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Lech Inequalities for Deformations of Singularities Defined by Power Products of Degree 2

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Abstract. Using a result from Herzog [2] we prove the following. Let $(B_0, \mathfrak{n}_0)$ be an artinian local algebra of embedding dimension v over some field L with tangent cone $\mathrm{gr}(B_0) \cong L[X_1, \dots, X_v]/I_0$. Suppose the ideal I_0 is generated by power products of degree 2. Then for every residually rational flat local homomorphism $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ of local L-algebras that has a special fiber isomorphic to B_0 the $(v+1)$th sum transforms of the local Hilbert series of A and B satisfy the Lech inequality $H_A^{v+1} \leq H_B^{v+1}$.

1. Notation

Throughout we fix a field L, an integer $v \geq 2$, indeterminates $\underline{X} = X_1, \dots, X_v$ and write $R := L[[\underline{X}]]$ for the ring of formal power series and $R_0 := L[\underline{X}]$ for the polynomial ring.

Note that R_0 and all the R_0-modules that will occur are canonically graded and furthermore admit a canonical $\mathbb{Z}^v$-(multi)grading that refines the grading. If M is such an R_0-module , $n \in \mathbb{Z}$ and $\mu \in \mathbb{Z}^v$, we let $M(n)$ and $M(\mu)$ denote the homogeneous parts of degree n and multidegree μ (e.g., $R_0(\mu) = L \cdot X^\mu$). We will write $M(< n) := \bigoplus_{m<n} M(m)$, $M(\geq \mu) := \bigoplus_{\nu \geq \mu} M(\nu)$ and similarly.

We use the term "local L-algebra" for a noetherian local L-algebra $(A, \mathfrak{m})$ such that $L \to A/\mathfrak{m}$ is an isomorphism. A deformation of a local L-algebra B_0 is a flat local homomorphism of local L-algebras with special fiber isomorphic to B_0. In particular, any deformation will be residually rational, i.e., it induces a trivial extension of the residue fields.

If $(A, \mathfrak{m})$ is a local L-algebra, $\mathrm{gr}(A)$ denotes the tangent cone of A, which is the graded ring associated with the natural filtration of A by the powers of the maximal ideal. H_A^i is the ith sum transform of the local Hilbert series of A, i.e.,

$$H_A^i = (1-T)^{-i} \sum_{j=0}^{\infty} \dim_L(\mathfrak{m}^j/\mathfrak{m}^{j+1}) T^j = (1-T)^{-i} \sum_{j=0}^{\infty} \dim_L(\mathrm{gr}(A)(j)) T^j .$$

We understand inequalities between formal power series in the "total" sense, i.e., $\sum_{i=0}^{\infty} a_i T^i \leq \sum_{i=0}^{\infty} b_i T^i \iff a_i \leq b_i \quad \forall i$.

2. The Lech problem

In [3] C. Lech asks whether the multiplicities of any two local rings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$, connected by a flat local homomorphism $A \to B$, satisfy the inequality $e_0(A) \leq e_0(B)$. A generalization is the question whether the analogous inequality

$$H_A^{d+i} \leq H_B^i, \tag{1}$$

always holds for some i, where d denotes the dimension of the special fiber $B_0 := B/\mathfrak{m}B$. (1) has been shown to hold true for $i = 1$ if B_0 has dimension zero and corresponds to a smooth point of the Hilbert scheme (see [1]) or, also for $i = 1$, if $A \to B$ is tangentially flat, i.e., induces a flat homomorphism of tangent cones (see [1], [2]). Little is known in between these two somewhat extreme situations. However, in ([2] Cor. 8.3) Herzog proved the following estimation:

Theorem. (Herzog) *Let $A \to B$ be a deformation of a local L-algebra B_0. Then it holds*

$$H_A^1 \cdot H_{B_0}^0 \leq H_B^1 \cdot \prod_{l=2}^{\infty} \left(\frac{1-T^l}{1-T} \right)^{\dim_L(T^1_{\mathrm{gr}(B_0)}(-l))}, \tag{2}$$

where $T^1_{\mathrm{gr}(B_0)}(-l)$ denotes the homogeneous part of degree $-l$ of Schlessinger's T^1 of the tangent cone of B_0.

Remark. If $T^1_{\mathrm{gr}(B_0)}(< -1) = 0$ one immediately obtains $H_A^1 \leq H_B^1$; this is the tangentially flat situation. If the product on the right of (2) is not trivial, there are situations where it is small enough to allow for the conclusion of a Lech-type inequality from (2) (compare [2], 9.3). If $\mathrm{gr}(B_0) \cong R_0/I_0$ for I_0 generated by power products, $\mathrm{gr}(B_0)$ and hence $T^1_{\mathrm{gr}(B_0)}$ are $\mathbb{Z}^v$-graded and the determination of the dimensions of $T^1_{\mathrm{gr}(B_0)}(-l)$ becomes a combinatorial problem. This problem does not appear to have an elegant solution. Therefore, we restrict ourselves to estimating $\dim_L T^1_{\mathrm{gr}(B_0)}(-l)$ in terms of the power products that generate I_0 (see the Lemma below).

3. Estimates for $\dim_L T^1_{R_0/I_0}(-l)$

Let S be a set of power products of degree d in the indeterminates $\underline{X}$.

Definition. *For every $k \in \mathbb{N}, k > 0$ we define an equivalence relation $\equiv^k$ ("k-connectivity") on S: we call $s, t \in S$ k-connected (and write $s \equiv^k t$) iff there exists a sequence $s = s_1, \ldots, s_m = t$ of elements of S such that*

$$\deg(\gcd(s_j, s_{j+1})) \geq d - k \qquad \text{for} \qquad j = 1, \ldots, m-1.$$

S splits into equivalence classes $S = S_{k,1} \cup \ldots \cup S_{k,\mathrm{n}(S,k)}$ (called k-components in the following). We define

$$d^S_{k,j} := \deg(\gcd(S_{k,j})) \qquad \text{for} \qquad j = 1, \ldots, \mathrm{n}(S,k).$$

Whenever $n < m$, let the binomial coefficient $\binom{n}{m}$ be zero.

Lemma. *Let I_0 be the ideal generated by S in R_0. Then with the above notation for all $k \in \mathbb{N}$ it holds*

$$\dim_L(T^1_{R_0/I_0}(-(k+1))) \leq \sum_{j=1}^{\mathrm{n}(S,k)} \binom{v + d^S_{j,k} - k - 2}{v - 1}.$$

Proof. By the exact sequence of graded modules and homomorphisms

$$\mathrm{Der}_L(R_0, R_0/I_0) \to \mathrm{Hom}_{R_0}(I_0, R_0/I_0) \to T^1_{R_0/I_0} \to 0$$

and $\mathrm{Der}_L(R_0, R_0/I_0)(l) = 0$ if $l < -1$ we conclude

$$\mathrm{Hom}_{R_0}(I_0, R_0/I_0)(< -1) \cong T^1_{R_0/I_0}(< -1).$$

For $j = 1, \ldots, \mathrm{n}(S,k)$ let $I_{j,k} \subseteq I_0$ denote the ideal generated by the elements of $S_{j,k}$ in R_0. By applying $\mathrm{Hom}_{R_0}(\cdot, R/I_0)$ to the surjection

$$\bigoplus_{j=1}^{\mathrm{n}(S,k)} I_{j,k} \to I_0 \quad : \quad (\alpha_1, \ldots, \alpha_{\mathrm{n}(S,k)}) \mapsto \sum_{j=1}^{\mathrm{n}(S,k)} \alpha_j$$

we obtain an injection $\mathrm{Hom}_{R_0}(I_0, R_0/I_0) \to \bigoplus_{j=1}^{\mathrm{n}(S,k)} \mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)$. Thus it remains to prove that

$$\dim_L(\mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)(-(k+1))) \leq \binom{v + d^S_{j,k} - k - 2}{v - 1} \tag{3}$$

for all $k > 0, j = 1, \ldots, \mathrm{n}(S,k)$. Considering $\mathbb{Z}^v$-gradings, there are vector space isomorphisms

$$\mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)(-(k+1)) \cong \bigoplus_{\substack{\mu \in \mathbb{Z}^v \\ |\mu| = -(k+1)}} \mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)(\mu)$$

for all j,k. Since there are exactly $\binom{v+d^S_{j,k}-k-2}{v-1}$ multi-indices μ satisfying $|\mu| = -(k+1)$, $\underline{X}^\mu \cdot \gcd(S_{j,k}) \in R_0(\geq (0,\dots,0))$, equality will hold in (3) if for each $\mu \in \mathbb{Z}^v$ satisfying $|\mu| = -(k+1)$ it holds

$$\dim_L(\mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)(\mu)) = \begin{cases} 1 & \text{if } \underline{X}^\mu \cdot \gcd(S_{j,k}) \in R_0(\geq (0,\dots,0)) \\ 0 & \text{if } \underline{X}^\mu \cdot \gcd(S_{j,k}) \notin R_0(\geq (0,\dots,0)). \end{cases} \tag{4}$$

To prove (4), note that the relation module of $I_{j,k}$ is generated by the pairwise relations

$$R_0^t \ni r^{q,p} = (r_1^{q,p}, \dots, r_t^{q,p}), \quad 1 \leq q < p \leq t$$

$$r_l^{q,p} = \begin{cases} s_p/\gcd(s_q,s_p) & \text{if } l = q, \\ -s_q/\gcd(s_q,s_p) & \text{if } l = p, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

of the elements $s_1, \dots, s_t$ of $S_{j,k}$.
Thus we can identify $g \in \mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)$ with $(g_1, \dots, g_t) \in R_0^t$ (g_p arbitrary liftings of $g(s_p)$ to R_0) for which

$$\sum_{l=1}^{t} g_l r_l^{p,q} \in I_0 \qquad \text{for all the } r^{p,q}. \tag{6}$$

We fix j,k and $\mu \in \mathbb{Z}^v$ such that $|\mu| = -(k+1)$ and let $g \in \mathrm{Hom}_{R_0}(I_{j,k}, R_0/I_0)(\mu)$. Then g corresponds to

$$(a_1 \cdot s_1 \cdot \underline{X}^\mu, \dots, a_t \cdot s_t \cdot \underline{X}^\mu)$$

with $a_p \in L$ and $a_p = 0$ if $s_p \cdot \underline{X}^\mu \notin R_0(\geq (0,\dots,0))$. Furthermore, we have

$$a_1 = \dots = a_t. \tag{7}$$

Indeed, by k-connectivity of $S_{j,k}$ there is $u \in \{1,\dots,t\}$ with $\deg(\gcd(s_1,s_u)) \geq d-k$. By substituting $q=1$, $p=u$ into (6) we obtain

$$(a_1 - a_u) \cdot \frac{s_1 \cdot s_u}{\gcd(s_1,s_u)} \cdot \underline{X}^\mu \in I_0.$$

This monomial has degree less than d. Since I_0 is generated by monomials of degree d, we obtain $a_1 = a_u$. We use k-connectivity and iterate to obtain (7). Equation (4) follows, since $\underline{X}^\mu \cdot \gcd(S_{j,k}) \in R_0(\geq (0,\dots,0))$ iff for every element s_p of $S_{j,k}$ it holds $s_p \cdot \underline{X}^\mu \in R_0(\geq (0,\dots,0))$. $\square$

4. The estimates imply Lech-inequalities in the case I_0 generated in degree 2

Theorem. *Let B_0 be an artinian local L-algebra of embedding dimension v such that* $\mathrm{gr}(B_0) \cong R_0/I_0$, *where I_0 is generated by power products of degree* 2. *Then for every deformation $A \to B$ of B_0 Lech's inequality holds with $i = v+1$,*

$$H_A^{v+1} \leq H_B^{v+1}.$$

Proof. Let S be the set of generators of I_0. Since B_0 is artinian, $\{X_1^2, \dots, X_v^2\} \subseteq S$. One verifies that S is 2-connected. Therefore $d_{1,k} = 0$ for all $k \geq 2$ and thus by the lemma $\dim_L(T^1_{R_0/I_0}(\leq -2)) = 0$. Herzog's theorem now reads (let $\omega(2) := dim_L(T^1_{\mathrm{gr}(B_0)}(-2))$):

$$H_A^1 \cdot H_{B_0}^0 \leq H_B^1 \cdot (1+T)^{\omega(2)}$$

It remains to show that

i. $H_{B_0}^0 \geq (1+T)^{\omega(2)}$

ii. $\omega(2) \leq v$,

since i. implies

$$H_A^1 \cdot (1+T)^{\omega(2)} \leq H_B^1 \cdot (1+T)^{\omega(2)}$$

and multiplying with $(1-T^2)^{-\omega(2)} \cdot (1-T)^{-(v-\omega(2))}$ (which has positive coefficients by ii.) yields $H_A^{v+1} \leq H_B^{v+1}$.

Any 1-component of S either consists of a square of an indeterminate or it contains two or more squares of indeterminates, in which case its gcd is equal to 1. We may assume that the 1-components that have a gcd of degree 2 are $\{X_1^2\}, \dots, \{X_w^2\}$. The lemma gives

$$\omega(2) \leq w \cdot \binom{v+2-3}{v-1} + (\mathrm{n}(S,1) - w) \cdot \binom{v+0-3}{v-1} = w \leq v$$

and proves ii. The residue classes of all the squarefree monomials in $X_1, \dots, X_w$ are linearly independent elements of $\mathrm{gr}(B_0)$. Hence

$$H_{B_0}^0(n) \geq \binom{w}{n} \qquad \text{for} \quad 1 \leq n \leq w$$

and by the binomial theorem $H_{B_0} \geq (1+T)^w$ which implies i. $\square$

References

[1] Herzog, Bernd: *Local singularities such that all deformations are tangentially flat.* Trans. Amer. Math. Soc **324**(2) (1991), 555–601. Zbl 0728.13004

[2] Herzog, Bernd: *Kodaira-Spencer maps in local algebra.* Lecture Notes in Mathematics **1597**. Springer-Verlag, Berlin Heidelberg 1994. Zbl 0809.13011

[3] Lech, Christer: *Note on multiplicities of ideals.* Ark. Mat. **4** (1960), 63–86. Zbl 0192.13901

[4] Schlessinger, Michael: *Functors of artin rings.* Trans. Amer. Math. Soc. **130** (1968), 205–222. Zbl 0167.49503

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