G_2) \ge \dim_k \mathcal{O}_{\mathbb{A}^2(k),P}/(F, G_1, G_2) = \dim_k \mathcal{O}_{C,P}/(g_1, g_2) = \min\{\mathrm{ord}_P(g_1),\ \mathrm{ord}_P(g_2)\} = \min\{i(C, C_1; P),\ i(C, C_2; P)\}$. □

Theorem 2.4. *Let $P \in C \subset \mathcal{S}_{mn}$ be a non-singular r-ramification point and let T be the line of the ruling passing through P. If $F \sim sE + tL$ is a divisor of $\mathcal{S}_{mn}$ such that $s < r$ and $i(C, F; P) \ge r$ then $F = T + G$ and $G \sim sE + (t-1)L$.*

Proof. From the above Lemma $i(F, T; P) \ge \min\{i(C, T; P), i(C, F; P)\} \ge r$ but $\deg(F \cdot T) = s\deg(E \cdot T) + t\deg(L \cdot T) = s < r$. Then (cf. [8, page 360]) F and T must have a common irreducible component so $F = T + G$ for some divisor $G \subset \mathcal{S}_{mn}$ and $G \sim sE + (t-1)L$. □

Corollary 2.5. *Let $P \in C \subset \mathcal{S}_{mn}$ be an r-ramification point of $C \sim \ell D + d_\ell L$ where $r \in \{\ell, \ell-1\}$. If $ir + s$ is a Weierstrass gap at P, with s and i positive integers, then $\{(i-1)r+s, \dots, r+s, s\}$ are also Weierstrass gaps at P.*

Proof. Since $ir+s$ is a Weierstrass gap at P there exists an adjoint curve F such that $i(C,F;P) = ir+s-1$. Let T be the line of the ruling passing through P, from the above Theorem we get that $F = T + G$ where $G \sim (\ell-2)E + (d_\ell + (\ell-1)(n-m) - 3)L$ and $i(C,G;P) = (i-1)r+s-1$. From $\deg(C\cdot T) = \ell$ and $r \in \{\ell, \ell-1\}$ we get that T intersects C at most at another non-singular point so if T' is a line of the ruling not containing P then the curve $G+T'$ is an adjoint curve. From $i(C,G+T';P) = i(C,G;P)$ we get that the integer $(i-1)r+s$ is a Weierstrass gap at P. Thus the corollary follows from repeated applications of the above theorem. □

In view of the above proof, if $ir+s \in WG(P)$ one would expect iT to be a component of an adjoint curve that yields this gap. The result below shows that if $m < n$ then the curve E is also a component of many adjoint curves.

Theorem 2.6. *Let $C \sim \ell D + d_\ell L$ be a singular curve on $\mathcal{S}_{mn}$, where $m < n$ and let T be the line of the ruling passing through the r-ramification point P, with $r \in \{\ell, \ell-1\}$. If F is an adjoint curve such that $i(C,F;P) = ir+j$ or $i(C,F;P) = ir+\ell-2-j$, where $j \in \{0,\dots,\ell-3\}$ and $i \in \{d_\ell+(n-m)-1,\dots,d_\ell+(\ell-1-j)(n-m)-2\}$, then $F = iT+sE+H$, where s is the integer satisfying $d_\ell + s(n-m)-2 < i \le d_\ell+(s+1)(n-m)-2$ and H is an effective divisor of $\mathcal{S}_{mn}$.*

Proof. Let F be an adjoint curve such that $i(C,F;P) = ir+j$ or $i(C,F;P) = ir+\ell-2-j$, with i and j as in the theorem. After successive applications of Theorem 2.4 we get $F = iT + G$, where $G \sim (\ell-2)E + (d_\ell + (\ell-1)(n-m) - 2 - i)L \sim (\ell-2)D + (d_\ell + (n-m) - 2 - i)L$. As we remarked in Section 1, a curve that does not have E or L_∞ as a component is linearly equivalent to a divisor $aD+bL$ of $\mathcal{S}_{mn}$, with $a \ge 0$ and $b \ge 0$. Since $d_\ell + (n-m) - 2 - i < 0$ the curve G must have E as a component, so $G = E + G_1$ where $G_1 \sim (\ell-3)D + (d_\ell + 2(n-m) - 2 - i)L$. We repeat this argument s times to obtain $G = sE + G_s$ with $G_s \sim (\ell-2-s)D + (d_\ell + (s+1)(n-m) - 2 - i)L$. □

Taking into account that C_{Sing} is contained in every adjoint curve, we may interpret geometrically the greatest possible integers in $WG(P)$ as follows.

Corollary 2.7. *Let $C \sim \ell D + d_\ell L$ be a singular curve on $\mathcal{S}_{mn}$ and let P be a non-singular r-ramification point, where $r \in \{\ell, \ell-1\}$.*

a) *If $m < n$ and $(d_\ell+(\ell-1)(n-m)-2)r+1 \in WG(P)$ or $(d_\ell+(\ell-1)(n-m)-2)r+\ell-1 \in WG(P)$ then $C_{Sing} \subset E$.*

b) *If $m = n$ and $(d_\ell-2)r+\ell-1 \in WG(P)$ then C_{Sing} is contained in a curve linearly equivalent to D that also contains P.*

Proof. To prove (a) we take $i = (d_\ell + (\ell-1)(n-m) - 2)$ and $j = 0$ in the above theorem and get $s = \ell-2$. Thus $F = (d_\ell + (\ell-1)(n-m) - 2)T + (\ell-2)E$ and we must have $C_{Sing} \subset E$. To prove (b) we use Theorem 2.4 to obtain an adjoint curve $F = (d_\ell-2)T + G$ where $G \sim (\ell-2)D$ and $C_{Sing} \subset G$. Since $n = m$ the curves linearly equivalent to D are exactly E and the curves given in $\mathbb{A}^2(k) \simeq U_0$ by the equations $Y - b = 0$, with $b \in k$; it is easy to check that these curves do not intersect each other. Since $i(C,G;P) = \ell-2$ we must have $G = (\ell-2)D'$, where D' is a curve linearly equivalent to D that contains P and C_{Sing}. □

The following result determines $WG(P)$ for curves satisfying certain restrictions on the singularities.

Proposition 2.8. *Let $C \sim \ell D + d_\ell L$ be a curve of singularity degree δ on the scroll $\mathcal{S}_{mn}$. Let $P \in C$ be an r-ramification point, where $r \in \{\ell, \ell-1\}$. Suppose that $C_{Sing} \subset E$ if $m < n$, or that C_{Sing} is contained in a curve linearly equivalent to D, if $m = n$. Suppose also that the singularities of C are either simple nodes or simple cusps.*

a) *If $m < n$ and $P \notin E$, or if $m = n$ and P and C_{Sing} are not contained in a curve linearly equivalent to D, then $WG(P) = \{ir + j + 1 \,|\, 0 \le j \le \ell - 3;\ 0 \le i \le d_\ell + (\ell - 1 - j)(n - m) - 2\} \cup \{ur + \ell - 1 \,|\, 0 \le u \le d_\ell + (n - m) - 2 - \delta\}$.*

b) *If $m < n$ and $P \in E$, or if $m = n$ and P and C_{Sing} are contained in a curve linearly equivalent to D, then $WG(P) = \{ir + \ell - 1 - j \,|\, 0 \le j \le \ell - 3;\ 0 \le i \le d_\ell + (\ell - 1 - j)(n - m) - 2\} \cup \{(d_\ell + (n - m) - 2 - \delta - u)r + 1 \,|\, 0 \le u \le d_\ell + (n - m) - 2 - \delta\}$.*

Proof. Let $\mathcal{F}$ be the conductor divisor of C. We recall that C is a Gorenstein curve, for it lies on a surface, and thus the degree of singularity of a point $Q \in C$ is equal to $\dim_k(\mathcal{O}_Q/\mathcal{F}_Q)$. From $\mathcal{F}_Q \subset \mathcal{M}_Q$, where $\mathcal{M}_Q$ is the maximal ideal of $\mathcal{O}_Q$, and the hypothesis on the singularities we get $\mathcal{F}_Q = \mathcal{M}_Q$ for all singular points of C (thus $\mathcal{M}_Q$, as $\mathcal{F}_Q$, is not only an $\mathcal{O}_Q$-module but also an $\widetilde{\mathcal{O}}_Q$-module). If F is a curve intersecting C at a singular point Q and f_Q defines F locally on an open set of $\mathcal{S}_{mn}$ containing Q then $f_Q \in \mathcal{M}_Q$ and hence $\widetilde{\mathcal{O}}_P f_Q \subset \mathcal{F}_Q$, i.e. $\widetilde{\mathcal{O}}_P \subset (F \cdot C)_Q \mathcal{F}_Q$. This shows that any curve $F \sim (\ell - 2)E + (d_\ell + (\ell - 1)(n - m) - 2)L$ passing through all the singular points of C is an adjoint curve.

If $m = n$ then there exists an automorphism of the scroll taking a given curve linearly equivalent to D onto E (cf. [12]), so we may assume that $C_{sing} \subset E$. Let T be the line of the ruling that contains P. If $P \notin E$, let D' be a curve linearly equivalent to D containing P, then $P = T \cap D'$; if $P \in E$ then $P = T \cap E$. Let $L_1, \ldots, L_\delta$ be the lines of the ruling passing through the points in C_{Sing}. Let L be a line of the ruling different from T. To obtain the Weierstrass gaps listed in the theorem it suffices to calculate the local degree at P of the intersection divisor of C and the adjoint curves $iT + (\ell - 2 - j)E + jD' + (d_\ell + (\ell - 1 - j)(n - m) - 2 - i)L$, where $0 \le j \le \ell - 3$, $0 \le i \le d_\ell + (\ell - 1 - j)(n - m) - 2$ and $(\ell-2)D' + (d_\ell + (n-m) - 2 - \delta - u)T + (u+1)L_1 + L_2 + \cdots + L_\delta$, where $0 \le u \le d_\ell + (n-m) - 2 - \delta$. Using $r \in \{\ell, \ell-1\}$ one may check that we get $(\ell-1)(2d_\ell + \ell(n-m) - 2)/2 - \delta = g - \delta$ distinct numbers (where g is the arithmetic genus of C) and this is the cardinality of $WG(P)$. $\square$

The next result follows from the above proposition and Corollary 2.7.

Corollary 2.9. *Let $C \sim \ell D + d_\ell L$ be a curve on $\mathcal{S}_{mn}$, whose singularities are only simple nodes or simple cusps. Let P be a non-singular r-ramification point of C, where $r \in \{\ell, \ell-1\}$.*

a) *If $m < n$ and $P \notin E$ then $C_{Sing} \subset E$ if and only if $(d_\ell + (\ell - 1)(n - m) - 2)r + 1$ is a Weierstrass gap at P.*

b) *If $m < n$ and $P \in E$ then $C_{Sing} \subset E$ if and only if $(d_\ell + (\ell - 1)(n - m) - 2)r + \ell - 1$ is a Weierstrass gap at P.*

c) *If $m = n$ then P and C_{Sing} are on a curve that is linearly equivalent to D if and only if $(d_\ell - 2)r + \ell - 1$ is a Weierstrass gap at P.*

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