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A Classification of Contact Metric 3-Manifolds with Constant ξ-sectional and ϕ-sectional Curvatures

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Abstract. We study the 3-dimensional contact metric manifolds equipped with constant ξ-sectional curvature and ϕ-sectional curvature or constant norm of the Ricci operator.

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1. Introduction

D. E. Blair in [2], [3] constructed a family of examples of $(3-\tau)$-manifolds which do not satisfy the condition $Q\phi = \phi Q$. The existence of these examples depends on the constancy of the ξ-sectional curvature. After this remark the following question raises:

Question 1: Does every $(3-\tau)$-manifold with constant ξ-sectional curvature satisfy the condition $Q\phi = \phi Q$?

S. Tanno in [16] stated the problem about the existence of $(2n+1)$-dimensional contact metric manifolds of constant ϕ-sectional curvature, which are not Sasakian. Positive answers have been given by D. E. Blair, Th. Koufogiorgos and R. Sharma in [5], for 3-dimensional contact metric manifolds satisfying $Q\phi = \phi Q$, Th. Koufogiorgos in [14], for (κ, μ)-contact metric manifolds of dimension greater than 3 and D. E. Blair, Th. Koufogiorgos and B. Papantoniou in [4] for (κ, μ)-contact metric manifolds of dimension 3. In [4] the authors, extending the Tanno's problem showed that there exist (κ, μ)-contact metric manifolds of dimension 3 which do not belong to the class of the manifolds satisfying $Q\phi = \phi Q$.

Extending Tanno's problem and the result of [4] we can state the following:

Question 2: Do there exist 3-dimensional contact metric manifolds of constant ϕ-sectional curvature, which do not belong to the class of (κ, μ)-contact metric manifolds?

Combination of the above mentioned questions leads us to the study of 3-dimensional contact metric manifolds of constant ξ-sectional and ϕ-sectional curvature.

The main goal of the present paper (Theorem 15) is the proof of the existence of two new classes of 3-dimensional contact metric manifolds with constant ξ-sectional and constant ϕ-sectional curvatures, which do not belong to the up to date well known classes ([4], [5]).

D. E. Blair, Th. Koufogiorgos and R. Sharma in [5] proved that a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$ is flat or Sasakian or a manifold with constant ϕ-sectional curvature k and constant ξ-sectional curvature $-k$. In the present paper we prove the converse and so we can state the argument: A non-flat, non-Sasakian 3-dimensional contact metric manifold satisfies $Q\phi = \phi Q$ if and only if it has constant ϕ-sectional curvature k and constant ξ-sectional curvature $-k$.

Complete, conformally flat Riemannian manifolds with constant scalar curvature and the norm of the Ricci tensor bounded (respectively constant) were classified by Goldberg ([8]) in general dimension (respectively, by Cheng, Ishikawa and Shiohama [7] in dimension 3). On the other hand the first author and R. Sharma in [10] proved that a conformally flat, contact metric 3-manifold with Ricci curvature vanishing along the characteristic vector field ξ and the norm of its Ricci tensor being constant, is flat. Therefore, it is interesting to study 3-dimensional contact metric manifolds equipped with more general conditions: constant ξ-sectional curvature and constant norm of the Ricci operator along ξ.

2. Preliminaries

A contact metric manifold $M^{2n+1} \equiv M^{2n+1}(\phi, \xi, \eta, g)$ is a $(2n+1)$-dimensional Riemannian manifold on which has been defined globally a $(1,1)$ tensor field ϕ, a vector field ξ (*characteristic vector field*), a 1-form η (*contact form*) and a Riemannian metric g (*associated metric*) which satisfy:

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi), \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad d\eta(X, Y) = g(X, \phi Y) \end{aligned}$$

for all vector fields X and Y on M^{2n+1}. The structure (ϕ, ξ, η, g) is called *contact metric structure.*

Denoting by L and R the Lie derivation and the curvature tensor respectively, we define the operators l and h by

$$l := R(., \xi)\xi, \quad \eta := \frac{1}{2} L_\xi \phi.$$

The tensors l and h are self-adjoint and satisfy

$$h\xi = l\xi = 0, \quad \eta \circ h = 0, \quad Trh = Trh\phi = 0, \quad h\phi + \phi h = 0.$$

On every contact metric manifold M^{2n+1} the following formulas hold

$$\begin{aligned} \eta \circ \phi &= 0, \quad \phi\xi = 0, \quad d\eta(\xi, X) = 0, \quad \nabla_\xi \phi = 0, \\ \nabla_X \xi &= -\phi X - \phi h X \quad (\Rightarrow \nabla_\xi \xi = 0), \quad \phi l \phi - l = 2(\phi^2 + h^2), \\ \nabla_\xi h &= \phi - \phi l - \phi h^2, \quad Trl = g(Q\xi, \xi) = 2n - trh^2, \end{aligned} \tag{1}$$

where ∇ is the Riemannian connection. On $M^{2n+1} \times \mathbf{R}$ we can define an almost complex structure J by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where f is a real-valued function. If J is integrable, then the contact metric structure is said to be normal and M^{2n+1} is called *Sasakian.* A 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$, ([1]).

The sectional curvature $K(X, \xi)$ of a plain section spanned by ξ and a vector field X orthogonal to ξ is called *ξ-sectional curvature.* The sectional curvature $K(X, \phi X)$ of a plain section spanned by the vector field X (orthogonal to ξ) and ϕX is called *ϕ-sectional curvature.*

It is well known that on every 3-dimensional Riemannian manifold the curvature tensor $R(X, Y)Z$ is given by

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y \\ &\quad - \frac{S}{2}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2}$$

where Q is the Ricci operator, $S(= TrQ)$ is the scalar curvature and X, Y and Z are arbitrary vector fields.

A 3-dimensional contact metric manifold satisfing $\nabla_\xi \tau = 0$, $(\tau = L_\xi g)$ is called $(3-\tau)$-*manifold*, ([11]).

A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called (κ, μ)-*contact metric manifold* ([4]) if it satisfies the condition

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

where κ and μ are real constants and X, Y are vector fields on M^{2n+1}.

3. Auxiliary results

Let M^3 be a 3-dimensional contact metric manifold. If $e \in \ker(\eta)$ is a unit eigenvector of h with eigenvalue λ, then ϕe is also an eigenvector of h with eigenvalue $-\lambda$. Hence, $(e, \phi e, \xi)$ is an orthonormal frame on M^3.

Since e and ϕe are unit vector fields orthogonal to ξ, we see that

$$\nabla_\xi e = a\phi e, \quad \nabla_\xi \phi e = -\alpha e,$$

for some function a on M^3. The orthogonality of $e, \phi e$ and ξ implies

$$\nabla_e e = b\phi e, \quad \nabla_{\phi e}\phi e = ce, \quad \nabla_e \phi e = -be + (\lambda+1)\xi, \quad \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi,$$

where b and c are functions on M^3. Finally, from (1) we have

$$\nabla_e \xi = -(1+\lambda)\phi e, \quad \nabla_{\phi e}\xi = (1-\lambda)e.$$

Therefore, we can state the following

Lemma 1. *Let M^3 be 3-dimensional contact metric manifold. Then, the following formulas hold:*

$$\begin{aligned} \nabla_\xi e &= a\phi e, \quad \nabla_\xi \phi e = -\alpha e, \quad \nabla_e e = b\phi e, \quad \nabla_{\phi e}\phi e = ce, \\ \nabla_e \phi e &= -be + (\lambda+1)\xi, \quad \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi, \\ \nabla_e \xi &= -(1+\lambda)\phi e, \quad \nabla_{\phi e}\xi = (1-\lambda)e, \end{aligned} \tag{3}$$

where a, b and c are functions on M^3.

Proposition 2. *Let M^3 be 3-dimensional contact metric manifold of constant ξ-sectional curvature k. Then, M^3 is $(3-\tau)$-manifold with constant Trl.*

Proof. By straightforward computation using (3) and $\nabla_\xi \xi = 0$ we obtain

$$le = (1-\lambda^2 - 2\alpha\lambda)e + (\xi\cdot\lambda)\phi e, \quad l\phi e = (1-\lambda^2+2\alpha\lambda)\phi e + (\xi\cdot\lambda)e,$$

and hence

$$1-\lambda^2-2\alpha\lambda = k, \quad 1-\lambda^2+2\alpha\lambda = k.$$

Adding the above two relations we obtain $2(1-\lambda^2) = 2k$. Because of $Trl = 2(1-\lambda^2)$ ([5]) we have Trl =constant. Subtracting the same relations we obtain $\alpha\lambda = 0$, that is $\alpha = 0$ or $\lambda = 0$.

If $\lambda = 0$, then M^3 is Sasakian, which is trivially $(3-\tau)$-manifold ([5]).

Suppose that $a = 0$. Taking into account that Trl = *constant* we obtain that $\nabla_\xi h = 0$. This relation and ([11]) complete the proof. □

Proposition 2 and Theorem 3.2 of [12] imply the following

Corollary 3. *Let M^3 be a 3-dimensional, conformally flat, contact metric manifold of constant ξ-sectional curvature. Then, M^3 is either flat or a Sasakian space form.*

Proposition 2 and Theorem 3.1 of [14] imply the following

Corollary 4. *Let M^3 be a 3-dimensional contact metric manifold of constant ξ-sectional curvature satisfing $R(e,\xi)\cdot R = 0$. Then, M^3 is either flat or a Sasakian manifold.*

Proposition 2 and Theorem 3.1 of [13] imply the following

Corollary 5. *Let M^3 be a 3-dimensional contact metric manifold of constant ξ-sectional curvature satisfing $R(e,\xi)\cdot C = 0$. Then, M^3 is either flat or a Sasakian manifold.*

Proposition 2 and Theorem 5.1 of [11] imply the following

Corollary 6. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature and η-parallel Ricci tensor. Then, M^3 is either flat or a Sasakian space form.*

Proposition 2 and Theorem 6.2 of [11] imply the following

Corollary 7. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature and cyclic η-parallel Ricci tensor. Then, M^3 is either flat or a Sasakian manifold with constant scalar curvature or of constant ξ-sectional curvature $k < 1$ and constant ϕ-sectional curvature $-k$.*

Lemma 1, Proposition 2 and [11] imply:

Lemma 8. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature. Then, the following formulas hold:*

$$\begin{aligned} \nabla_\xi e &= \nabla_\xi \phi e = 0, \quad \nabla_e e = b\phi e, \quad \nabla_{\phi e}\phi e = ce, \\ \nabla_e \phi e &= -be + (\lambda+1)\xi, \quad \nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi, \\ \nabla_e \xi &= -(1+\lambda)\phi e, \quad \nabla_{\phi e}\xi = (1-\lambda)e. \end{aligned} \tag{4}$$

where a, b and c are functions on M^3 and λ is a constant.

Proposition 2 and [6] (relations 2.16) yield

Lemma 9. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature. Then, the following formulas hold:*

$$Qe = (\lambda^2 + \frac{S}{2} - 1)e + 2\lambda b\xi, \quad \eta(Qe) = 2\lambda b,$$

$$Q\phi e = (\lambda^2 + \frac{S}{2} - 1)\phi e + 2\lambda c\xi, \quad \eta(Q\phi e) = 2\lambda c, \tag{5}$$

$$Q\xi = 2\lambda be + 2\lambda c\phi e + 2(1-\lambda^2)\xi.$$

Lemma 10. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature. Then, either $l = 0$, or the following relations are equivalent: $b = 0$, $c = 0$.*

Proof. Suppose that l is not identically equal to zero on M^3. Let $\lambda^2 \neq 1$ on an open neighborhood U at a point $p \in M^3$, where $l \neq 0$. Applying the Jacobi's identity for the vector fields e, ϕe, ξ and taking into account the relation (4) we obtain

$$\xi \cdot b = (\lambda - 1)c, \quad \xi \cdot c = (\lambda+1)b. \tag{6}$$

Let $b = 0$ (or $c = 0$) on M^3. Then, from the first (or the second) of (6) we conclude that $c = 0$ (or $b = 0$) on U. So, $c = 0$, ($b = 0$) on M^3. □

Remark 11. *On a 3-dimensional contact metric manifold M^3, we have $b = c = 0$ if and only if $Q\phi = \phi Q$,* ([11]).

4. Main results

Theorem 12. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature. Then, either M^3 is Sasakian or*

$$\xi \cdot \xi \cdot \xi \cdot S = 4(\lambda^2 - 1)(\xi \cdot S). \tag{7}$$

Proof. If $l = 0$ on M^3, then $\lambda^2 = 1$ and $\xi \cdot \xi \cdot \xi \cdot S = 0$ ([9]).

Suppose that M^3 is not Sasakian and l is not identically equal to zero. So, let $\lambda^2 \neq 0, 1$ on an open neighborhood U of a point $p \in M^3$. Applying the second Bianchi's identity for the vector fields e, ϕe and ξ we obtain

$$e \cdot b + \phi e \cdot c - \frac{1}{4\lambda} \xi \cdot S = 2bc. \tag{8}$$

Differentiating the above equation along ξ and taking into account (6) we obtain

$$\xi \cdot e \cdot b + \xi \cdot \phi e \cdot c - \frac{1}{4\lambda} \xi \cdot \xi \cdot S = 2(\lambda - 1)c^2 + 2(\lambda + 1)b^2.$$

Next, differentiating the first and the second equations of (6) with respect to e and ϕe respectively and adding the results we get

$$e \cdot \xi \cdot b + \phi e \cdot \xi \cdot c = (\lambda - 1)e \cdot c + (\lambda + 1)\phi e \cdot b.$$

Hence,

$$[\xi, e]b + [\xi, \phi e]c = \frac{1}{4\lambda} \xi \cdot \xi \cdot S + 2(\lambda - 1)c^2 + 2(\lambda + 1)b^2 + (1 - \lambda)e \cdot c - (\lambda + 1)\phi e \cdot b.$$

The above equation using (4) yields

$$(\lambda + 1)\phi e \cdot b + (\lambda - 1)e \cdot c = \frac{1}{8\lambda} \xi \cdot \xi \cdot S + \lambda(b^2 + c^2) + b^2 - c^2. \tag{9}$$

Differentiating again (9) along ξ and taking into account (6) and (8) we obtain

$$(\lambda + 1)\xi \cdot \phi e \cdot b + (\lambda - 1)\xi \cdot e \cdot c = \frac{1}{8\lambda} \xi \cdot \xi \cdot \xi \cdot S + 4(\lambda^2 - 1)bc. \tag{10}$$

As $\lambda^2 \neq 1$ on U we obtain from (6) and (8)

$$(\lambda + 1)\phi e \cdot \xi \cdot b + (\lambda - 1)e \cdot \xi \cdot c = (\lambda^2 - 1)[\frac{1}{4\lambda} \xi \cdot S + 2bc].$$

Subtracting the above equation from (10) and using (4) the seeking formula follows at once. □

Theorem 13. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature. If the norm of the Ricci operator is constant along ξ, then either $Q\phi = \phi Q$ or $l = 0$ with constant scalar curvature and $\eta(QX) = 0$ for all eigenvectors $X \in \ker(\eta)$ of h with eigenvalue 1.*

Proof. The square of the norm of the Ricci operator Q is $TrQ^2 = g(Q^2e, e) + g(Q^2\phi e, \phi e) + g(Q^2\xi, \xi)$ and is computed using (5) and turns out to be

$$(\lambda^2 + \frac{S}{2} - 1)^2 + 4\lambda^2(b^2 + c^2) + 2(1 - \lambda^2)^2 = \psi, \tag{11}$$

where ψ is a smooth function on M^3 being constant along ξ.

Suppose that $l = 0$. Then, $\lambda^2 = 1$ and (11) yields

$$\frac{S^2}{4} + 4(b^2 + c^2) = \psi. \tag{12}$$

Differentiating three times the equation (12) along ξ and taking into account (6) and (7) for $\lambda = 1$ we obtain respectively

$$\begin{gathered} S(\xi \cdot S) + 32bc = 0, \\ S(\xi \cdot \xi \cdot S) + (\xi \cdot S)^2 + 64b^2 = 0, \\ (\xi \cdot S)(\xi \cdot \xi \cdot S) = 0. \end{gathered} \tag{13}$$

Therefore, $\xi \cdot S = 0$. or $\xi \cdot \xi \cdot S = 0$.

Supposing $\xi \cdot S = 0$ from the first of (13) we have $b = 0$ or $c = 0$.

If $b = 0$, from (5) we obtain $\eta(Qe) = 0$.

If $c = 0$ then (6) implies $b = 0$ that is $Q\phi = \phi Q$. In this case the manifold is flat.

If $\xi \cdot \xi \cdot S = 0$ then from (13) we have $\xi \cdot S = 0$ and $b = 0$.

If M^3 is Sasakian then it is known that we have $Q\phi = \phi Q$.

Suppose that M^3 is not Sasakian with l not identically equal to zero. So, let be $\lambda^2 \neq 0, 1$ on an open neighborhood U of a point $p \in M^3$. Hence, we can write the equation (11) in the form

$$b^2 + c^2 = \frac{\psi}{4\lambda^2} + \frac{(\lambda^2 - 1)^2}{2\lambda^2} - \frac{(\lambda^2 + \frac{S}{2} - 1)^2}{4\lambda^2}.$$

Differentiating the above equation along ξ and taking into account (6) we obtain

$$bc = -\frac{1}{16\lambda^2}(\lambda^2 + \frac{S}{2} - 1)(\xi \cdot S). \tag{14}$$

Differentiating two times the relation (14) with respect to ξ and using (6) and (14) we have

$$(\xi \cdot S)[8(1 - \lambda^2)(\lambda^2 + \frac{S}{2} - 1) - 1 - \xi \cdot \xi \cdot S] = 0.$$

Hence,

$$\xi \cdot S = 0 \quad \text{or} \quad \xi \cdot \xi \cdot S = 8(1 - \lambda^2)(\lambda^2 + \frac{S}{2} - 1) - 1. \tag{15}$$

Supposing $\xi \cdot S = 0$, the equation (14) yields $b = 0$ or $c = 0$ on U and hence $b = 0$ or $c = 0$ on M^3. Both cases using (6) imply $Q\phi = \phi Q$.

If the second of (15) holds on U, differentiating this relation along ξ and using Theorem 12 we obtain $\xi \cdot S = 0$ and therefore $Q\phi = \phi Q$. □

Proposition 14. *Let M^3 be a 3-dimensional non-Sasakian contact metric manifold with constant ξ-sectional curvature. If l is not identically equal to zero then the following formulas hold:*

$$e \cdot b = \frac{1}{8\lambda}\xi \cdot S + bc + \Phi, \tag{16}$$

$$\phi e \cdot b = \frac{1}{16\lambda}\xi \cdot \xi \cdot S + \frac{1}{2}(1-\lambda)(\lambda^2 + \frac{S}{2} - 1) + b^2, \tag{17}$$

$$e \cdot c = -\frac{1}{16\lambda}\xi \cdot \xi \cdot S + \frac{1}{2}(1+\lambda)(\lambda^2 + \frac{S}{2} - 1) + c^2, \tag{18}$$

$$\phi e \cdot c = \frac{1}{8\lambda}\xi \cdot S + bc - \Phi. \tag{19}$$

where Φ is a smooth function on M^3 such that

$$\xi \cdot \Phi = 0, \tag{20}$$

$$\begin{aligned} e \cdot \Phi \;&=\; \frac{1}{16\lambda}[\phi e \cdot \xi \cdot \xi \cdot S - 2b(\xi \cdot \xi \cdot S) + 2(e \cdot \xi \cdot S) - 4c(\xi \cdot S) - \\ &\quad - 4\lambda(\lambda+1)(\phi e \cdot S)] + (\lambda+1)(\lambda^2 + \frac{S}{2} - 3)b + 4c\Phi, \end{aligned} \tag{21}$$

$$\begin{aligned} \phi e \cdot \Phi \;&=\; \frac{1}{16\lambda}[e \cdot \xi \cdot \xi \cdot S - 2c(\xi \cdot \xi \cdot S) - 2(\phi e \cdot \xi \cdot S) + 4b(\xi \cdot S) + \\ &\quad + 4\lambda(1-\lambda)(e \cdot S)] + (\lambda-1)(\lambda^2 + \frac{S}{2} - 3)c + 4b\Phi. \end{aligned} \tag{22}$$

Proof. Calculating $R(e, \phi e)\xi$ firstly by straightforward computation using Lemma 8 and secondly from the relation (2) we obtain

$$\phi e \cdot b + e \cdot c = b^2 + c^2 + \lambda^2 - 1 + \frac{S}{2}. \tag{23}$$

From (23) and (9) the relations (17) and (18) follow at once.

Differentiating (17) first with respect to ξ (respectively with respect to e) and secondly with respect to e (respectively with respect to ξ) and using (6) we have

$$\xi \cdot e \cdot \phi e \cdot b = \frac{\lambda - 1}{4\lambda} e \cdot \xi \cdot S + 2(\lambda - 1)[e \cdot (bc)] \tag{24}$$

respectively

$$\begin{aligned} e \cdot \xi \cdot \phi e \cdot b \;&=\; \frac{1}{16\lambda}(\xi \cdot e \cdot \xi \cdot \xi \cdot S) + \frac{1-\lambda}{4}(\xi \cdot e \cdot S) + \\ &\quad + 2b(\xi \cdot e \cdot b) + 2(\lambda-1)c(e \cdot b). \end{aligned} \tag{25}$$

Differentiation of the relation (7) along e implies

$$\frac{1}{16\lambda}(e\cdot\xi\cdot\xi\cdot\xi\cdot S)=\frac{\lambda^2-1}{4\lambda}(e\cdot\xi\cdot S). \tag{26}$$

Adding (25) and (26) and using Lemma 8 we obtain

$$\begin{aligned}\xi\cdot e\cdot\phi e\cdot b \;=\; & \frac{\lambda+1}{16\lambda}(\phi e\cdot\xi\cdot\xi\cdot S)+\frac{\lambda^2-1}{4\lambda}(e\cdot\xi\cdot S)+\\ & +\frac{1-\lambda}{4}(\xi\cdot e\cdot S)+2b(\xi\cdot e\cdot b)+2(\lambda-1)c(e\cdot b).\end{aligned} \tag{27}$$

Subtraction of (24) from (27) yields

$$\begin{aligned}(\lambda+1)\phi e\cdot\phi e\cdot b \;=\; & \frac{\lambda+1}{16\lambda}(\phi e\cdot\xi\cdot\xi\cdot S)+\frac{1-\lambda^2}{4}(\phi e\cdot S)+\\ & +2b(\xi\cdot e\cdot b)+2(1-\lambda)b(e\cdot c).\end{aligned} \tag{28}$$

On the other hand differentiation of (17) with respect to ϕe using $\lambda^2\neq 1$ (since $l\neq 0$) implies

$$(\lambda+1)(\phi e\cdot\phi e\cdot b)=\frac{\lambda+1}{16\lambda}(\phi e\cdot\xi\cdot\xi\cdot S)+\frac{1-\lambda^2}{4}(\phi e\cdot S)+2(\lambda+1)b(\phi e\cdot b).$$

Comparing the above relation with (28) we obtain

$$b=0,\quad \xi\cdot e\cdot b=(\lambda+1)\phi e\cdot b+(\lambda-1)e\cdot c. \tag{29}$$

If $b=0$ Lemma 10 implies $c=0$, therefore from Remark 11 we obtain $Q\phi=\phi Q$. In this case it has been proved ([5]) that $S=constant$, which means that (16) and (19) are trivial ($\Phi=0$).

Differentiating (18) first with respect to ξ (respectively to ϕe) and secondly with respect to ϕe (respectively to ξ) and following the technique used to prove the relation (29) we can show that either $Q\phi=\phi Q$ or

$$\xi\cdot\phi e\cdot c=(\lambda+1)\phi e\cdot b+(\lambda-1)e\cdot c. \tag{30}$$

We suppose that the second of (29) and (30) hold on M^3.

Using (6), (17) and (18) we obtain

$$\xi\cdot e\cdot b=\xi\cdot\phi e\cdot c=\frac{1}{8\lambda}(\xi\cdot\xi\cdot S)+\xi\cdot(bc).$$

From the above relation and (23) the relations (16) and (19) follow at once.

Now we compute $[e,\phi e]b$ (respectively $[e,\phi e]c$) in two ways, first using (16) and (17) (respectively (18), (19)) as $e\cdot\phi e\cdot b-\phi e\cdot e\cdot b$ (respectively $e\cdot\phi e\cdot c-\phi e\cdot e\cdot c$), and secondly through (4), (6), (16) and (17) as $(\nabla_e\phi e-\nabla_{\phi e}e)b$ (respectively (4), (6), (18) and (19) as $(\nabla_e\phi e-\nabla_{\phi e}e)c$). Comparing the two resulting expressions we obtain (22) (respectively (21)). □

Theorem 15. *Let M^3 be a 3-dimensional contact metric manifold with constant ξ-sectional curvature k and constant ϕ-sectional curvature m. Then, one of the following conditions holds:*

(i) M^3 *is Sasakian,* (ii) $Q\phi = \phi Q$, *and* $m = -k$,
(iii) $l = 0$, (iv) $k + m = \frac{2}{3}$, (v) $k + m = -2$.

Proof. We suppose that M^3 is a non-Sasakian manifold with l being not identically equal to zero.

It is known ([5]) that on every 3-dimensional contact metric manifold $K(e, \phi e) = \frac{S}{2} - Trl$. Hence, this relation and Proposition 2 imply that $S = constant$. In this case the relations (16), (17), (18), (19), (21) and (22) take the form:

$$e \cdot b = bc + \Phi, \tag{31}$$

$$\phi e \cdot b = b^2 + \frac{1-\lambda}{2}(\lambda^2 + \frac{S}{2} - 1), \tag{32}$$

$$e \cdot c = c^2 + \frac{1+\lambda}{2}(\lambda^2 + \frac{S}{2} - 1), \tag{33}$$

$$\phi e \cdot c = bc - \Phi, \tag{34}$$

$$e \cdot \Phi = (\lambda + 1)(\lambda^2 + \frac{S}{2} - 3)b + 4c\Phi, \tag{35}$$

$$\phi e \cdot \Phi = (\lambda - 1)(\lambda^2 + \frac{S}{2} - 3)c + 4b\Phi. \tag{36}$$

Computing $[e, \phi e]\Phi$ in two different ways (as in the last part of the proof of Proposition 14), using (4), (20), (35) and (36) we obtain

$$8\Phi^2 = (\lambda^2 + \frac{S}{2} - 3)[-4(\lambda+1)b^2 + 4(\lambda-1)c^2 + (1-\lambda^2)(\lambda^2 + \frac{S}{2} - 1)]. \tag{37}$$

Differentiating (37) with respect to e (respectively to ϕe) and taking into account (31), (33), (35) and (37) (respectively (32), (34), (36) and (37)) we have

$$(\lambda^2 + \frac{S}{2} - 3)[-(\lambda+1)b^2c + (\lambda-1)c^3 + \frac{1-\lambda^2}{2}(\lambda^2 + \frac{S}{2} - 1)c + (\lambda+1)b\Phi] = 0,$$

$$(\lambda^2 + \frac{S}{2} - 3)[-(\lambda+1)b^3 + (\lambda-1)bc^2 + \frac{1-\lambda^2}{2}(\lambda^2 + \frac{S}{2} - 1)b + (\lambda-1)c\Phi] = 0.$$

Hence, either

$$\lambda^2 + \frac{S}{2} - 3 = 0,$$

or

$$(\lambda+1)b\Phi = c[(\lambda+1)b^2 + (1-\lambda)c^2 + \frac{\lambda^2-1}{2}(\lambda^2 + \frac{S}{2} - 1)] = 0 \tag{38}$$

and

$$(\lambda-1)c\Phi = b[(\lambda+1)b^2 + (1-\lambda)c^2 + \frac{\lambda^2-1}{2}(\lambda^2 + \frac{S}{2} - 1)] = 0. \tag{39}$$

Suppose that $\lambda^2+\frac{S}{2}-3=0$, then using $K(e,\phi e) = \frac{S}{2} - Trl$, $Trl = 2(1-\lambda^2)$ and $K(e,\xi) = \frac{Trl}{2}$, we obtain $k+m=-2$.

In this case using [16] we conclude that if $k=-3$ and $m=1$, then M^3 is Sasakian. Also, for $k+m=-2$ and $m>1$ we obtain a new class of contact metric 3-manifolds, which does not belong to the (κ,μ)-contact metric manifolds, ([4]).

Suppose now that (38) and (39) hold. If $b=0$ (respectively $c=0$), then (6) implies $c=0$ (respectively $b=0$) and therefore $Q\phi=\phi Q$. In this case using [5] we have $m=-k$. If $bc\neq 0$, multiplying (38) with b and (39) with c we obtain

$$\Phi[(\lambda+1)b^2 + (1-\lambda)c^2] = 0.$$

Case A: $\Phi = 0$.

The relation (37) yields

$$(\lambda+1)b^2 + (1-\lambda)c^2 + \frac{\lambda^2-1}{4}(\lambda^2 + \frac{S}{2} - 1) = 0.$$

On the other hand the relation (38) yields

$$(\lambda+1)b^2 + (1-\lambda)c^2 + \frac{\lambda^2-1}{2}(\lambda^2 + \frac{S}{2} - 1) = 0.$$

Comparing the last two relations we obtain either $\lambda^2=1$, a contradiction because of the assumption that l is not identically equal to zero on M^3, or

$$\lambda^2 + \frac{S}{2} - 1 = 0.$$

From $\Phi=0$, (31), (32), (33) and (34) we obtain

$$e\cdot b = \phi e\cdot c = bc, \quad \phi e\cdot b = b^2, \quad \phi e\cdot c = c^2. \tag{40}$$

Computing $[e,\phi e]b$ in two ways (by use of (4) and (40)) and comparing the results we obtain $\xi\cdot b=0$. Hence, from the assumption $\lambda^2\neq 1$ and (6) we obtain $b=c=0$, a contradiction.

Case B:

$$\Phi \neq 0 \quad and \quad (\lambda+1)b^2 + (1-\lambda)c^2 = 0. \tag{41}$$

The relations (38), (39) and (41) with the assumption $\lambda^2 \neq 1$ yield

$$b\Phi = \frac{\lambda-1}{2}(\lambda^2 + \frac{S}{2} - 1)c, \tag{42}$$

$$c\Phi = \frac{\lambda+1}{2}(\lambda^2 + \frac{S}{2} - 1)b. \tag{43}$$

On the other hand (37) and (41) imply

$$8\Phi^2 = (\lambda^2 + \frac{S}{2} - 3)(1-\lambda^2)(\lambda^2 + \frac{S}{2} - 1).$$

Hence, Φ =constant. This conclusion and the relations (35) and (36) yield

$$4b\Phi = (1-\lambda)(\lambda^2 + \frac{S}{2} - 3)c, \tag{44}$$

$$4c\Phi = -(\lambda+1)(\lambda^2 + \frac{S}{2} - 3)b. \tag{45}$$

Comparing (42) with (44) or (43) with (45) we obtain

$$\lambda^2 + \frac{S}{2} = \frac{5}{3}.$$

Taking into account the last relation, $K(e,\phi e) = \frac{S}{2} - Trl$, $Trl = 2(1-\lambda^2)$ and $K(e,\xi) = \frac{Trl}{2}$, we obtain $k + m = \frac{2}{3}$.

In this case using [16] we conclude that if $k = 1$ and $m = -\frac{1}{3}$, then M^3 is Sasakian. Also, for $k + m = \frac{2}{3}$ and $m > -\frac{1}{3}$ we obtain a new class of contact metric 3-manifolds, which does not belong to the (κ,μ)-contact metric manifolds, ([4]). □

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