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On the Characterization of some Families of Closed Convex Sets*

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Abstract. This paper deals with the characterization of the sums of compact convex sets with linear subspaces, simplices, sandwiches (convex hulls of pairs of parallel affine manifolds) and parallelotopes in terms of the so-called internal and conical representations, topological and geometrical properties. In particular, it is shown that a closed convex set is a sandwich if and only if its relative boundary is unconnected. The characterizations of families of closed convex sets can be useful in different fields of applied mathematics. For instance, it is proved that a bounded linear semi-infinite programming problem whose feasible set is the sum of a compact convex set with a linear subspace is necessarily solvable and has zero duality gap.

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1. Introduction

The characterization of families of closed convex sets can be useful from different perspectives. A linear semi-infinite programming (LSIP) problem consists of the minimization of a linear functional on a closed convex set in $\mathbb{R}^n$ which is described by means of infinitely many linear inequalities. If the feasible set is the sum of a compact convex set with a linear subspace, we shall see that the boundedness of the LSIP problem entails its solvability. On the other hand,

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under suitable assumptions on the constraints system, it is possible to obtain an extreme point of the feasible set from any feasible solution without loss in the objetive (LSIP purification algorithms can be found in [2] and [5]) and then, starting at this initial extreme point, it is possible to construct a polygonal of linked edges along which the optimal functional decreases (an LSIP simplex method has been proposed in [1]). Obviously, the viability of an algorithm progressing on the boundary of the feasible region requires its connectivity by arcs. This paper characterizes the class of closed convex sets whose (relative) boundary is non-empty and connected by arcs.

On the other hand, typical geometric combinatorial problems are the characterization of those convex bodies (full dimensional closed convex sets) for which the minimum number of points (or directions) illuminating them in a certain sense has a given expression (see the survey in [9]).

This paper deals with the different ways of characterizing families of closed convex sets, focussing the attention on the sums of compact convex sets with linear subspaces and on three particular subfamilies of this class: simplices, sandwiches and parallelotopes.

Now, let us introduce some notation. The zero vector in $\mathbb{R}^n$ will be represented by 0_n and the Euclidean open ball by B_n. Given a set X, $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\operatorname{conv} X$, $\operatorname{cone} X$, $\operatorname{span} X$, $\operatorname{aff} X$ and $X^{\perp}$ the convex hull of X, the conical convex hull of X, the linear subspace of $\mathbb{R}^n$ spanned by X, the affine hull of X and $\{y \in \mathbb{R}^n \mid x'y = 0 \text{ for all } x \in X\}$, respectively. Moreover, we define $\operatorname{cone} \emptyset = \{0_n\}$.

Given a convex set X, $\dim X$ represents its dimension (i.e., the dimension of $\operatorname{aff} X$), O^+X the recession cone of X and, given $x \in X$, $D(X;x)$ denotes the (convex) cone of feasible directions at x.

From the topological side, given $X \subset \mathbb{R}^n$, $\operatorname{cl} X$, $\operatorname{int} X$, $\operatorname{bd} X$, $\operatorname{rint} X$ and $\operatorname{rbd} X$ denote the closure of X, the interior of X, the boundary of X, the relative interior of X and the relative boundary of X, respectively. We shall use the following result.

Lemma 1.1. (2.1 and 2.5 in [4]) *Let $\emptyset \neq X \subset \mathbb{R}^n$. $z \in \operatorname{rint} \operatorname{cone} X$ if, and only if, there exist points $x^i \in X$, $i = 1, \ldots, p$, and corresponding positive scalars λ_i, $i = 1, \ldots, p$, such that* $\operatorname{span}\{x^1, \ldots, x^p\} = \operatorname{span} X$ *and* $z = \sum_{i=1}^{p} \lambda_i x^i$.

Any non-empty closed convex set $C \subset \mathbb{R}^n$ admits different representations. First, C can be decomposed as the sum of its lineality space, $L(C)$, with $C \cap L(C)^{\perp}$ (this is the pointed cone of C if C is a convex cone). The last set turns out to be the sum of the convex hull of its set of extreme points, $E(C) \neq \emptyset$, with the convex conical hull of its set of extreme directions, $D(C)$ ([10], Th. 18.5), so that $C = L(C) + E(C) + D(C)$. The triple $(L(C), E(C), D(C))$ constitutes the internal representation of C.

On the other hand, C is the intersection of all the supporting half-spaces to C, so that C is the solution set of a certain linear system $\sigma = \{a_t'x \geq b_t,\ t \in T\}$, where $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T$, the index set T being possibly infinite. Such a system σ is called an external representation of C. The non-homogeneous Farkas Lemma [15] establishes that $a'x \geq b$ is a consequence of the consistent system σ if, and only if,

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl} \operatorname{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

From here, it can be easily shown that the right hand side cone,

$$K(C) := \operatorname{cl}\operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\},$$

is the same for all the external representations of $C \neq \emptyset$, so that, the so-called reference cone, $K(C)$, can be seen as a conical representation of C. From $K(C)$ it is possible to obtain different external representations of C (e.g., $\left\{a'x \geq b, \begin{pmatrix} a \\ b \end{pmatrix} \in I\right\}$, where the index set I is an arbitrary dense subset of $K(C)$). There exists a one-to-one correspondence between non-empty closed convex sets in $\mathbb{R}^n$ and closed convex cones in $\mathbb{R}^{n+1}$ containing $\begin{pmatrix} 0_n \\ -1 \end{pmatrix}$ but not containing $\begin{pmatrix} 0_n \\ 1 \end{pmatrix}$ (their corresponding reference cones). The interest of the conical representation derives from the fact that $K(C)$ captures all the relevant information on C. For example, $\dim C = n - \dim L[K(C)]$ (Theorem 5.8 in [5]) and the value of the optimization problem $P(c)$: $\operatorname{Min} c'x$ $s.t.$ $x \in C$, where $c \in \mathbb{R}^n$, is $\sup\left\{\alpha \in \mathbb{R} \mid \begin{pmatrix} c \\ \alpha \end{pmatrix} \in K(C)\right\}$ (Theorem 8.1 (ii) in [5]), so that the properties of $K(C)$ and $P(c)$ are closely related to each other. Moreover, two closed convex sets, C_1 and C_2, can be separated by a hyperplane if, and only if, $K(C_1) \cap [-K(C_2)]$ contains at least one ray.

Uniqueness is a useful feature of both internal and conical representations, so that large families of non-empty closed convex sets can be characterized by means of the properties of their corresponding internal and conical representations (see the table below which comes from Ths. 5.8 and 5.13 in [5]).

C	**Internal representation**	**Conical representation**
affine manifold	$E(C)$ singleton, $D(C) = \{0_n\}$	The pointed cone of $K(C)$ is $\operatorname{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$
polyhedral convex set	$E(C)$ polytope, $D(C)$ polyhedral	$K(C)$ polyhedral
convex body	$\dim L(C) + \dim[E(C) + D(C)] = n$	$K(C)$ pointed

In Section 6 we shall make use of the following illumination concept which was introduced by Valentine [14]: given a convex body C and $z \notin C$, $x \in \operatorname{bd} C$ is visible from z if $]x, z[\cap C = \emptyset$. We denote by $\operatorname{vis}(C; z)$ the set of boundary points of C visible from $z \notin C$. If C is compact, $\operatorname{vis}(C; z) \neq \operatorname{bd} C$ [12].

2. Characterizing the sums of compact convex sets with linear subspaces

Proposition 2.1. *Given a closed convex set $C \neq \emptyset$, the following conditions are equivalent to each other:*

(i) *C is the sum of a compact convex set with a linear subspace;*

(ii) *$E(C)$ is compact and $D(C) = \{0_n\}$;*

(iii) $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K(C)$; *and*

(iv) $L(C)^{\perp} = \Pi(K(C))$, *where* Π *denotes the vertical projection* $\Pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$.

Proof. (i) $\Rightarrow$ (ii) If $C = E + L$, with E compact convex set and L linear subspace, then $L(C) = L$ and $C \cap L^{\perp} = (E + L) \cap L^{\perp}$ is the orthogonal projection of E onto $L^{\perp}$, so that it is the continuous image of a compact set. Hence $E(C) = C \cap L^{\perp}$ is compact.

(ii) $\Rightarrow$ (iii) We assume that $C = E(C) + L(C)$, with $E(C)$ compact, $E(C) \subset L(C)^{\perp}$. If $L(C) = \mathbb{R}^n$, $C = \mathbb{R}^n$ and $K(C) = \operatorname{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$, so that (iii) holds.

Let $\{u^1, \ldots, u^p\}$ be an orthonormal basis of $L(C)^{\perp} \neq \{0_n\}$ and let $\rho > 0$ such that $\|y\| \leq \rho$ for all $y \in E(C)$.

Given $x \in C$, we can write $x = y + z$, $y \in E(C)$ and $z \in L(C)$, so that $|\pm x'u^j| = |\pm y'u^j| \leq \rho$ and $\pm x'u^j \geq -\rho$. Hence $\begin{pmatrix} \pm u^j \\ -\rho \end{pmatrix} \in K(C)$, $j = 1, \ldots, p$ (Farkas Lemma).

Given $\begin{pmatrix} a \\ b \end{pmatrix} \in K(C)$, $a'x \geq b$ for all $x \in C$. If $z \in L(C)$, taking an arbitrary point $\overline{x} \in E(C)$, we have $\overline{x} + \alpha z \in C$ for all $\alpha \in \mathbb{R}$, so that $a'(\overline{x} + \alpha z) \geq b$ for all $\alpha \in \mathbb{R}$, and this entails $a'z = 0$. Hence $a \in L(C)^{\perp}$ and we can write $a = \sum_{j=1}^{p} \alpha_j u^j$ for certain scalars $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, p$.

Since

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} = (2p\rho)^{-1} \sum_{j=1}^{p} \left[\begin{pmatrix} u^j \\ -\rho \end{pmatrix} + \begin{pmatrix} -u^j \\ -\rho \end{pmatrix}\right],$$

we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \sum_{j=1}^{p} \alpha_j \begin{pmatrix} u^j \\ -\rho \end{pmatrix} - \left(b + \rho \sum_{j=1}^{p} \alpha_j\right) \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix} \pm u^j \\ -\rho \end{pmatrix},\ j = 1, \ldots, p\right\},$$

and the last set turns out to be $\operatorname{span} K(C)$. Hence we can apply Lemma 1.1 to conclude that $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K(C)$.

(iii) $\Rightarrow$ (iv) Since $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is linear and $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K(C)$, we have $0_n \in \operatorname{rint} \Pi[K(C)]$ (Th. 6.6 in [10]), with $\Pi[K(C)]$ being a convex cone. Then $\Pi[K(C)]$ is a linear subspace.

If $\Pi[K(C)] = \{0_n\}$, then $K(C) = \operatorname{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$ and $C = \mathbb{R}^n$, so that $L(C) = \mathbb{R}^n$ and $L(C)^{\perp} = \Pi[K(C)]$. We can assume without loss of generality that $\Pi[K(C)] \neq \{0_n\}$.

Let $\{v^1, \ldots, v^q\}$ an orthonormal basis of the linear subspace $\Pi[K(C)]$. Given $k \in \{1, \ldots, q\}$, there exist scalars α_k and β_k such that $\begin{pmatrix} v^k \\ \alpha_k \end{pmatrix} \in K(C)$ and $\begin{pmatrix} -v^k \\ -\beta_k \end{pmatrix} \in K(C)$, so that $\alpha_k \leq x'v^k \leq \beta_k$ for all $x \in C$.

Now we shall prove that $L(C) = [\Pi[K(C)]]^{\perp}$.

In fact, if $z \in L(C)$, taking an arbitrary $\overline{x} \in C$, we get, for $k \in \{1, \ldots, q\}$, $\alpha_k \leq (\overline{x} + \alpha z)' v^k \leq \beta_k$ for all $\alpha \in \mathbb{R}$, and this entails $z'v^k = 0$. Hence, $L(C) \subset \{v^1, \ldots, v^q\}^\perp = [\Pi[K(C)]]^\perp$.

Conversely, if $z \in [\Pi[K(C)]]^\perp$, then $a'z = 0$ for all $a \in \mathbb{R}^n$ such that $\begin{pmatrix} a \\ b \end{pmatrix} \in K(C)$ for a certain $b \in \mathbb{R}$. Since $\left\{ a'x \geq b \mid \begin{pmatrix} a \\ b \end{pmatrix} \in K(C) \right\}$ is a linear representation of C, $\pm z$ is a solution of the corresponding homogeneous system, so that $\pm z \in O^+C$ and $z \in L(C)$. Hence, $[\Pi[K(C)]]^\perp = L(C)$, so that (iv) holds.

(iv) $\Rightarrow$ (i) If $\Pi[K(C)] = \{0_n\}$, $C = \mathbb{R}^n$. Assume $\dim \Pi[K(C)] = q$, $1 \leq q \leq n$. Let $\{v^1, \ldots, v^q\}$ be an orthonormal basis of $\Pi[K(C)]$ and let $\alpha_1, \ldots, \alpha_q$; $\beta_1, \ldots, \beta_q$ be scalars such that $\alpha_k \leq x'v^k \leq \beta_k$, for all $x \in C$, $k = 1, \ldots, q$. If $q = n$, C is compact. Otherwise $L(C)^\perp = \Pi[K(C)] \neq \mathbb{R}^n$ and we can select vectors $v^k \in L(C)$, $k = q+1, \ldots, n$, such that $\{v^1, \ldots, v^n\}$ is an orthonormal basis of $\mathbb{R}^n$. Given $x \in C \cap L(C)^\perp$, we have $\alpha_k \leq x'v^k \leq \beta_k$, $k = 1, \ldots, q$ (since $x \in C$) and $x'v^k = 0$, $k = q+1, \ldots, n$ (since $x \in L(C)^\perp$). Hence $C \cap L(C)^\perp$ is compact and $C = \left[C \cap L(C)^\perp\right] + L(C)$ is the aimed decomposition. $\square$

Proposition 2.1 provides the next algebraic characterization of the sums of compact convex sets with linear subspaces from which we shall obtain nice properties of this class of feasible sets in LSIP.

Corollary 2.1. *Let $C = \{x \in \mathbb{R}^n \mid a_t'x \geq b_t,\ t \in T\} \neq \emptyset$ and let $M = \operatorname{cone}\{a_t,\ t \in T\}$. Then C is the sum of a compact convex set with a linear subspace if and only if M is a linear subspace.*

Proof. Assume that C is the sum of a compact convex set with a linear subspace. According to Proposition 2.1 we can write $C = E(C) + L(C)$, with $E(C)$ compact and $\Pi[K(C)] = L(C)^\perp$.

Given $t \in T$, we have $a_t = \Pi(a_t', b_t) \in \Pi[K(C)]$, so that $M \subset \Pi[K(C)]$. On the other hand, if $z \in \Pi[K(C)]$, there exists a sequence

$$\{z^r\} \subset \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$$

such that $z_i = \lim_{r\to\infty} z_i^r$, $i = 1, \ldots, n$. For each z^r there exist a function $\lambda^r : T \longrightarrow \mathbb{R}_+$ such that $\lambda_t^r = 0$ for all $t \in T$ except for a finite number of indices and a non-negative real number μ^r such that

$$z^r = \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \mu^r \begin{pmatrix} 0_n \\ -1 \end{pmatrix}.$$

Since $\sum_{t\in T} \lambda_t^r a_t \in M$, $r = 1, 2, \ldots$, we have

$$z = \lim_{r\to\infty} (z_1^r, \ldots, z_n^r)' = \lim_{r\to\infty} \sum_{t\in T} \lambda_t^r a_t \in \operatorname{cl} M.$$

We have shown that $M \subset L(C)^{\perp} \subset \operatorname{cl} M$, so that $\operatorname{cl} M = L(C)^{\perp}$. Hence,

$$L(C)^{\perp} = \operatorname{rint} L(C)^{\perp} = \operatorname{rint} \operatorname{cl} M = \operatorname{rint} M \subset M \subset L(C)^{\perp},$$

which proves that $M = L(C)^{\perp}$ is a linear subspace.

Conversely, if $M = \mathbb{R}^n$, then C is compact (by Th. 9.3 in [5]). So, we assume that M is a linear subspace and $\operatorname{cone}\left\{a_t,\ t \in T;\ \pm v^k,\ k = 1, \ldots, q\right\} = \mathbb{R}^n$, where $\{v^1, \ldots, v^q\}$ is a basis of $M^{\perp}$. Since $L(C) = \{y \in \mathbb{R}^n \mid a_t' y = 0,\ t \in T\} = \{a_t,\ t \in T\}^{\perp}$, $L(C)^{\perp} = \{a_t,\ t \in T\}^{\perp\perp} = \operatorname{span}\{a_t,\ t \in T\} = M$ (recall that M is a linear subspace). Then the assumption entails that

$$C \cap L(C)^{\perp} = \left\{x \in \mathbb{R}^n \mid a_t' x \geq b_t,\ t \in T;\ x' v^k = 0,\ k = 1, \ldots, q\right\}$$

is compact, so that $C = \left[C \cap L(C)^{\perp}\right] + L(C)$ is the sum of a compact convex set with a linear subspace. □

Now assume that the LSIP problem $P(c)$ has a finite value, $v(c)$, and its feasible set C is the sum of a compact convex set with a linear subspace. If $c \notin L(C)^{\perp}$, then there exists a vector $y \in L(C)$ such that either $c'y < 0$ or $c'y > 0$, so that either y or $-y$ is a recession direction of C forming an acute angle with c, with $v(c) = -\infty$ (contradiction). Hence $c \in L(C)^{\perp} = M = \operatorname{rint} M$ and this entails the solvability of $P(c)$, its discretizability (an optimal solution can be obtained as the limit of optimal solutions of a sequence of finite subproblems) and the zero duality gap for any external representation of C (by Theorems 8.1, part (v), and 8.2 in [5]).

Corollary 2.2. *Given a closed convex set $C \neq \emptyset$, the following statements are equivalent to each other:*

(i) *C is a compact set;*

(ii) *$E(C)$ is compact and $L(C) = D(C) = \{0_n\}$;*

(iii) *$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{int} K(C)$;*

(iv) *$\Pi(K(C)) = \mathbb{R}^n$; and*

(v) *$P(c)$ is solvable for all $c \in \mathbb{R}^n$.*

Proof. The equivalence between statements from (i) to (iv) is straightforward consequence of Proposition 2.1 and the arguments therein. Since (i) $\Rightarrow$ (v) is trivial, we have just to prove that (v) $\Rightarrow$ (i). In fact, if (i) fails there exists $y \in O^+C$, $y \neq 0_n$, then $P(-y)$ is not even bounded. Then (v) fails too. □

Hence the compact convex bodies are those closed convex sets for which the reference cone is pointed and a neighbourdhood of $\begin{pmatrix} 0_n \\ -1 \end{pmatrix}$. Significant geometric properties of this family can be proved by means of LSIP theory ([6], [7] and [8]).

3. Characterization of simplices

C is a k-simplex if it is the convex hull of $k+1$ points affinely independent. Obviously, any k-simplex is compact.

Proposition 3.1. *Given a closed convex set $C \neq \emptyset$, the following statements are equivalent to each other:*

(i) *C is a k-simplex;*

(ii) *$E(C)$ has $k+1$ extreme points, $\dim E(C) = k$ and $L(C) = D(C) = \{0_n\}$; and*

(iii) *$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{int} K(C)$, $\dim L[K(C)] = n-k$ and the pointed cone of $K(C)$ has $k+1$ extreme rays.*

Proof. (i) $\Leftrightarrow$ (ii) is trivial.

(i) $\Rightarrow$ (iii) Since every k-simplex is compact, we have $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{int} K(C)$. On the other hand, according to the dimensional formula,

$$\dim L[K(C)] = n - \dim C = n - k. \tag{3.1}$$

Now, since C is a full-dimensional simplex in the affine manifold $V := \operatorname{aff} C$, with $\dim V = k$, there exist non-zero vectors $\{a_i,\ i = 1,\dots,k+1\} \subset V - V$ and corresponding scalars $\{b_i,\ i = 1,\dots,k+1\}$, such that $\{x \in \mathbb{R}^n \mid a_i'x = b_i\}$ is a supporting hyperplane at the relative interior points of the i-th facet, $i = 1,\dots,k+1$. We can assume without loss of generality $a_i'x \geq b_i$ for all $x \in C$, $i = 1,\dots,k+1$, so that $C = \{x \in V \mid a_i'x \geq b_i,\ i = 1,\dots,k+1\}$. Moreover, since $\dim V = k$, we can write $V = \{x \in \mathbb{R}^n \mid a_i'x = b_i,\ i = k+2,\dots,n+1\}$, with $\left\{\begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i = k+2,\dots,n+1\right\}$ linearly independent (if C is an n-simplex, then $V = \mathbb{R}^n$ and this part of the proof can be simplified).

Then,

$$K(C) = \operatorname{cone}\left\{\begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i = 1,\dots,k+1;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\} + \operatorname{span}\left\{\begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i = k+2,\dots,n+1\right\},$$

and we shall show that $\begin{pmatrix} 0_n \\ -1 \end{pmatrix}$ can be eliminated in this expression. To do this, we shall appeal to a well-known characterization of the interior points of a convex cone (Lemma 1.1).

Since $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{int} K(C)$, we can write

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} = \sum_{i\in I} \lambda_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ -1 \end{pmatrix},\ \lambda_i > 0 \text{ if } i \leq k+1,\ \mu \geq 0, \tag{3.2}$$

for a certain set $I \subset \{1,\dots,n+1\}$, with

$$\operatorname{span}\left\{\begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i \in I\right\} = \mathbb{R}^{n+1} \text{ if } \mu = 0$$

and
$$\operatorname{span}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i\in I;\ \begin{pmatrix}0_n\\-1\end{pmatrix}\right\}=\mathbb{R}^{n+1} \text{ if } \mu>0.$$
If $0\le\mu<1$, from (3.2) we get
$$\begin{pmatrix}0_n\\-1\end{pmatrix}\in\operatorname{cone}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=1,\dots,k+1\right\}+\operatorname{span}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=k+2,\dots,n+1\right\}. \tag{3.3}$$
We shall prove that this always happens when $\mu\ge 0$ by means of the following discussion.

If $\mu>1$, then $(\mu-1)\begin{pmatrix}0_n\\1\end{pmatrix}=\sum_{i\in I}\lambda_i\begin{pmatrix}a_i\\b_i\end{pmatrix}\in K(C)$, and this implies $C=\emptyset$. Alternatively, if $\mu=1$, then we get $\sum_{i\in I}\lambda_i\begin{pmatrix}a_i\\b_i\end{pmatrix}=0_{n+1}$ and there will exist a $j\le k+1$ such that $\lambda_j>0$ (otherwise $I\subset\{k+2,\dots,n+1\}$ and $\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=k+2,\dots,n+1\right\}$ is linearly dependent). Then,
$$-\begin{pmatrix}a_j\\b_j\end{pmatrix}=\sum_{i\in I\setminus\{j\}}\lambda_j^{-1}\lambda_i\begin{pmatrix}a_i\\b_i\end{pmatrix}\in K(C),$$
so that $a_j'x=b_j$ for all $x\in C$. Hence $a_j\in(V-V)\cap(V-V)^{\perp}=\{0_n\}$, i.e., $a_j=0_n$. This is a contradiction.

From (3.3) we get
$$K(C)=\operatorname{cone}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=1,\dots,k+1\right\}+\operatorname{span}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=k+2,\dots,n+1\right\}.$$
Comparing $\dim\operatorname{span}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=k+2,\dots,n+1\right\}=n-k$ with (3.1) we conclude that the pointed cone of $K(C)$ is
$$\widehat{K}(C):=\operatorname{cone}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=1,\dots,k+1\right\}.$$
If $\operatorname{cone}\left\{\begin{pmatrix}a_j\\b_j\end{pmatrix}\right\}$, $j\in\{1,\dots,k+1\}$, is not an extreme ray of $\widehat{K}(C)$, then we can write
$$\begin{pmatrix}a_j\\b_j\end{pmatrix}=\sum_{\substack{i=1\\i\neq j}}^{k+1}\gamma_i\begin{pmatrix}a_i\\b_i\end{pmatrix},\ \gamma_i\ge 0,\ i=1,\dots,k+1,\ i\neq j,$$
so that $\widehat{K}(C)=\operatorname{cone}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix},\ i=1,\dots,k+1,\ i\neq j\right\}$ and $\dim\widehat{K}(C)\le k$. Then $\dim K(C)=\dim\widehat{K}(C)+\dim L[K(C)]\le n$ and so $\operatorname{int}K(C)=\emptyset$. Hence $\left\{\operatorname{cone}\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix}\right\},\ i=1,\dots,k+1\right\}$ is the set of extreme rays of $\widehat{K}(C)$.

(iii) $\Rightarrow$ (i) The assumptions $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{int} K(C)$ and $\dim L[K(C)] = n - k$ guarantee that C is compact and $\dim C = k$, respectively. Let $\left\{ \operatorname{cone}\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right\},\ i = 1, \ldots, k+1 \right\}$ be the set of extreme rays of $\widehat{K}(C)$. According to the representation theorem, $\widehat{K}(C) = \operatorname{cone}\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i = 1, \ldots, k+1 \right\}$.

Let $\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i = k+2, \ldots, n+1 \right\}$ be a basis of $L[K(C)]$. Then $C = \{x \in V \mid a_i' x \geq b_i\ ,\ i = 1, \ldots, k+1\}$, where $V = \{x \in \mathbb{R}^n \mid a_i' x = b_i,\ i = k+2, \ldots, n+1\}$. So, the number of extreme points of C is $p \leq \binom{k+1}{k} = k+1$. Assume that $p < k+1$ and let $\{x^1, \ldots, x^p\}$ be the set of extreme points of C. The representation theorem yields $C = \operatorname{conv}\{x^1, \ldots, x^p\}$, so that $\dim C \leq p - 1 < k$. Hence $p = k+1$ and $\{x^1, \ldots, x^{k+1}\}$ is affinely independent (otherwise, $\dim C < k$). This completes the proof. $\square$

4. Characterization of sandwiches

Two affine manifolds in $\mathbb{R}^n$ (also called flats) of the same dimension, U_1 and U_2, are parallel if $U_1 - U_1 = U_2 - U_2$ and $U_1 \cap U_2 = \emptyset$. We say that a set is a k-sandwich when it is the convex hull of the union of two parallel affine manifolds of dimension $k-1$. The next result establishes some elementary properties of the k-sandwiches that will be used later.

Proposition 4.1. *Let* $C = \operatorname{conv}(U_1 \cup U_2)$, *where* U_1 *and* U_2 *are parallel affine manifolds, with* $\dim U_i = k-1$, $i = 1, 2$. *Let* $V := U_1 - U_1 = U_2 - U_2$, $U_i \cap V^\perp = \{x^i\}$, $i = 1, 2$, *and* $w = x^2 - x^1$. *Then the following statements hold:*

(i) $C = V + [x^1, x^2]$ *(and so C is the sum of a compact convex set with a linear subspace);*

(ii) $\dim C = k$;

(iii) $\operatorname{aff} C = U_i + \operatorname{span}\{w\}$, $i = 1, 2$;

(iv) $U_i = \{x \in \operatorname{aff} C \mid w'(x - x^i) = 0\}$, $i = 1, 2$; *and*

(v) $\operatorname{rint} C = V +]x^1, x^2[$ *and* $\operatorname{rbd} C = U_1 \cup U_2$.

Proof. The assumptions on U_1 and U_2 guarantee that x^1 and x^2 are well defined and $w = x^2 - x^1 \neq 0_n$. Obviously, $U_i = x^i + V$, $i = 1, 2$.

(i) It is trivial.

(ii) Obviously, for $i = 1, 2$, we have

$$U_i + \operatorname{span}\{w\} = x^i + V + \operatorname{span}\{w\} = V + \left(x^i + \operatorname{span}\{w\}\right) = V + \operatorname{aff}\left(\left[x^1, x^2\right]\right),$$

the last set being an affine manifold containing U_1 and U_2. Hence, if $\{i, j\} = \{1, 2\}$,

$$\operatorname{conv}\left[U_i \cup \left\{x^j\right\}\right] \subset C \subset U_i + \operatorname{span}\{w\}. \tag{4.1}$$

Since $x^j \notin U_i$ and $w \in V^\perp$, we get from (4.1)

$$k \leq \dim \operatorname{conv}\left[U_i \cup \left\{x^j\right\}\right] \leq \dim C \leq \dim\left[U_i + \operatorname{span}\{w\}\right] =$$

$$= \dim [V + \operatorname{span}\{w\}] = k.$$

Hence (ii) holds.

(iii) It follows from the second inclusion in (4.1) and the equation $\dim C = \dim [U_i + \operatorname{span}\{w\}]$ which has proved above.

(iv) Given $x \in U_i = x^i + V$, $w'x = w'x^i$ because $w \in V^{\perp}$. Conversely, if $x \in \operatorname{aff} C$ satisfies $w'(x - x^i) = 0$, then we can write (recall (iii)) $x = x^i + v + \alpha w$, $v \in V$ and $\alpha \in \mathbb{R}$, with $w'(v + \alpha w) = \alpha \|w\|^2 = 0$. This entails $\alpha = 0$, i.e., $x = x^i + v \in U_i$.

(v) It is a straightforward consequence of Cor. 6.6.2 in [10] applied to (i). □

Next we give three different characterizations of the sandwiches. Another topological characterization will be given in Section 6.

Proposition 4.2. *Let C be a non-empty closed convex set and let $K(C)$ be its reference cone. The following statements are equivalent to each other:*

(i) *C is a k-sandwich.*

(ii) *$D(C) = \{0_n\}$, $E(C)$ is a proper closed segment and* $\dim L(C) = k - 1$.

(iii) *There exists a linear subspace $V \subset (\operatorname{aff} C) - C$ with $\dim V = k - 1$, a non-zero vector $w \in V^{\perp} \backslash [(\operatorname{aff} C) - C]^{\perp}$ and two real numbers α_1 and α_2, such that $\alpha_1 < \alpha_2$ and*

$$C = \{x \in \operatorname{aff} C \mid \alpha_1 \leq w'x \leq \alpha_2\}.$$

(iv) *$K(C) = K + W$, where K is a pointed closed convex cone and W is a linear subspace such that* $\dim K = 2$, $\dim W = n - k$, $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K$ *and*

$$K \cap \left(W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right) = \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}. \tag{4.2}$$

Proof. We shall prove that (ii) ⇔ (i) ⇒ (iii) ⇒ (iv) ⇒ (i).

(ii) ⇒ (i) If $E(C) + D(C) = [x^1, x^2]$, with $x^1 \neq x^2$, defining $U_i = x^i + L(C)$, it is easy to prove that $C = \operatorname{conv}(U_1 \cup U_2)$, U_1 and U_2 being parallel manifolds, such that $\dim U_i = k - 1$, $i = 1, 2$.

(i) ⇒ (ii) It is straightforward consequence of statement (i) in Proposition 4.1.

(i) ⇒ (iii) Let $C = \operatorname{conv}(U_1 \cup U_2)$, where U_1 and U_2 are parallel affine manifolds. Let V, x^1, x^2 and w be defined as in Proposition 4.1, whose statements (ii) and (iii) show that $\dim V = \dim C - 1$ and $V \subset V + \operatorname{span}\{w\} = (\operatorname{aff} C) - C$, respectively. Recalling the definition of w, we have $w \in V^{\perp} \backslash [(\operatorname{aff} C) - C]^{\perp}$.

Let $\alpha_i = w'x^i$, $i = 1, 2$. Obviously, $\alpha_2 - \alpha_1 = \|w\|^2 > 0$.

Since $U_i = \{x \in \operatorname{aff} C \mid w'x = \alpha_i\}$, $i = 1, 2$, according to statement (iv) in Proposition 4.1, we obtain

$$C = \operatorname{conv}(U_1 \cup U_2) = \{x \in \operatorname{aff} C \mid \alpha_1 \leq w'x \leq \alpha_2\}.$$

(iii) ⇒ (iv) Let $d = \dim C$. We shall distinguish the cases $d = n$ and $d < n$.

Assume $d = n$. Since $C = \{x \in \mathbb{R}^n \mid \alpha_1 \leq w'x \leq \alpha_2\}$, $\alpha_1 < \alpha_2$, we have

$$K(C) = \operatorname{cone}\left\{ \begin{pmatrix} w \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

Moreover,

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} = \frac{1}{\alpha_2 - \alpha_1} \left[\begin{pmatrix} w \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix} \right],$$

so that $K(C) = \operatorname{cone}\left\{ \begin{pmatrix} w \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix} \right\}$ and Lemma 1.1 yields $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K(C)$.

Even more, since $\alpha_1 \neq \alpha_2$, $\begin{pmatrix} w \\ \alpha_1 \end{pmatrix}$ and $\begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix}$ are linearly independent and $K(C)$ is a two-dimensional pointed cone.

We shall finish this part of the proof showing that $K(C) = K(C) + \{0_{n+1}\}$ is the aimed decomposition. In fact, if $z \in K(C) \cap \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$, it is possible to write

$$z = \rho_1 \begin{pmatrix} w \\ \alpha_1 \end{pmatrix} + \rho_2 \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix} = \gamma \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \ \rho_1 \geq 0,\ \rho_2 \geq 0,\ \gamma \in \mathbb{R}.$$

This entails $\rho_1 = \rho_2$ and $\gamma = \rho_1 (\alpha_2 - \alpha_1) \geq 0$, so that $z \in \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$. This proves that $K(C) \cap \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \subset \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$, whereas the reverse inclusion holds trivially. Hence (4.2) holds.

Now assume $d < n$. Let $\operatorname{aff} C = \{x \in \mathbb{R}^n \mid a_i'x = b_i,\ i = 1, \ldots, n-d\}$, with $\{a_i,\ i = 1, \ldots, n-d\}$ a linearly independent subset of $\mathbb{R}^n$ and $b_i \in \mathbb{R}$, $i = 1, \ldots, n-d$.

Since $C = \{x \in \operatorname{aff} C \mid \alpha_1 \leq w'x \leq \alpha_2\}$, we have now

$$K(C) = \operatorname{cone}\left\{ \pm \begin{pmatrix} a_i \\ b_i \end{pmatrix}, i = 1, \ldots, n-d; \begin{pmatrix} w \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} =$$

$$= \operatorname{span}\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix}, i = 1, \ldots, n-d \right\} + \operatorname{cone}\left\{ \begin{pmatrix} w \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

Let $W := \operatorname{span}\left\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix},\ i = 1, \ldots, n-d \right\}$ and $K := \operatorname{cone}\left\{ \begin{pmatrix} w \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$. W is a linear subspace of $\mathbb{R}^{n+1}$, with $\dim W = n - d$, and K (the same cone as in the case $d = n$) is a pointed closed convex cone, with $\dim K = 2$ and $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K$. Moreover, it is obvious that

$$\operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \subset K \cap \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \subset K \cap \left(W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right). \tag{4.3}$$

Now consider an arbitrary $z \in K \cap \left(W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right)$. We can write

$$z = \rho_1 \begin{pmatrix} w \\ \alpha_1 \end{pmatrix} + \rho_2 \begin{pmatrix} -w \\ -\alpha_2 \end{pmatrix} = \sum_{i=1}^{n-d} \beta_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \gamma \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \tag{4.4}$$

with $\rho_1 \geq 0$, $\rho_2 \geq 0$, $\beta_i \in \mathbb{R}$, $i = 1, \ldots, n-d$, and $\gamma \in \mathbb{R}$. From (4.4),

$$(\rho_2 - \rho_1)\, w + \sum_{i=1}^{n-d} \beta_i a_i = 0_n,$$

with $\{w;\, a_i, i = 1, \ldots, n-d\}$ being a linearly independent set because $w \neq 0_n$ and

$$w \notin [(\operatorname{aff} C) - C]^{\perp} = \operatorname{span}\{a_i, i = 1, \ldots, n-d\}.$$

Hence $\rho_1 = \rho_2$ and $\beta_i = 0$, $i = 1, \ldots, n-d$, so that (4.4) reads

$$z = \rho_1 \begin{pmatrix} 0_n \\ \alpha_1 - \alpha_2 \end{pmatrix} = \gamma \begin{pmatrix} 0_n \\ -1 \end{pmatrix}$$

and we get $\gamma = \rho_1 (\alpha_2 - \alpha_1) \geq 0$. Thus $z = \gamma \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$.

We have proved that

$$K \cap \left(W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right) \subset \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\},$$

which together with (4.3) shows that (4.2) holds.

(iv) $\Rightarrow$ (i) Any two-dimensional pointed closed convex cone is the conical convex hull of two extreme directions (i.e., a plane acute angle). Let $K = \operatorname{cone}\left\{ \begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \right\}$, where $\left\{ \begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \right\}$ is a linearly independent set in $\mathbb{R}^{n+1}$. Since we are assuming that $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \operatorname{rint} K$, we can write (by Lemma 1.1),

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} = \rho_1 \begin{pmatrix} a \\ \alpha \end{pmatrix} + \rho_2 \begin{pmatrix} b \\ \beta \end{pmatrix}, \ \rho_1 > 0, \ \rho_2 > 0, \tag{4.5}$$

so that $a \neq 0_n$ (otherwise $a = b = 0_n$ and $\left\{ \begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \right\}$ is linearly dependent).

Defining $w = \rho_1 a \neq 0_n$ and $\gamma = \rho_1 \alpha$, we get from (4.5)

$$K = \operatorname{cone}\left\{ \begin{pmatrix} w \\ \gamma \end{pmatrix}, \begin{pmatrix} -w \\ -\gamma - 1 \end{pmatrix} \right\}.$$

Let $\left\{ \begin{pmatrix} a_i \\ \alpha_i \end{pmatrix}, i = 1, \ldots, n-d \right\}$ a basis of W, with $d = \dim C$. Since we are assuming that $K(C) = K + W$, we have

$$K(C) = \operatorname{cone}\left\{ \pm \begin{pmatrix} a_i \\ \alpha_i \end{pmatrix}, i = 1, \ldots, n-d; \begin{pmatrix} w \\ \gamma \end{pmatrix}, \begin{pmatrix} -w \\ -\gamma - 1 \end{pmatrix} \right\},$$

so that

$$C = \{ x \in \mathbb{R}^n \mid a_i' x = \alpha_i, \ i = 1, \ldots, n-d; \ \gamma \leq w' x \leq \gamma + 1 \}. \tag{4.6}$$

Let $\gamma_1 = \gamma$, $\gamma_2 = \gamma + 1$ and $U_j = \{x \in \mathbb{R}^n \mid a_i'x = \alpha_i,\ i = 1, \ldots, n-d;\ w'x = \gamma_j\}$, $j = 1, 2$. We shall prove that $U_j \neq \emptyset$, $j = 1, 2$.

If $U_j = \emptyset$,

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K(U_j) = \operatorname{span}\left\{ \begin{pmatrix} a_i \\ \alpha_i \end{pmatrix},\ i = 1, \ldots, n-d;\ \begin{pmatrix} w \\ \gamma_j \end{pmatrix} \right\}$$

and we can write

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \sum_{i=1}^{n-d} \beta_i \begin{pmatrix} a_i \\ \alpha_i \end{pmatrix} + \beta_0 \begin{pmatrix} w \\ \gamma_j \end{pmatrix},\ \beta_i \in \mathbb{R},\ i = 0, \ldots, n-d. \tag{4.7}$$

From (4.7), $\beta_0 \begin{pmatrix} w \\ \gamma_j \end{pmatrix} \in W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$ and we shall discuss the sign of β_0.

If $\beta_0 > 0$, then $\begin{pmatrix} w \\ \gamma_j \end{pmatrix} \in W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$. If $j = 1$,

$$\begin{pmatrix} w \\ \gamma_1 \end{pmatrix} \in K \cap \left[W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right] = \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\},$$

according to (4.2), contradicting $w \neq 0_n$. Alternatively, if $j = 2$, then

$$-\begin{pmatrix} w \\ \gamma_2 \end{pmatrix} \in K \cap \left[W + \operatorname{span}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right] = \operatorname{cone}\left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\},$$

and we get again $w = 0_n$.

If $\beta_0 < 0$ we obtain $w = 0_n$ in the same way.

Finally, if $\beta_0 = 0$, (4.7) entails that $\{a_i'x = \alpha_i,\ i = 1, \ldots, n-d\}$ is inconsistent, in contradiction with (4.6) (because $C \neq \emptyset$).

We conclude that U_1 and U_2 are parallel affine manifolds and it can be easily shown that $C = \operatorname{conv}(U_1 \cup U_2)$.

This completes the proof. □

5. A topological characterization of *k*-sandwiches

From statement (v) in Proposition 4.1, it is clear that the relative boundary of any k-sandwich is not even connected. In order to prove the converse statement we shall use the following lemma.

Lemma 5.1. *Let C be a closed convex set and let $u \in \mathbb{R}^n \backslash O^+C$. If $x^i \in C$ and $u \notin D(C; x^i)$, $i = 1, 2$, then x^1 and x^2 can be connected through a certain arc contained in* $\operatorname{bd} C$.

Proof. The statement is trivially true when $n = 1$ because the assumption on x^1 and x^2 entails $x^1 = x^2$, so we assume $n \geq 2$.

Since $u \notin O^+C$ and C is closed, for every $x \in C$ there exists a unique non-negative real number $\varphi(x)$ such that

$$\varphi(x) = \max\{t \in \mathbb{R} \mid x + tu \in C\}.$$

$u \notin D(C; x^i)$ implies $\varphi(x^i) = 0$, and this for $i = 1, 2$. On the other hand, if $x \in C$, $x + \varphi(x) u \in C$ whereas $x + \gamma u \notin C$ for all $\gamma > \varphi(x)$, so that $x + \varphi(x) u \in \operatorname{bd} C$ for all $x \in C$. We shall prove the continuity of $\varphi\big|_{[x^1,x^2]}$, so that $\{x + \varphi(x) u \mid x \in [x^1, x^2]\}$ will be the aimed arc connecting x^1 with x^2 and contained in $\operatorname{bd} C$.

In fact, given two points of C, z^1 and z^2, and a scalar $\lambda \in [0,1]$, since $z^i + \varphi(z^i) u \in C$, $i = 1, 2$, we have

$$\begin{aligned}&(1-\lambda) z^1 + \lambda z^2 + \left[(1-\lambda)\varphi\left(z^1\right) + \lambda\varphi\left(z^2\right)\right] u = \\ &= (1-\lambda)\left[z^1 + \varphi\left(z^1\right) u\right] + \lambda\left[z^2 + \varphi\left(z^2\right) u\right] \in C.\end{aligned}$$

Hence $(1-\lambda)\varphi(z^1) + \lambda\varphi(z^2) \leq \varphi[(1-\lambda) z^1 + \lambda z^2]$ and this means that φ is concave on C. In particular $\varphi\big|_{[x^1,x^2]}$ will be concave and so $\varphi\big|_{[x^1,x^2]}$ will be continuous on $]x^1, x^2[$.

It remains to prove the continuity of φ at the extreme points of $[x^1, x^2]$. We assume the contrary.

If $\varphi\big|_{[x^1,x^2]}$ is not continuous at x^i, $i \in \{1,2\}$, since $\varphi(x^i) = 0$ and $\varphi(x) \geq 0$ for all $x \in [x^1, x^2]$, there exists $\varepsilon > 0$ and a sequence $\left\{z^k\right\}_{k=1}^{\infty} \subset\,]x^1, x^2[$ such that $\lim\limits_{k\to\infty} z^k = x^i$ and $\varphi\left(z^k\right) \geq \varepsilon$ for all $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we have

$$z^k + \varepsilon u \in \left[z^k, z^k + \varphi\left(z^k\right) u\right] \subset C,$$

so that $x^i + \varepsilon u = \lim\limits_{k\to\infty}\left(z^k + \varepsilon u\right) \in C$ and $u \in D(C; x^i)$, contradicting the assumption. This completes the proof. $\square$

Proposition 5.2. *Let C be a non-empty closed convex set in $\mathbb{R}^n$ that it is not an affine manifold, with* $\dim C = k$. *Then the following statements are equivalent to each other:*

(i) $\operatorname{rbd} C$ *is not connected;*

(ii) $\operatorname{rbd} C$ *is not connected by arcs; and*

8iii) C *is a k-sandwich.*

Proof. (i) $\Rightarrow$ (ii) It is trivial.

(ii) $\Rightarrow$ (iii) First we assume that C is full-dimensional. Thus the hypothesis will be that $\operatorname{bd} C$ is not connected by arcs.

If $n = 1$, since $\dim C = 1$ and $C \neq \mathbb{R}$, C will be either a closed half-line (and this is impossible because any singleton set is connected by arcs) or a closed proper segment in $\mathbb{R}$. Hence C is a sandwich.

So, we can assume without loss of generality that $n \geq 2$.

Let $x^i \in \operatorname{bd} C$, $i = 1, 2$, points that cannot be connected by means of any curve entirely contained in $\operatorname{bd} C$.

Let $c_i \neq 0_n$, such that $c_i'(x - x^i) \geq 0$ for all $x \in C$, $i = 1, 2$. We denote by $H_i := \{x \in \mathbb{R}^n \mid c_i'(x - x^i) = 0\}$, $i = 1, 2$, the corresponding supporting hyperplanes to C. We shall prove that $H_1 \cap H_2 = \emptyset$ by assuming the contrary. So, let $z \in H_1 \cap H_2$.

For $\{i, j\} = \{1, 2\}$, since $z \in H_i$ and $x^j \in C$, we have

$$c_i'\left(z - x^j\right) = c_i'\left(z - x^i + x^i - x^j\right) = c_i'\left(x^i - x^j\right) \leq 0.$$

If $c_i'(z-x^j)=0$ then $x^j\in H_i$ (because $z\in H_i$), so that $[x^i,x^j]\subset C\cap H_i\subset \operatorname{bd} C$ (because H_i is supporting hyperplane to C), and this contradicts the assumption on x^1 and x^2. Hence,

$$c_i'\left(z-x^j\right)<0 \text{ if } \{i,j\}=\{1,2\}. \tag{5.1}$$

Now consider the vector $u=2z-(x^1+x^2)$. From (5.1), we get

$$c_i'u=c_i'\left(z-x^i\right)+c_i'\left(z-x^j\right)<0, \text{ if } \{i,j\}=\{1,2\}. \tag{5.2}$$

(5.2) is incompatible with $u\in O^+C$ because $c_i'(x-x^i)\geq 0$ for all $x\in C$, $i=1,2$. Even more, also from (5.2), if $t>0$, $c_i'[(x^i+tu)-x^i]=t(c_i'u)<0$, so that $x^i+tu\notin C$. This means that $u\notin D(C;x^i)$, $i=1,2$. Then we can apply Lemma 4.1 in order to obtain the aimed contradiction. Hence we have proved that $H_1\cap H_2=\emptyset$, so that $\operatorname{span}\{c_1\}=\operatorname{span}\{c_2\}$. Next we shall prove that $C=\operatorname{conv}(H_1\cup H_2)$.

Since $x^1\notin H_2$ and $x^2\notin H_1$ (otherwise $H_1=H_2$), $c_2'(x^1-x^2)>0$ and $c_1'(x^1-x^2)<0$, so that c_2 is a negative multiple of c_1 and we can write $H_i=\{x\in\mathbb{R}^n \mid c'x=\alpha_i\}$, $i=1,2$, where $c\neq 0_n$, $c'x^i=\alpha_i$, $i=1,2$, and $\alpha_1<\alpha_2$. Since H_i is a supporting hyperplane to C at x^i, $i=1,2$, we have

$$C\subset\{x\in\mathbb{R}^n \mid \alpha_1\leq c'x\leq\alpha_2\}=\operatorname{conv}(H_1\cup H_2). \tag{5.3}$$

In order to prove the reverse inclusion, let us consider an arbitrary vector $v\in V:=H_1-H_1=H_2-H_2$.

If $v\notin O^+C$, there exist non-negative real numbers

$$\lambda_i:=\max\left\{t\in\mathbb{R}\mid x^i+tv\in C\right\},\ i=1,2.$$

Then $[x^i,x^i+\lambda_i v]\subset C\cap H_i\subset\operatorname{bd} C$, $i=1,2$. On the other hand, there exists an arc connecting $x^1+\lambda_1 v$ with $x^2+\lambda_2 v$ which is completely contained in $\operatorname{bd} C$ (because $v\notin D(C;x^i+\lambda_i v)$, $i=1,2$, and so Lemma 5.1 applies again), and this means that x^1 can be connected with x^2 by means of an arc contained in $\operatorname{bd} C$, composed by three linked arcs. This is a contradiction, so that $V\subset O^+C$.

Hence $H_i=x^i+V\subset C+O^+C=C$, $i=1,2$, and we obtain

$$\operatorname{conv}(H_1\cup H_2)\subset C. \tag{5.4}$$

From (5.3) and (5.4), we conclude that $C=\operatorname{conv}(H_1\cup H_2)$ is a full-dimensional sandwich.

Now we assume that $k<n$. Then C is a full-dimensional closed convex set, in $\operatorname{aff} C$, such that its boundary, in the topology of $\operatorname{aff} C$, is not connected by arcs. Applying the previous argument, C turns out to be a full-dimensional sandwich in $\operatorname{aff} C$, i.e., a k-sandwich.

(iii) $\Rightarrow$ (i) It is straightforward consequence of statement (v) in Proposition 4.1. $\square$

6. Characterization of n-simplices, n-sandwiches and parallelotopes

In a recent paper of Martini and Soltan [9], it has been proved that, given a compact convex body C, C is an n-simplex if, and only if, for all $z^1\notin C$ there exists another point $z^2\notin C$ such that $\operatorname{vis}(C;z^1)\cup\operatorname{vis}(C;z^2)=\operatorname{bd} C$, i.e., the whole set C can be seen from $\{z^1,z^2\}$.

The compactness of C is essential in this characterization, even reinforcing the above condition with $\operatorname{vis}(C;z) \neq \operatorname{bd} C$ for all $z \notin C$ (a consequence of the boundedness of C). In fact, given an n-sandwich $C = \{x \in \mathbb{R}^n \mid \alpha_1 \leq a'x \leq \alpha_2\}$, with $a \neq 0_n$ and $\alpha_1 < \alpha_2$, and $z \notin C$, we have

$$\operatorname{vis}(C;z) = \begin{cases} \{x \in \mathbb{R}^n \mid a'x = \alpha_1\} & if \quad a'z < \alpha_1 \\ \\ \{x \in \mathbb{R}^n \mid a'x = \alpha_2\} & if \quad a'z > \alpha_2 \end{cases}, \tag{6.1}$$

so that the n-sandwiches satisfy the above conditions. Unfortunately, they are not the only unbounded convex bodies satisfying these conditions. This is the case of every closed convex set C such that $E(C)$ is an $(n-1)$-simplex, $L(C)$ is a line through the origin and $D(C) = \{0_n\}$ (e.g., the cartesian product $S \times \mathbb{R}$ where S is an $(n-1)$-simplex in $\mathbb{R}^{n-1}$). Nevertheless, the n-sandwiches (the class of convex bodies with non-empty unconnected boundary) can also be characterized in terms of visibilities.

Proposition 6.1. *Let $C \neq \mathbb{R}^n$ be a convex body. Then C is an n-sandwich if, and only if, $\operatorname{vis}(C;z)$ is a hyperplane and $\operatorname{vis}(C;z) \neq \operatorname{bd} C$ for all $z \notin C$.*

Proof. The direct statement is a straightforward consequence of (6.1). Assume that $\operatorname{vis}(C;z)$ is a hyperplane different of $\operatorname{bd} C$ for all $z \notin C$. Since $\{\operatorname{vis}(C;z),\ z \notin C\}$ is a covering of $\operatorname{bd} C$, $\operatorname{bd} C$ turns out to be the union of at least two hyperplanes. Assuming that $\operatorname{bd} C$ contains two non-parallel hyperplanes, we shall obtain a contradiction. In fact, let $H_i = \{x \in \mathbb{R}^n \mid a_i'x = b_i\}$, with $a_i'x \geq b_i$ for all $x \in C$, $i = 1, 2$, with $\operatorname{span}\{a_1\} \neq \operatorname{span}\{a_2\}$. Since H_1 is not contained in $\{x \in \mathbb{R}^n \mid a_2'x \geq b_2\}$, there exists $x^1 \in \mathbb{R}^n$ such that $a_1'x^1 = b_1$ and $a_2'x^1 < b_2$. Then $x^1 \in H_1 \backslash C$, contradicting $H_1 \subset \operatorname{bd} C \subset C$.

Since $\operatorname{bd} C$ is the union of at least two parallel hyperplanes, $\operatorname{bd} C$ is unconnected, and so C is an n-sandwich by Proposition 5.2. □

A parallelotope can be defined as the intersection of a family of n "independent" n-sandwiches, i.e., a set of the form $C = \bigcap_{i=1}^{n} C_i$, with $C_i = \{x \in \mathbb{R}^n \mid \alpha_i \leq a_i'x \leq \beta_i\}$, $\alpha_i < \beta_i$ for $i = 1, \ldots, n$, and $\{a_i,\ i = 1, \ldots, n\}$ linearly independent. Consequently,

$$K(C) = \operatorname{cone}\left\{\begin{pmatrix} a_i \\ \alpha_i \end{pmatrix},\ i = 1, \ldots, n;\ -\begin{pmatrix} a_i \\ \beta_i \end{pmatrix},\ i = 1, \ldots, n;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$$

whereas $E(C)$ is the sum of n segments, $[x^i, y^i]$, $i = 1, \ldots, n$, such that $\{y^i - x^i,\ i = 1, \ldots, n\}$ is linearly independent, and $L(C) = D(C) = \{0_n\}$. A parallelotope is a particular class of n-zonotope (convex body that can be expressed as the sum of finitely many compact segments). As the n-simplices and the n-sandwiches, the parallelotopes not only can be characterized by means of their conical and internal representations but also through their geometric combinatorial properties. In fact, a given n-zonotope C is a parallelotope if, and only if, the minimum number of smaller homothetic copies of C covering C (or directions, or points, illuminating C) is exactly 2^n ([11] and [13]).

Parallelotopes, n-simplices and n-sandwiches are only some of the families of convex bodies enjoying nice combinatorial, optimality and separability properties (see, e.g., [9], [5] and [3], respectively). The characterization of all these families in terms of their internal and conical representations and their illumination properties are challenging open problems.

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