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Polytopal Linear Algebra

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Abstract. We investigate similarities between the category of vector spaces and that of polytopal algebras, containing the former as a full subcategory. In Section 2 we introduce the notion of a polytopal Picard group and show that it is trivial for fields. The coincidence of this group with the ordinary Picard group for general rings remains an open question. In Section 3 we survey some of the previous results on the automorphism groups and retractions. These results support a general conjecture proposed in Section 4 about the nature of arbitrary homomorphisms of polytopal algebras. Thereafter a further confirmation of this conjecture is presented by homomorphisms defined on Veronese singularities.
This is a continuation of the project started in [3, 4, 5]. The higher K-theoretic aspects of polytopal linear objects will be treated in [6, 7].

1. Introduction

The present work is a continuation of our study of the similarities between the categories $\mathrm{Vect}(k)$ – the category of finitely generated vector spaces over a field k, and its natural extension $\mathrm{Pol}(k)$ – the *polytopal linear category* over k, started in the series of papers [3, 4, 5]. The category $\mathrm{Pol}(k)$ was first introduced explicitly in [5] where we studied a special class of morphisms, the retractions.

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We recall that the objects of $\mathrm{Pol}(k)$ are by definition polytopal k-algebras (discussed in detail in [8]), i. e. the standard graded k-algebras $k[P]$ associated to arbitrary finite convex lattice polytopes $P \subset \mathbb{R}^d$ in the following way: the lattice points $\mathrm{L}_P = \mathbb{Z}^d \cap P$ form degree one generators of $k[P]$ and they are subject to the binomial relations coming from the affine dependencies inside P.

Alternatively, $k[P]$ is the semigroup ring $k[S_P]$ of the semigroup $S_P \subset \mathbb{Z}^d$ generated by $\{(x, 1) \mid x \in \mathrm{L}_P\}$.

The embedding $\mathrm{Vect}(k) \subset \mathrm{Pol}(k)$, resulting from viewing a vector space V as the degree one component of its symmetric algebra $\mathrm{S}_k(V) = k[X_1, \ldots, X_{\dim_k V}]$, makes $\mathrm{Vect}(k)$ a full subcategory of $\mathrm{Pol}(k)$. Obviously, the latter category is far from being additive but it reveals many surprising similarities with $\mathrm{Vect}(k)$.

This work provides further results supporting the analogy. In particular, in Section 2 we show that *polytopal Picard groups* defined as the groups of certain autoequivalences of $\mathrm{Pol}(k)$, are trivial (i. e. coincide with $\mathrm{Pic}(k)$). We work exclusively over a field k, which is often assumed to be algebraically closed. But in Remark 2.1 we indicate an approach that enables one to use arbitrary commutative rings. In the general definition of the categories $\mathrm{Vect}(k)$ and $\mathrm{Pol}(k)$ the hom-sets are replaced by appropriate affine schemes. This definition must already be used for fields in general. The description of $\mathrm{Pic}^{\mathrm{Pol}}(R)$ for general (commutative) rings R remains an open question.

In order to present a coherent picture of polytopal linear algebra and to ease references throughout the text, we recall some of the results from [3] and [4] in Section 3; they concern the automorphism groups and the retractions in $\mathrm{Pol}(k)$. In Section 4 we propose a conjecture describing arbitrary homomorphisms in $\mathrm{Pol}(k)$. Roughly, it says that the homomorphisms are obtained from the trivial ones by a sequence of standard procedures encoded in the shapes of the underlying polytopes. In particular, the arithmetic of k is irrelevant in the description of $\mathrm{Pol}(k)$ because everything is determined on the combinatorial level.

The results obtained so far [3, 4] can be viewed as a confirmation of refined versions of this conjecture for special classes of morphisms, namely automorphisms and retractions.

Thereafter in Section 4 we provide further evidence towards our conjecture by an explicit description of homomorphisms from Veronese subalgebras of polynomial rings. This result, in conjunction with the results from [4], provides a complete description of the variety of idempotent endomorphisms of $k[P]$ when k is algebraically closed and P is a lattice polygon (i. e. $\dim P = 2$).

The approach developed in [5] suggests a further generalization to the even more general category of *polyhedral algebras* and their graded homomorphisms. This corresponds to the passage from single polytopes to *lattice polyhedral complexes* in the sense of [5]. However, in this article we do not pursue such level of generality and only remark that even the subclass of simplicial complexes (i. e. the category of Stanley-Reisner rings and their graded homomorphisms) provides interesting possibilities for the generalization of linear algebra.

The arguments in Section 2 below use results presented in Section 3. But we resort to this order of exposition in analogy with the classical hierarchy – Picard groups, retractions, automorphisms. The objects just listed constitute the subject of 'classical' algebraic K-theory (Bass [2]). In [6, 7] we consider higher K-theoretical aspects of the polytopal generalization of vector spaces.

For the standard terminology in category theory we refer to MacLane [15].

As usual, *standard graded k-algebra* means a graded k-algebra $k \oplus A_1 \oplus A_2 \oplus \cdots$, generated by A_1. In what follows *homomorphism* always means *graded homomorphism*.

For a semigroup S its group of differences (the universal group of S) will be denoted by $\mathrm{gp}(S)$.

Finally, we are grateful to the referee for pointing out to us the references [13] and [17].

2. Polytopal Picard groups

Assume k is a field. Then the group of covariant k-linear autoequivalences of $\mathrm{Vect}(k)$, modulo functor isomorphisms, is a trivial group. Here a functor $F : \mathrm{Vect}(k) \to \mathrm{Vect}(k)$ is called *k-linear* if the mappings

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W)), \quad V, W \in |\,\mathrm{Vect}(k)|,$$

are k-linear homomorphisms of vector spaces. This triviality follows from the fact that the mentioned group is naturally isomorphic to the Picard group $\mathrm{Pic}(k)$ (=0) – an observation valid for any commutative ring R. More precisely, the assignments $F \mapsto F(R)$ and $L \mapsto L \otimes -$ for $F : \mathbb{M}(R) \to \mathbb{M}(R)$ and $L \in \mathrm{Pic}(R)$ establish an isomorphism between the group of R-linear covariant autoequivalences of $\mathbb{M}(R)$ – the category of finitely generated R-modules, modulo functor isomorphisms, and $\mathrm{Pic}(R)$ – the group of invertible R-modules up to isomorphism [1].

If k is an algebraically closed field then the condition on a functor $F : \mathrm{Vect}(k) \to \mathrm{Vect}(k)$ to be k-linear is equivalent to the requirements that the mappings between affine spaces

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W)), \quad V, W \in |\,\mathrm{Vect}(k)|,$$

are algebraic and, simultaneously, k^*-equivariant with respect to the action

$$k^* \times \mathrm{Hom}(V, W) \mapsto \mathrm{Hom}(V, W), \quad (a, \varphi) \mapsto a\varphi.$$

In the category $\mathrm{Pol}(k)$ both these requirements make sense (under the assumption k is algebraically closed). In fact, the sets $\mathrm{Hom}(k[P], k[Q])$ carry natural k-variety structures as follows. An element $f \in \mathrm{Hom}(k[P], k[Q])$ can be identified with the corresponding matrix

$$(a_{ij}) \in M_{m \times n}(k), \quad f(x_i) = \sum_{j=1}^{n} a_{ij} y_j, \qquad x_i \in \mathrm{L}_P,\ y_j \in \mathrm{L}_Q\,.$$

Then the equations, defining the Zariski closed subset

$$\mathrm{Hom}(k[P], k[Q]) \subset \mathbb{A}_k^{mn}, \qquad m = \#\,\mathrm{L}_P,\ n = \#\,\mathrm{L}_Q,$$

are derived by the following procedure. The binomial relations between the x_i are preserved by the $f(x_i) \in k[Q]_1$ (the degree 1 component of $k[Q]$). After passing to the canonical k-linear expansions as linear forms of monomials in the y_j and comparing corresponding coefficients (at this point the binomial dependencies between the y_j are used) we get the desired system of homogeneous equalities $F_s = 0$, $F_s \in k[X_1, \ldots, X_{mn}]$.

Observation. Clearly, we could derive similarly certain homogeneous polynomials $G_t \in \mathbb{Z}[X_1, \dots, X_{mn}]$ by substituting $\mathbb{Z}$ for the field k. Then the polynomials F_s are just specializations of the G_t under the canonical ring homomorphism $\mathbb{Z}[X_1, \dots, X_{mn}] \to k[X_1, \dots, X_{mn}]$. In particular, the varieties $\operatorname{Hom}(k[P], k[Q])$ are defined over $\mathbb{Z}$ and the corresponding defining integral equations only depend on the polytopes P and Q.

As for the k^*-equivariant structure, we observe that any object $A \in |\operatorname{Pol}(k)|$ is naturally equipped with the following k^*-action:

$$(a_0 \oplus a_1 \oplus a_2 \oplus \cdots)^\xi = a_0 \oplus \xi a_1 \oplus \xi^2 a_2 \oplus \cdots,$$

which induces the algebraic action

$$k^* \times \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B)$$

as follows:

$$f^\xi(a) = f(a^\xi), \qquad f \in \operatorname{Hom}(A, B),\ \xi \in k^*,\ a \in A.$$

Notice that we obtain the same action on $\operatorname{Hom}(A, B)$ by requiring

$$f^\xi(a) = \xi f(a) \text{ for all } f \in \operatorname{Hom}(A, B),\ \xi \in k^*,\ a \in A_1.$$

It is natural to ask whether the group of the covariant autoequivalences (up to functor isomorphism) of $\operatorname{Pol}(k)$, for which the mappings

$$\operatorname{Hom}(A, B) \to \operatorname{Hom}(F(A), F(B)), \quad A, B \in |\operatorname{Pol}(k)|$$

respect the k-variety structures and are k^*-equivariant, is trivial. For short, is the 'polytopal Picard group' $\operatorname{Pic}^{\operatorname{Pol}}(k)$ trivial?

Remark 2.1. We can define the category $\operatorname{Pol}(R)$ for any ring R as follows. It is the category enriched on the (symmetric) monoidal category of affine $\operatorname{Spec}(R)$-schemes whose objects are the polytopal algebras over R and the hom-schemes $\operatorname{Hom}(R[P], R[Q])$ are the affine schemes $\operatorname{Spec}(R^{mn}/(G_t))$. Here the Q_t are the same polynomials as in the observation above. (For the generalities on enriched categories see [12].)

In order to simplify the notation we will consider the underlying rings instead of the affine schemes. Thus $\operatorname{Hom}(R[P], R[Q]) = R^{mn}/(G_t)$. The equivariant structure on the hom-schemes is encoded into the ring homomorphisms

$$\operatorname{Hom}(R[P], R[Q]) \to \operatorname{Hom}(R[P], R[Q])[X, X^{-1}], \qquad a \mapsto Xa,$$

and the composition operation is given by the naturally defined ring homomorphisms

$$\operatorname{Hom}(R[P], R[L]) \to \operatorname{Hom}(R[P], R[Q]) \otimes \operatorname{Hom}(R[Q], R[L]).$$

Clearly, if $R = k$ is an algebraically closed field, the two definitions of $\operatorname{Pol}(R)$ are equivalent.

Now we can define $\operatorname{Pic}^{\operatorname{Pol}}(R)$ as the group of those covariant autoequivalences of $\operatorname{Pol}(R)$ which on the hom-rings induce ring homomorphisms respecting the $\operatorname{Spec}(\mathbb{Z}[X, X^{-1}])$-equivariant structures. In particular, we have defined $\operatorname{Pic}^{\operatorname{Pol}}(k)$ for general fields.

The proof we present below for algebraically closed fields yields the equality $\mathrm{Pic}^{\mathrm{Pol}}(k) = 0$ for arbitrary fields. We leave this to the interested reader and only remark that the crucial fact is that the main result of [3] (Theorem 3.2 below) has been proved for general fields.

The lack of an analogous description of the automorphism groups over a general ring of coefficients is the obstacle in describing the group $\mathrm{Pic}^{\mathrm{Pol}}(R)$. We expect that this is a trivial group, being a polytopal counterpart of the group of R-linear autoequivalences (up to functor isomorphism) of the category of finitely generated free R-modules – a trivial group.

Remark 2.2. The last step in the proof of Theorem 2.3 uses the fact from [8] that for a polytope P and a field k the polytopal algebra $k[cP]$ is quadratically defined (i. e. by degree 2 equations) whenever $c \geq \dim P$. This however does not create an additional difficulty in generalizing the result on polytopal Picard groups from fields to arbitrary rings. It is an elementary fact that the condition on a polytopal ring to be quadratically defined depends only on the combinatorial structure of the polytope.

In the remaining part of this section k is an algebraically closed field.

Theorem 2.3. $\mathrm{Pic}^{\mathrm{Pol}}(k) = 0$.

Proof. Let $[F] \in \mathrm{Pic}^{\mathrm{Pol}}(k)$. We want to show that there are isomorphisms $\big(\sigma_A : A \to F(A)\big)_{|\,\mathrm{Pol}(k)|}$ such that

$$F(f) = \sigma_B \circ f \circ \sigma_A^{-1}$$

for every homomorphism $f : A \to B$ in $\mathrm{Pol}(k)$.

First of all notice that we can work on an arbitrarily fixed skeleton of $\mathrm{Pol}(k)$: all the notions we are dealing with are invariant under such a passage. Henceforth $\mathrm{Pol}(k)$ is the fixed skeleton. By [14] one knows that $k[P]$ determines (up to an affine integral isomorphism) the polytope P (this is so even in the category of all commutative k-algebras). Therefore, we assume that for each object $A \in |\,\mathrm{Pol}(k)|$ there is a unique polytope P such that $A = k[P]$ and different polytopal algebras determine non-isomorphic polytopes. By a suitable choice of the skeleton we can also assume that the objects of $\mathrm{Vect}(k)$ are of the type $k[\Delta_n]$ for the unit n-simplices Δ_n, $n = -1, 0, 1, 2, \ldots$ (by convention $k[\Delta_{-1}] = k$). Also, we will use the notation $k[t] = k[\Delta_0]$.

Step 1. We claim that

(i) $\dim A = \dim F(A)$ (Krull dimension),

(ii) $\dim_k A_1 = \dim_k F(A)_1$ (k-ranks of the degree 1 components),

(iii) $F(f)$ is surjective (the degree 1 component $F(f)_1$ is injective) if and only if f is surjective (f_1 is injective).

In fact, it follows from Theorem 3.2 below that for any polytopal algebra $k[P]$ its Krull dimension is the dimension of a maximal torus of the linear subgroup

$$\Gamma_k(P) = \mathrm{Aut}_{\mathrm{Pol}(k)}(k[P]) \subset \mathrm{Aut}_k k[P]_1;$$

it is certainly an invariant of F – hence (i).

The claims (ii) and (iii) follow from the observation that both the injectivity and surjectivity conditions can be reformulated in purely categorical terms by using morphisms originating from $k[t]$.

Step 2. Observe that F restricts to an autoequivalence of $\mathrm{Vect}(k)$. This follows from the claims (i) and (ii) in Step 1 and the fact that polynomial algebras are the only polytopal algebras whose Krull dimensions coincide with the k-ranks of the degree 1 components.

Next we correct F on $\mathrm{Vect}(k)$ in such a way that $F|_{\mathrm{Vect}(k)} = \mathbf{1}_{\mathrm{Vect}(k)}$.

We know that $F|_{\mathrm{Vect}(k)} \approx \mathbf{1}_{\mathrm{Vect}(k)}$. This means that there are elements $\rho_V \in \mathrm{Aut}(V)$ for all $V \in |\,\mathrm{Vect}(k)|$ such that

$$\forall\, U, V \in |\,\mathrm{Vect}(k)| \quad (f : U \to V) \mapsto (\rho_V \circ f \circ \rho_U^{-1} : U \to V).$$

For each $A \in |\,\mathrm{Pol}(k)| \setminus |\,\mathrm{Vect}(k)|$ put $\rho_A = \mathbf{1}_A$ and let $H : \mathrm{Pol}(k) \to \mathrm{Pol}(k)$ be the autoequivalence determined as follows: it is the identity on the objects and

$$H(f) = \rho_B^{-1} \circ f \circ \rho_A$$

for a morphism $f : A \to B$ in $\mathrm{Pol}(k)$. Then H represents the neutral element of $\mathrm{Pic}^{\mathrm{Pol}}(k)$. Now the functor $G = H \circ F$ is isomorphic to F and it restricts to the identity functor on $\mathrm{Vect}(k)$.

Without loss of generality we can therefore assume $F|_{\mathrm{Vect}_k} = \mathbf{1}_{\mathrm{Vect}_k}$.

Step 3. For any positive dimensional object $k[P] \in |\,\mathrm{Pol}(k)|$ we fix a bijective mapping Δ_P from the set of vertices of Δ_{n-1}, $n = \#\,\mathrm{L}_P$ to the set of lattice points of P. The resulting surjective homomorphism $k[\Delta_{n-1}] \to k[P]$ will also be denoted by Δ_P.

Every matrix $\alpha \in \mathrm{GL}_n(k)$ gives rise to a graded k-surjective homomorphism

$$\alpha\Delta_P : k[\Delta_{n-1}] \to k[P]$$

whose degree 1 component is given by

$$(\alpha\Delta_P)_1 = (\Delta_P)_1 \circ \alpha_*,$$

where α_* is the linear transformation of the k-vector space $k[\Delta_{n-1}]_1$ determined by α, and $(\Delta_P)_1$ is the degree 1 component of Δ_P.

By Step 2 $F(k[\Delta_{n-1}]) = k[\Delta_{n-1}]$ for all $n \in \mathbb{N}$. Therefore, for a polytope P, satisfying the condition $F(k[P]) = k[P]$, the claim (iii) in Step 1 implies the following commutative square

$$\begin{array}{ccc} \mathrm{Hom}(k[\Delta_{n-1}], k[P]) & \xrightarrow{F} & \mathrm{Hom}(k[\Delta_{n-1}], k[P]) \\ \uparrow & & \uparrow \\ \mathrm{Sur}(k[\Delta_{n-1}], k[P]) & \xrightarrow{F} & \mathrm{Sur}(k[\Delta_{n-1}], k[P]) \end{array}$$

where $\mathrm{Sur}(k[\Delta_{n-1}], k[P])$ denotes the set of surjective homomorphisms. By the definition of $\mathrm{Pic}^{\mathrm{Pol}}(k)$ the horizontal mappings are k^*-equivariant automorphisms of k-varieties.

We have the following obvious equalities:

$$\begin{aligned}\operatorname{Hom}(k[\Delta_{n-1}], k[P]) &= \{\gamma\Delta_P \mid \gamma \in M_{n\times n}(k)\},\\ \operatorname{Sur}(k[\Delta_{n-1}], k[P]) &= \{\gamma\Delta_P \mid \gamma \in \mathrm{GL}_n(k)\}.\end{aligned}$$

After the appropriate identifications with $M_{n\times n}(k)$ and $\mathrm{GL}_n(k)$ respectively we arrive at the commutative square of k-varieties

$$\begin{array}{ccc} M_{n\times n}(k) & \xrightarrow{F} & M_{n\times n}(k) \\ \uparrow & & \uparrow \\ \mathrm{GL}_n(k) & \xrightarrow{F} & \mathrm{GL}_n(k) \end{array}$$

whose horizontal arrows are algebraic k^*-equivariant automorphisms for the diagonal k^*-actions. Thus the upper horizontal mapping is a linear non-degenerate transformation of the k-vector space $M_{n\times n}(k)$, which leaves the subset $\mathrm{GL}_n(k) \subset M_{n\times n}(k)$ invariant, i. e. the matrix degeneracy locus $M_{n\times n}(k) \setminus \mathrm{GL}_n(k)$ is invariant under F. Then by Proposition 3.3 below there are only two possibilities:

(a) there are $\alpha, \beta \in \mathrm{GL}_n(k)$ such that $F(\gamma\Delta_P) = (\beta\gamma\alpha)\Delta_P$ for all $\gamma \in M_{n\times n}(k)$, or

(b) there are $\alpha, \beta \in \mathrm{GL}_n(k)$ such that $F(\gamma\Delta_P) = (\beta\gamma^T\alpha)\Delta_P$ for all $\gamma \in M_{n\times n}(k)$, where $-^T$ is the transposition.

Consider the commutative square

$$\begin{array}{ccc} k[\Delta_{n-1}] & \xrightarrow{\gamma_*} & k[\Delta_{n-1}] \\ {\scriptstyle \gamma\Delta_P}\downarrow & & \downarrow{\scriptstyle \Delta_P}, \\ k[P] & = & k[P], \end{array}$$

where $\gamma \in M_{n\times n}(k)$ is arbitrary matrix and the degree 1 component of the upper horizontal mapping is the linear transformation given by γ. By applying the functor F to this square and using the equality $F|_{\mathrm{Vect}(k)} = \mathbf{1}_{\mathrm{Vect}(k)}$ (Step 2) we arrive at the following equalities in the corresponding cases:

$$\text{(a)}\quad (\beta\gamma\alpha)\Delta_P = (\beta\alpha)\Delta_P \circ \gamma_*, \qquad \text{(b)}\quad (\beta\gamma^T\alpha)\Delta_P = (\beta\alpha)\Delta_P \circ \gamma_*.$$

Identifying $\mathrm{L}_{\Delta_{n-1}}$ and L_P along Δ_P we get the matrix equalities:

$$\text{(a)}\quad \beta\gamma\alpha = \gamma\beta\alpha, \qquad \text{(b)}\quad \beta\gamma^T\alpha = \gamma\beta\alpha.$$

First notice that case (b) is excluded, i. e. there is no matrix $\beta \in \mathrm{GL}_n(k)$ for which the following holds:

$$\forall\, \gamma \in M_{n\times n}(k) \quad \beta\gamma\beta^{-1} = \gamma^T.$$

This follows from running γ through the set of standard basic matrices (i. e. the matrices with only one entry 1 and 0s elsewhere).

For case (a) we have

$$\forall\ \gamma \in M_{n\times n}(k) \quad \beta\gamma\beta^{-1} = \gamma.$$

Then β is in the center of $M_{n\times n}(k)$ (in particular, it is a scalar matrix). So we can write

$$F(\gamma\Delta_P) = (\gamma\beta\alpha)\Delta_P$$

We arrive at the

Claim. For each lattice polytope P with $F(k[P]) = k[P]$, there is $\alpha_P \in \mathrm{GL}_n(k)$, $n = \#\,\mathrm{L}_P$, such that

$$\forall\ \gamma \in\ M_{n\times n}(k) \quad F(\gamma\Delta_P) = (\gamma\alpha_P)\Delta_P.$$

Step 4. Now we show that for every polytope P the linear automorphism $(\alpha_P)_*$ of the k-vector space $k[P]_1$, determined by the matrix α_P (Step 3), belongs to the closed subgroup

$$\Gamma_k(P) \subset \mathrm{Aut}_k(k[P]_1),$$

provided $F(k[P]) = k[P]$.

The functor F induces a k^*-equivariant automorphism of the k-variety $\mathrm{Hom}(k[P], k[t])$. On the other hand we have the natural identification

$$\mathrm{Hom}(k[P], k[t]) = \mathrm{maxSpec}(k[P]),$$

where the right hand side denotes the variety of closed points of $\mathrm{Spec}(k[P])$. Therefore, there exists an automorphism ψ of the k-algebra $k[P]$ – *not a priori* graded, such that the mapping

$$F : \mathrm{Hom}(k[P], k[t]) \to \mathrm{Hom}(k[P], k[t])$$

is given by $\varphi \mapsto \varphi\circ\psi$, and, moreover, it is k^*-equivariant. It follows that ψ is a k^*-equivariant automorphism of $k[P]$. It is easily seen that a k^*-equivariant automorphism is graded (and conversely). Therefore $\psi_1 \in \Gamma_k(P)$. (Here we identify elements of $\Gamma_k(P)$ with their degree one components, which are linear automorphisms of $\bigoplus_{\mathrm{L}_P} k$.)

We let $P^* : k[P] \to k[t]$ denote the homomorphism which sends L_P to t. For any toric automorphism $\tau \in \mathbb{T}_k(P)$ (i.e. an automorphism for which any element of L_P is an eigenvector, $\mathbb{T}_k(P)$ is the group of such automorphisms, see Section 3) we have the commutative diagram

$$\begin{array}{ccc}
k[\Delta_{n-1}] & \xrightarrow{(\Delta_{n-1})^*} & k[t] \\
{\scriptstyle \tau^*\Delta_P}\downarrow & & \uparrow{\scriptstyle P^*} \\
k[P] & \xrightarrow[\tau^{-1}]{} & k[P]
\end{array}$$

where τ^* refers to the diagonal $n \times n$-matrix corresponding to the degree 1 component of τ.

In view of what has been said above and of Step 3, an application of F to the last commutative diagram yields the equality

$$(\Delta_{n-1})^* = (P^* \circ \tau^{-1} \circ \psi) \circ \bigl((\tau^* \alpha_P)\Delta_P\bigr).$$

Let $\psi^* \in M_{n\times n}(k)$ be the matrix of the degree 1 component of ψ, and put $\omega = \alpha_P \psi^*$. Then the equality can be reformulated into the condition:

(∗) for any $\tau \in \mathbb{T}_k(P)$ the sum of the entries of each row in the matrix $\tau^*\omega(\tau^*)^{-1}$ is 1, that is
$$\bigl((\tau^{-1} \circ \omega_* \circ \tau)(x_i) = \textstyle\sum_{j=1}^n c_{ij} x_j\bigr) \implies \bigl(\textstyle\sum_{j=1}^n c_{ij} = 1\bigr),$$
where ω_* is the linear transformation of $k[P]_1$ determined by ω, and $\mathrm{L}_P = \{x_1, \dots, x_n\}$.

Now we derive from (∗) that $\omega_* = \mathbf{1}_{k[P]_1}$, that is $(\alpha_P)_* = \psi^{-1} \in \Gamma_k(P)$. Fix $i \in [1, n]$ and $\tau \in \mathbb{T}_k(P)$. We put

$$\omega_*(x_i) = \sum_{j=1}^n a_j x_j$$

and

$$\tau(x_j) = b_j x_j, \qquad j \in [1, n].$$

Then

$$(\tau^{-1} \circ \omega_* \circ \tau)(x_i) = \sum_{j=1}^n \frac{b_i a_j}{b_j} x_j.$$

Consider the Laurent polynomial

$$l_i = \sum_{j=1}^n a_j \frac{x_j}{x_i} \in k[\mathbb{Z}^d] \ (= k[z_1, z_1^{-1}, \dots, z_d, z_d^{-1}]).$$

Without loss of generality we assume that $d = \dim P$ and $\mathrm{gp}(S_P) = \mathbb{Z}^{d+1}$.

Observe that the assignment

$$\frac{x_j}{x_i} \mapsto \frac{b_i a_j}{b_j} \cdot \frac{x_j}{x_i}, \qquad j \in [1, n],$$

extends uniquely to a toric automorphism of $k[\mathbb{Z}^d]$ (the torus $(k^*)^d$ acts tautologically on its coordinate ring $k[\mathbb{Z}^d]$). Conversely, any toric automorphism of $k[\mathbb{Z}^d]$ can be obtained in this way from some element of $\mathbb{T}_k(P)$. Therefore

$$\bigl\{\xi(l_i) \mid \xi \in (k^*)^d\bigr\} = \bigl\{x_i^{-1}((\tau^{-1} \circ \omega_* \circ \tau)(x_i)) \mid \tau \in \mathbb{T}_k(P)\bigr\}.$$

In particular, the sum of the coefficients of the Laurent polynomial $\xi(l_i)$ is 1 for any $\xi \in (k^*)^d$, in other words

$$l_i(\xi_1, \dots, \xi_d) = 1$$

for all $\xi_1, \dots, \xi_d \in k^*$. Taking into account the infinity of k we conclude $l_i = 1$, that is $\omega_* = \mathbf{1}_{k[P]_1}$.

Step 5. Let $\mathrm{Pol}(k)' \subset \mathrm{Pol}(k)$ denote the full subcategory whose objects are those polytopal algebras $k[P]$ for which $F(k[P]) = k[P]$. Now we show that there is $G \in \mathrm{Pic}^{\mathrm{Pol}}(k)$ such that $F \approx G$ and $G|_{\mathrm{Pol}(k)'} = \mathbf{1}_{\mathrm{Pol}(k)'}$.

As in Step 2, this means that we have to show the existence of elements

$$\sigma_P \in \Gamma_k(P), \qquad k[P] \in |\,\mathrm{Pol}(k)'|,$$

such that $F(f)_1 = \sigma_Q \circ f_1 \circ \sigma_P^{-1}$ for any morphism $f : k[P] \to k[Q]$ in $\mathrm{Pol}(k)'$. Consider the commutative diagram

$$\begin{array}{ccc} k[\Delta_{m-1}] & \xrightarrow{\tilde f} & k[\Delta_{n-1}] \\ \Big\downarrow{\scriptstyle \Delta_P} & & \Big\downarrow{\scriptstyle \Delta_Q} \\ k[P] & \xrightarrow[f]{} & k[Q] \end{array}$$

where f is any morphism in $\mathrm{Pol}(k)'$, $m = \#\,\mathrm{L}_P$, $n = \#\,\mathrm{L}_Q$, and $\tilde f$ is the unique lifting of f. By Step 3 we know that F transforms this square into the square

$$\begin{array}{ccc} k[\Delta_{n-1}] & \xrightarrow{\tilde f} & k[\Delta_{m-1}] \\ \Big\downarrow{\scriptstyle \alpha_P\Delta_P} & & \Big\downarrow{\scriptstyle \alpha_Q\Delta_Q} \\ k[P] & \xrightarrow[F(f)]{} & k[Q] \end{array}$$

Therefore, looking at the degree 1 components we conclude

$$F(f)_1 = (\alpha_Q)_* \circ f_1 \circ (\alpha_P)_*^{-1}.$$

So by Step (4) the system

$$\big\{(\alpha_P)_* \mid k[P] \in |\,\mathrm{Pol}(k)'|\big\}$$

is the desired one.

Step 6. By the previous step we can assume that $F|_{\mathrm{Pol}(k)'} = \mathbf{1}_{\mathrm{Pol}(k)'}$. Now we complete the proof by showing that this assumption implies $\mathrm{Pol}(k)' = \mathrm{Pol}(k)$, and therefore $F = \mathbf{1}_{\mathrm{Pol}(k)}$.

Assume P is a lattice polytope. We let $\square$ denote the unit square and consider all the possible integral affine mappings:

$$\mathcal{M} = \{\mu : \square \to P\}.$$

We define the diagram $\mathbb{D}(k, P)$ as follows. It consists of

(1) $\#(\mathrm{L}_P)$ copies of $k[t]$ indexed by the elements of $\mathrm{L}_P = \{x_1, \dots, x_n\}$,

(2) $\#(\mathcal{M})$ copies of $k[\square]$: $\{k[\square]_\mu\}_{\mathcal{M}}$,

and the following morphisms between them

$$k[t]_{x_i} \to k[\square]_\mu, \qquad t \mapsto z \in \mathrm{L}_\square, \text{ if } \mu(z) = x_i.$$

We claim that

$$\varinjlim \mathbb{D}(k,P) = k[P]$$

whenever the defining ideal of the toric ring $k[P]$ is generated by quadratic binomials, where the direct limit is considered in the category of all (commutative) k-algebras.

In fact, it is clear that the mentioned limit is always a standard graded k-algebra, whose degree 1 component has k-dimension equal to $\#\,\mathrm{L}_P$, and that there is a surjective graded k-homomorphism

$$\mathbb{L}(k,P) = \varinjlim \mathbb{D}(k,P) \to k[P].$$

This is so because of the cone over the diagram $\mathbb{D}(k,P)$ with vertex $k[P]$ where $t \in k[t]_{x_i}$ is mapped to x_i and the vertices of $\square$ from $k[\square]_\mu$ are mapped accordingly to μ. Therefore, we only need to make sure that $t_it_j - t_kt_l = 0$ in $\mathbb{L}(k,P)$ whenever $x_ix_j - x_kx_l = 0$ in $k[P]$, where t_i is the image of $t \in k[t]_{x_i}$ in $\mathbb{L}(k,P)$, and similarly for t_j, t_k and t_l. But if $x_ix_j - x_kx_l = 0$ in $k[P]$ then these four points in P belong to $\mathrm{Im}(\mu)$ for some $\mu \in \mathcal{M}$ and the desired equality is encoded into the diagram $\mathbb{D}(k,P)$.

Having established the equality above for quadratically defined polytopal rings we now show that $k[cP] \in |\,\mathrm{Pol}(k)'|$ for every lattice polytope P and all $c \geq \dim P$. (Here cP denotes the cth homothetic multiple of P.)

By Theorem 1.3.3(a) in [8] the defining ideal of the toric ring $k[cP]$ is generated by quadratic binomials for $c \geq \dim P$. Therefore,

$$\mathbb{L}(k,cP) = k[cP]$$

for such $c \in \mathbb{N}$. Now observe that the claims (i) and (ii) in Step 1 imply that either $F(k[\square]) = k[\square]$ or $F(k[\square]) = k[T]$ for a lattice triangle T with 4 lattice points.

Let us show that $F(k[\square])$ cannot be a 'triangle ring'. Up to isomorphism there are only two lattice triangles with 4 lattice points

$$\mathrm{conv}\big((1,0),(0,1),(-1,0)\big) \qquad \text{and} \qquad \mathrm{conv}\big((1,0),(0,1),(-1,-1)\big)$$

By Theorem 3.2 below the triangle T with the property $F(k[\square]) = k[T]$ must have the same

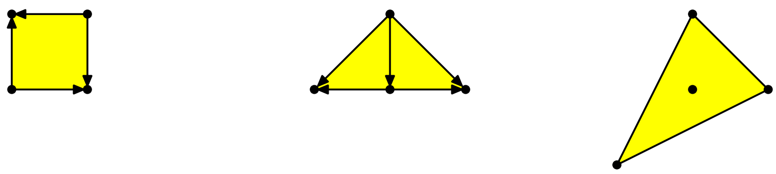

Figure 1

number of *column vectors* as $\square$ (see Section 3 for the definition). But this is not the case because the second triangle has no column vectors whereas the first has 5 of them, and the square only 4. (The column vectors are indicated in the figure.) Therefore $k[\square] \in |\operatorname{Pol}(k)'|$ as well.

By the assumption $F|_{\operatorname{Pol}(k)'} = \mathbf{1}_{\operatorname{Pol}(k)'}$ we get the natural homomorphism $k[cP] \to F(k[cP])$ (c as above). Arguing similarly for the functor F^{-1}, we get the natural homomorphism $k[cP] \to F^{-1}(k[cP])$, and applying F to the latter homomorphism we arrive at the natural homomorphism $F(k[cP]) \to k[cP]$. But every natural (that is compatible with $\mathbb{D}(k, cP)$) endomorphism $k[cP] \to k[cP]$ must be the identity mapping for reasons of universality. In particular, $k[cP]$ is a k-retract of $F(k[cP])$. By reasons of dimensions (and claim (i) in Step 1) we finally get the equality $F(k[cP]) = k[cP]$, as required.

Now we are ready to show the equality $\operatorname{Pol}(k)' = \operatorname{Pol}(k)$, completing the proof of Theorem 2.3.

Let $k[P] \in |\operatorname{Pol}(k)|$. Consider two coprime natural numbers $c, c' \geq \dim P$. We have the commutative square, consisting of embeddings in $\operatorname{Pol}(k)$,

$$\begin{CD} k[P] @>>> k[cP] \\ @VVV @VVV \\ k[c'P] @>>> k[cc'P] \end{CD} \tag{$**$}$$

where the horizontal (vertical) mappings send lattice points in the corresponding polytopes to their homothetic images (centered at the origin) with factor c and c' respectively. The key observation is that by restricting to the degree one components we get the pull back diagram of k-vector spaces

$$\begin{CD} k[P]_1 @>>> k[cP]_1 \\ @VVV @VVV \\ k[c'P]_1 @>>> k[cc'P]_1. \end{CD}$$

This follows from the fact that the following is a pull back diagram of finite sets

$$\begin{CD} \mathrm{L}_P @>>> \mathrm{L}_{cP} \\ @VVV @VVV \\ \mathrm{L}_{c'P} @>>> \mathrm{L}_{cc'P}\,. \end{CD}$$

Caution. $(**)$ is in general not a pull back diagram of k-algebras. Otherwise, by Theorem 1.3.3 in [8], any polytopal algebra would be normal, which is not the case.

That the latter square of finite sets is in fact a pull back diagram becomes transparent after thinking of it as the (isomorphic) diagram consisting of the inclusions:

$$\begin{array}{ccc}
\mathrm{L}_{P(\mathbb{Z}^d)} & \hookrightarrow & \mathrm{L}_{P(\frac{\mathbb{Z}^d}{c})} \\
\downarrow & & \downarrow \\
\mathrm{L}_{P(\frac{\mathbb{Z}^d}{c'})} & \hookrightarrow & \mathrm{L}_{P(\frac{\mathbb{Z}^d}{cc'})}
\end{array}$$

where $P(\mathcal{H})$ refers to the lattice polytope $P \subset \mathbb{R}^d$ with respect to the intermediate lattice $\mathbb{Z}^d \subset \mathcal{H} \subset \mathbb{R}^d$. Here again we assume that $d = \dim P$ and $\mathrm{gp}(S_P) = \mathbb{Z}^{d+1}$.

Since $k[cP], k[c'P], k[cc'P] \in \mathrm{Pol}(k)'$ and F is the identity functor on $\mathrm{Pol}(k)'$, an application of F to the square $(**)$ yields the square of graded homomorphisms

$$\begin{array}{ccc}
F(k[P]) & \longrightarrow & k[cP] \\
\downarrow & & \downarrow \\
k[c'P] & \longrightarrow & k[cc'P],
\end{array}$$

with the same arrows into $k[cc'P]$ as $(**)$. The same arguments as in the proof of (iii), Step 1 show that the degree 1 component of this square is a pull-back diagram, in other words $F(k[P])_1 = k[P]_1$. Therefore, $F(k[P])_1$ and $k[P]_1$ generate the same algebras, i. e. $k[P] = F(k[P])$. □

3. Automorphisms and retractions

In this section we survey those results of [3] and [4] that are related to polytopal linear algebra.

As far as automorphisms are concerned, k will be assumed to be a general, not necessarily algebraically closed field.

As remarked in Section 2, for a lattice polytope $P \subset \mathbb{R}^d$ the group $\Gamma_k(P) = \mathrm{gr.aut}(k[P])$ is a linear k-group in a natural way. It coincides with $\mathrm{GL}_n(k)$ in the case of the unit $(n-1)$-simplex $P = \Delta_{n-1}$. The groups $\Gamma_k(P)$ have been named *polytopal linear groups* in [3].

An element $v \in \mathbb{Z}^d$, $v \neq 0$, is a *column vector* (for P) if there is a facet $F \subset P$ such that $x + v \in P$ for every lattice point $x \in P \setminus F$. The facet F is called *the base facet* of v. The set of column vectors of P is denoted by $\mathrm{Col}(P)$. A pair (P, v), $v \in \mathrm{Col}(P)$, is called a *column structure*. Let (P, v) be a column structure and $P_v \subset P$ be the base facet for $v \in \mathrm{Col}(P)$. Then for each element $x \in S_P$ we set $\mathrm{ht}_v(x) = m$ where m is the largest non-negative integer for which $x + mv \in S_P$. Thus $\mathrm{ht}_v(x)$ is the 'height' of x above the facet of the cone $C(S_P)$ corresponding to P_v in direction $-v$. It is an easy observation that $x + \mathrm{ht}_v(x) \cdot v \in S_{P_v} \subset S_P$ for any $x \in S_P$.

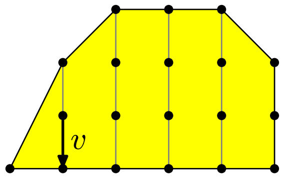

Figure 2. A column structure

Column vectors are just the dual objects to the roots of the normal fan $\mathcal{N}(P)$ in the sense of Demazure (see [16, Section 3.4]). (For the relationship of Theorem 3.2 with the results of Demazure [11] and Cox [10] on the automorphism groups of complete toric varieties see [3, Section 5].)

Let (P, v) be a column structure and $\lambda \in k$. We identify the vector v, representing the difference of two lattice points in P, with the degree 0 element $(v, 0) \in \operatorname{gp}(S_P) \subset k[\operatorname{gp}(S_P)]$. Then the assignment

$$x \mapsto (1 + \lambda v)^{\operatorname{ht}_v(x)} x.$$

gives rise to a graded k-algebra automorphism e_v^λ of $k[P]$. Observe that e_v^λ becomes an elementary matrix in the special case when $P = \Delta_{n-1}$, after the identifications $k[\Delta_{n-1}] = k[X_1, \dots, X_n]$ and $\Gamma_k(P) = \operatorname{GL}_n(k)$. Accordingly e_v^λ is called an *elementary automorphism*.

The following alternative description of elementary automorphisms is essential in showing that all automorphisms in $\operatorname{Pol}(k)$ are *tame* (see Remark 4.2(c) below). We may assume $\dim P = d$ and $\operatorname{gp}(S_P) = \mathbb{Z}^{d+1}$. By a suitable integral unimodular change of coordinates we may further assume that $v = (0, -1, 0, \dots, 0)$ and that P_v lies in the subspace $\mathbb{R}^{d-1}$ (thus P is in the upper halfspace). Consider the standard unimodular simplex Δ_{d-1} (i. e. the one with vertices at the origin and the standard coordinate unit vectors). Clearly, P is contained in a parallel integral shift of $c\Delta_{d-1}$ for a sufficiently large natural number c. Then we have a graded k-embedding $k[P] \to k[c\Delta_{d-1}]$, the latter ring being just the cth Veronese subring of the polynomial ring $k[X_1, \dots, X_d]$. Moreover, $v = X_1/X_2$. Now the automorphism of $k[X_1, \dots, X_d]$ mapping X_2 to $X_2 + \lambda X_1$ and leaving all the other variables fixed induces an automorphism of $k[c\Delta_{d-1}]$ and the latter restricts to an automorphism of $k[P]$, which is nothing but the elementary automorphism e_v^λ above.

As usual, $\mathbb{A}_k^s$ denotes the additive group of the s-dimensional affine space.

Proposition 3.1. *Let $v_1, \dots, v_s$ be pairwise different column vectors for P with the same base facet $F = P_{v_i}$, $i = 1, \dots, s$. The mapping*

$$\varphi : \mathbb{A}_k^s \to \Gamma_k(P), \qquad (\lambda_1, \dots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s},$$

is an embedding of algebraic groups. In particular, $e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute for all $i, j \in \{1, \dots, s\}$ and $(e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s})^{-1} = e_{v_1}^{-\lambda_1} \circ \cdots \circ e_{v_s}^{-\lambda_s}$.

The image of the embedding φ given by Lemma 3.1 is denoted by $\mathbb{A}(F)$. Of course, $\mathbb{A}(F)$ may consist only of the identity map of $k[P]$, namely if there is no column vector with base facet F.

Put $n = \dim(P) + 1$. The n-torus $\mathbb{T}_n = (k^*)^n$ acts naturally on $k[P]$ by restriction of its action on $k[\mathrm{gp}(S_P)]$ that is given by

$$(\xi_1, \ldots, \xi_n)(e_i) = \xi_i e_i, \quad i \in [1, n].$$

Here e_i is the i-th element of a fixed basis of $\mathrm{gp}(S_P) = \mathbb{Z}^n$. This gives rise to an algebraic embedding $\mathbb{T}_n \subset \Gamma_k(P)$, whose image we denote by $\mathbb{T}_k(P)$. It consists precisely of those automorphisms of $k[P]$ which multiply each monomial by a scalar from k^*.

The (finite) automorphism group $\Sigma(P)$ of the semigroup S_P is also a subgroup of $\Gamma_k(P)$. It is exactly the group of automorphisms of P as a lattice polytope.

Next we recapitulate the main result of [3]. It should be viewed a polytopal generalization of the standard linear algebra fact that any invertible matrix over a field can be reduced to a diagonal matrix (generalized to toric automorphisms in the polytopal setting) using elementary transformations on columns (rows). Moreover, we have normal forms for such reductions reflecting the fact that the elementary transformations can be carried out in an increasing order of the column indices.

Theorem 3.2. *Let P be a convex lattice n-polytope and k a field. Every element $\gamma \in \Gamma_k(P)$ has a (not uniquely determined) presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

where $\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_k(P)$, and $\alpha_i \in \mathbb{A}(F_i)$ such that the facets F_i are pairwise different and $\# \mathrm{L}_{F_i} \leq \# \mathrm{L}_{F_{i+1}}$, $i \in [1, r-1]$.

We have $\dim \Gamma_k(P) = \# \mathrm{Col}(P) + n + 1$ (the left hand side is the Krull dimension of the group scheme $\Gamma_k(P)$), and $\mathbb{T}_k(P)$ is a maximal torus in $\Gamma_k(P)$, provided k is infinite.

As an application beyond the theory of toric varieties we mention that Theorem 3.2 provides yet another proof of the classical description of the graded automorphisms of determinantal rings – a result which goes back to Frobenius [13, p. 99] and has been re-proved many times since then. See, for instance, [17] for a group-scheme theoretical approach which involves general commutative rings of coefficients and the classes of generic symmetric and alternating matrices.

Proposition 3.3. *Let k be any field, X an $m \times n$ matrix of indeterminates, and $R = k[X]/I_{r+1}(X)$ the residue class ring of the polynomial ring $k[X]$ in the entries of X modulo the ideal generated by the $(r+1)$-minors of X, $1 \leq r < \min(m, n)$. Let $G = \mathrm{gr.\,aut}(R)$ and G^0 denote the image of the mapping*

$$\psi : \mathrm{GL}_m(k) \times \mathrm{GL}_n(k) \to G$$

defined by

$$\forall M \in R_1 = \bigoplus_{i,j} kX_{ij} = M_{m \times n}(k) \quad \psi\big((\gamma_1, \gamma_2)\big)(M) = \gamma_1 M \gamma_2^{-1}.$$

Then $m \neq n$ implies $G^0 = G$. In case $m = n$ we have $G/G^0 = \mathbb{Z}_2$ where the other class is represented by the matrix transposition. Moreover, scheme theoretically G^0 is the k-rational locus of the unity component of G.

This is Corollary 3.4 in [3]. Here we just sketch how it follows from Theorem 3.2. In the case $r = 1$ the determinantal ring is just the polytopal ring corresponding to the polytope $\Delta_{m-1} \times \Delta_{n-1}$, that is the coordinate ring of the Segre embedding $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{mn-1}$ and Theorem 3.2 applies. For higher r we look at the singular locus of R which is exactly the coordinate ring of the locus of matrices with rank at most $r-1$. Since the singular locus is invariant under the automorphisms, the induction process goes through.

Now we describe those results from [4] that are relevant in Section 4 below. At this point we need to require that the field k is algebraically closed. The case of arbitrary fields remains open.

Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n and $F \subset P$ a face. Then there is a uniquely determined retraction

$$\pi_F : k[P] \to k[F], \quad \pi_F(x) = 0 \quad \text{for } x \in \mathrm{L}_P \setminus F.$$

Retractions of this type will be called *face retractions*, and *facet retractions* if F is a facet. In the latter case we write $\operatorname{codim}(\pi_F) = 1$.

Now suppose there are an affine subspace $H \subset \mathbb{R}^n$ and a vector subspace $W \subset \mathbb{R}^n$ with $\dim W + \dim H = n$, such that

$$\mathrm{L}_P \subset \bigcup_{x \in \mathrm{L}_P \cap H} (x + W).$$

(Observe that $\dim(H \cap P) = \dim H$.) The triple (P, H, W) is called a *lattice fibration of codimension* $c = \dim W$, whose *base polytope* is $P \cap H$; its *fibers* are the maximal lattice subpolytopes of $(x + W) \cap P$, $x \in \mathrm{L}_P \cap H$ (the fibers may have smaller dimension than W). P itself serves as a *total polytope* of the fibration. If $W = \mathbb{R}w$ is a line, then we call the fibration *segmental* and write (P, H, w) for it. Note that the column structures give rise to

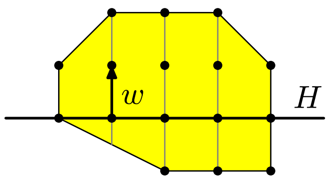

Figure 3. A lattice segmental fibration

lattice segmental fibrations in a natural way.

For a lattice fibration (P, H, W) let $L \subset \mathbb{Z}^n$ denote the subgroup spanned by L_P, and let H_0 be the translate of H through the origin. Then one has the direct sum decomposition

$$L = (L \cap W) \oplus (L \cap H_0).$$

Equivalently,

$$\operatorname{gp}(S_P) = L \oplus \mathbb{Z} = \big(\operatorname{gp}(S_P) \cap W_1\big) \oplus \operatorname{gp}(S_{P \cap H_1}).$$

where W_1 is the image of W under the embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$, $w \mapsto (w, 0)$, and H_1 is the vector subspace of $\mathbb{R}^{n+1}$ generated by all the vectors $(h, 1)$, $h \in H$.

For a fibration (P, H, W) one has the naturally associated retraction

$$\rho_{(P,H,W)} : k[P] \to k[P \cap H];$$

it maps L_P to $\mathrm{L}_{P\cap H}$ so that fibers are contracted to their intersection points with the base polytope $P \cap H$.

The following is a composition of Theorems 2.2 and 8.1 in [4]:

Theorem 3.4. (a) *If A is a retract of a polytopal algebra $k[P]$ and $\dim A \leq 2$ then A is polytopal itself, i. e. A is of type either k or $k[t]$ or $k[c\Delta_1]$ for some $c \in \mathbb{N}$.*

(b) *If $\dim P = 2$ and $f : k[P] \to k[P]$ is any idempotent endomorphism of codimension* 1 *($f^2 = f$ and $\dim \mathrm{Im}(f) = 2$) then either*

(i) *P admits a lattice segmental fibration (P, H, w) and $\gamma \circ f \circ \gamma^{-1} = \iota \circ \rho_{(P,H,w)}$ for some $\gamma \in \Gamma_k(P)$ and an embedding $\iota : k[P \cap H] \to k[P]$, or*

(ii) *$\gamma \circ f \circ \gamma^{-1} = \iota \circ \pi_E$ for some $\gamma \in \Gamma_k(P)$, an edge $E \subset P$ and an embedding $\iota : k[E] \to k[P]$.*

Remark 3.5. The conjectures (A) and (B) in [4] say that both statements (a) and (b) of Theorem 3.4 generalize to arbitrary dimensions. They should be thought of as analogues of the fact that idempotent matrices are conjugate to 'subunit' matrices (having diagonal entries 0 and 1 and entries 0 everywhere else). That one has to restrict oneself to codimension 1 idempotent endomorphisms in the higher analogue of (b) is explained by explicit examples in [4, Section 5].

The splitting embeddings ι will be described in the next section.

4. Tame homomorphisms

Assume we are given two lattice polytopes $P, Q \subset \mathbb{R}^d$ and a homomorphism $f : k[P] \to k[Q]$ in $\mathrm{Pol}(k)$. Under certain conditions there are several standard ways to derive new homomorphisms from it.

First assume we are given a subpolytope $P' \subset P$ and a polytope $Q' \subset \mathbb{R}^n$, $d \leq n$, such that $f(k[P']) \subset k[Q']$. Then f gives rise to a homomorphism $f' : k[P'] \to k[Q']$ in a natural way. (Notice that we may have $Q \subset Q'$.) Also if $P \approx \tilde{P}$ and $Q \approx \tilde{Q}$ are lattice polytope isomorphisms, then f induces a homomorphism $\tilde{f} : k[\tilde{P}] \to k[\tilde{Q}]$. We call these types of formation of new homomorphisms *polytope changes.*

Now consider the situation when $\mathrm{Ker}(f) \cap S_P = \emptyset$. Then f extends uniquely to a homomorphism $\bar{f} : k[\bar{S}_P] \to k[\bar{S}_Q]$ of the normalizations. Here $\bar{S}_P = \{x \in \mathrm{gp}(S_P) \mid x^m \in S_P \text{ for some } m \in \mathbb{N}\}$ and similarly for $\bar{S}_Q$. (It is well known that $k[\bar{S}] = \overline{k[S]}$ for any affine semigroup $S \subset \mathbb{Z}^d$ where on the right hand side we mean the normalization of the domain $k[S]$, [9, Ch. 6].) This extension is given by

$$\bar{f}(x) = \frac{f(y)}{f(z)}, \qquad x = \frac{y}{z},\ x \in \bar{S}_P,\ y \in S_P,\ z \in S_Q.$$

For any natural number c the subalgebra of $k[\bar{S}_P]$ generated by the homogeneous component of degree c is naturally isomorphic to the polytopal algebra $k[cP]$, and similarly for $k[\bar{S}_Q]$.

Therefore, the restriction of $\bar{f}$ gives rise to a homomorphism $f^{(c)} : k[cP] \to k[cQ]$. We call the homomorphisms $f^{(c)}$ *homothetic blow-ups* of f. (Note that $k[cP]$ is often a proper overring of the cth Veronese subalgebra of $k[P]$.)

One more process of deriving new homomorphisms is as follows. Assume that homomorphisms $f, g : k[P] \to k[Q]$ are given such that

$$\forall x \in \mathrm{L}_P \quad \mathrm{N}(f(x)) + \mathrm{N}(g(x)) \subset Q,$$

where $\mathrm{N}(-)$ denotes the Newton polytope and $+$ is the Minkowski sum in $\mathbb{R}^d$. Then we have $z^{-1}f(x)g(x) \in k[Q]$ where $z = (0, \dots, 0, 1) \in S_Q$. Clearly, the assignment

$$\forall x \in \mathrm{L}_P \quad x \mapsto z^{-1}f(x)g(x)$$

extends to a $\mathrm{Pol}(k)$-homomorphism $k[P] \to k[Q]$, which we denote by $f \star g$. We call this process *Minkowski sum* of homomorphisms.

All the three mentioned recipes have a common feature: the new homomorphisms are defined on polytopal algebras of dimension at most the dimension of the sources of the old homomorphisms. As a result we are not able to really create a non-trivial class of homomorphisms using only these three procedures. This possibility is provided by the fourth (and last in our list) process.

Suppose P is a pyramid with vertex v and basis P_0 such that $\mathrm{L}_P = \{v\} \cup \mathrm{L}_{P_0}$, that is $P = \mathrm{join}(v, P_0)$ in the terminology of [4, 5]. Then $k[P]$ is a polynomial extension $k[P_0][v]$. In particular, if $f_0 : k[P_0] \to k[Q]$ is an arbitrary homomorphism and $q \in k[Q]$ is any element, then f_0 extends to a homomorphism $f : k[P] \to k[Q]$ with $f(v) = q$. We call f a *free extension* of f_0.

Conjecture 4.1. Any homomorphism in $\mathrm{Pol}(k)$ is obtained by a sequence of taking free extensions, Minkowski sums, homothetic blow-ups, polytope changes and compositions, starting from the identity mapping $k \to k$. Moreover, there are normal forms of such sequences for idempotent endomorphisms.

Observe that for general homomorphisms we do not mean that the constructions mentioned in the conjecture are to be applied in certain order so that we get normal forms: we may have to repeat a procedure of the same type at different steps. However, the results mentioned in Section 3 and Theorem 4.3 below show that for special classes of homomorphisms such normal forms are possible.

We could call the homomorphisms obtained in the way described by Conjecture 4.1 just *tame*. Then we have the *tame* subcategory $\mathrm{Pol}(k)_{\mathrm{tame}}$ (with the same objects), and the conjecture asserts that actually $\mathrm{Pol}(k)_{\mathrm{tame}} = \mathrm{Pol}(k)$.

Remark 4.2. (a) The correctness of Conjecture 4.1 may depend on whether or not k is algebraically closed. For instance, some of the arguments in [4] only go through for algebraically closed fields.

(b) The current notion of tameness is weaker then the one for retractions and surjections in [4]. This follows from Example 5.2 and 5.3 of [4] in conjunction with the observation that all the explicitly constructed retractions in [4] are tame in the new sense.

(c) Theorems 3.2 and 3.4 can be viewed as substantial refinements of the conjecture above for the corresponding classes of homomorphisms. Observe that the tameness of elementary automorphisms follows from their alternative description in Section 3. We also need the tameness of the following classes of homomorphisms: automorphisms that map monomials to monomials, retractions of the type $\rho_{(P,H,w)}$ and π_F and the splitting embeddings ι as in Theorem 3.4. This follows from Theorem 4.3 and Corollary 4.4 below.

The next result shows that certain basic classes of morphisms in $\mathrm{Pol}(k)$ are tame.

Theorem 4.3. *Let k be a field (not necessarily algebraically closed). Then*

(a) *any homomorphism from $k[c\Delta_n]$, $c, n \in \mathbb{N}$, is tame,*

(b) *if $\iota : k[c\Delta_n] \to k[P]$ ($c \in \mathbb{N}$) splits either $\rho_{(P,H,W)}$ for some lattice fibration (P, H, W) or π_E for some face $E \subset P$ then there is a normal form for representing ι in terms of certain basic tame homomorphisms.*

Corollary 4.4. *For every field k the homomorphisms respecting monomial structures are tame, and we have the inclusion $\mathrm{Vect}_{\mathbb{N}}(k) \subset \mathrm{Pol}_{\mathrm{tame}}(k)$, where $\mathrm{Vect}_{\mathbb{N}}(k)$ is the full subcategory of $\mathrm{Pol}(k)$ spanned by the objects of the type $k[c\Delta_n]$, $c, n \in \mathbb{N}$.*

Proof. [Proof of Corollary 4.4] The inclusion $\mathrm{Vect}_{\mathbb{N}}(k) \subset \mathrm{Pol}_{\mathrm{tame}}(k)$ follows from Theorem 4.3(a).

Assume $f : k[P] \to k[Q]$ is a homomorphism respecting the monomial structures and such that $\mathrm{Ker}(f) \cap S_P = \emptyset$. By a polytope change we can assume $P \subset c\Delta_n$ for a sufficiently big natural number c, where $n = \dim P$ and Δ_n is taken in the lattice $\mathbb{Z}L_P$. In this situation there is a bigger lattice polytope $Q' \supset Q$ and a unique homomorphism $g : k[c\Delta_n] \to k[Q']$ for which $g|_{\mathrm{L}_P} = f|_{\mathrm{L}_P}$. By Theorem 4.3(a) f is tame.

Consider the situation when the ideal $I = (\mathrm{Ker}(f) \cap S_P)k[P]$ is a nonzero prime monomial ideal and there is a face $P_0 \subset P$ such that $\mathrm{Ker}(f) \cap \mathrm{L}_{P_0} = \emptyset$ and f factors through the face projection $\pi : k[P] \to k[P_0]$, that is $\pi(x) = x$ for $x \in \mathrm{L}_{P_0}$ and $\pi(x) = 0$ for $x \in \mathrm{L}_P \setminus \mathrm{L}_{P_0}$. In view of the previous case we are done once the tameness of face projections has been established.

Any face projection is a composite of facet projections. Therefore we can assume that P_0 is a facet of P. Let $(\mathbb{R}P)_+ \subset \mathbb{R}P$ denote the halfspace that is bounded by the affine hull of P_0 and contains P. There exists a unimodular (with respect to $\mathbb{Z}\,\mathrm{L}_P$) lattice simplex $\Delta \subset (\mathbb{R}P)_+$ such that $\dim \Delta = \dim P$, the affine hull of P_0 intersects Δ in one of its facets and $P \subset c\Delta$ for some $c \in \mathbb{N}$. But then π is a restriction of the corresponding facet projection of $k[c\Delta]$, the latter being a homothetic blow-up of the corresponding facet projection of the polynomial ring $k[\Delta_n]$ – obviously a tame homomorphism. □

Proof. [Proof of Theorem 4.3(a)] We will use the notation $\{x_0, \dots, x_n\} = \mathrm{L}_{\Delta_n}$. Any lattice point $x \in c\Delta_n$ has a unique representation $x = a_0x_0 + \cdots + a_nx_n$ where the a_i are nonnegative integer numbers satisfying the condition $a_0 + \cdots + a_n = c$. The numbers a_i are the *barycentric coordinates* of x in the x_i.

Let $f : k[c\Delta_n] \to k[P]$ be any homomorphism.

First consider the case when one of the points from $\mathrm{L}_{c\Delta_n}$ is mapped to 0. In this situation f is a composite of facet projections and a homomorphism from $k[c\Delta_m]$ with $m < n$. As observed in the proof of Corollary 4.4 facet projections are tame. Therefore we can assume that none of the x_i is mapped to 0. By a polytope change we can also assume $\mathrm{L}_P \subset \{X_1^{a_1} \cdots X_r^{a_r} Y^b Z \mid a_i, b \geq 0\}$, $r = \dim P - 1$.

Consider the polynomials $\varphi_x = Z^{-1} f(x) \in k[X_1, \dots, X_r, Y]$, $x \in \mathrm{L}_{c\Delta_n}$. Then the φ_x are subject to the same binomial relations as the x. One the other hand the multiplicative semigroup $k[X_1, \dots, X_r, Y] \setminus \{0\}/k^*$ is a free commutative semigroup and, as such, is an inductive limit of free commutative semigroups of finite rank. Therefore, by Lemma 4.5 below there exist polynomials $\psi, \eta_i \in k[X_1, \dots, X_r, Y]$, $i \in [0, n]$, and scalars $t_x \in k^*$, $x \in \mathrm{L}_{c\Delta_n}$, such that $\varphi_x = t_x \psi \eta_0^{a_0} \cdots \eta_n^{a_n}$ where the a_i are the barycentric coordinates of x. Clearly, t_x are subject to the same binomial relations as the $x \in \mathrm{L}_{c\Delta_n}$. Therefore, after the normalizations $\eta_i \mapsto t_{x_i} \eta_i$ ($i \in [0, n]$) we get $\varphi_x = \psi \eta_0^{a_0} \cdots \eta_n^{a_n}$. But the latter equality can be read as follows: f is obtained by a polytope change applied to $\Psi \star \Theta^{(c)}$, where

(i) $\Psi : k[c\Delta_n] \to k[Q]$, $\Psi(x) = \psi Z$, $x \in \mathrm{L}_{c\Delta_n}$,

(ii) $\Theta : k[\Delta_n] \to k[Q]$, $\Theta(x_i) = \eta_i Z$, $i \in [0, n]$,

and Q is a sufficiently large lattice polytope so that it contains all the relevant lattice polytopes. Now Ψ is tame because it can be represented as the composite map

$$k[c\Delta_n] \xrightarrow{\mathrm{L}_{c\Delta_n} \to t} k[t] \xrightarrow{t \mapsto \psi Z} k[Q]$$

(the first map is the cth homothetic blow-up of $k[\Delta_n] \to k[t]$, $x_i \mapsto t$ for all $i \in [0, c]$) and Θ is just a free extension of the identity embedding $k \to k[Q]$.

(b) First consider the case of lattice segmental fibrations.

Consider the rectangular prism $\Pi = (c\Delta_n) \times (m\Delta_1)$. By a polytope change (assuming m is sufficiently large) we can assume that $P \subset \Pi$ so that H is parallel to $c\Delta_n$: The lattice

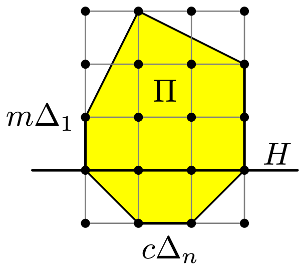

Figure 4

point $(x, b) \in \Pi$ will be identified with the monomial $X_1^{a_1} \cdots X_n^{a_n} Y^b Z$ whenever we view it as a monomial in $k[\Pi]$, where the a_i are the corresponding barycentric coordinates of x (see the proof of (a) above). (In other words, the monomial $X_1^{a_1} \cdots X_n^{a_n}$ is identified with the point $x = (c - a_1 - \cdots - a_n)x_0 + a_1 x_1 + \cdots + a_n x_n \in c\Delta_n$.)

Assume $A : k[c\Delta_n] \to k[m'\Delta_1]$ is a homomorphism of the type $A(x_i) = aZ$, $i \in [0, c]$ for some $a \in k[Y]$ satisfying the condition $a(1) = 1$. Consider any homomorphism $B : k[\Delta_n] \to k[\Pi']$, $\Pi' = \Delta_n \times (m'\Delta_1)$ that splits the projection $\rho' : k[\Pi'] \to k[\Delta_n]$, $\rho'(ZY^b) = Z$ and $\rho'(X_i Y^b Z) = X_i Z$ for $i \in [1, n]$, $b \in [0, m']$. The description of such homomorphisms is clear – they are exactly the homomorphisms B for which

$$B(x_0) = B_0 \in Z + (Y - 1)(Zk[Y] + X_1 Zk[Y] + \cdots + X_n Zk[Y]),$$

$B(x_i) = B_i \in X_iZ + (Y-1)(Zk[Y] + X_1Zk[Y] + \cdots + X_nZk[Y])$, $i \in [1,n]$,

$\deg_Y B_i \le m'$ for all $i \in [0,n]$.

Clearly, all such B are tame.

In case $m \ge \max\{m' + c\deg_Y B_i\}_{i=0}^n$ we have the homomorphism $A \star B^{(c)} : k[c\Delta_n] \to k[\Pi]$ which obviously splits the projection $\rho : k[\Pi] \to k[c\Delta_n]$ defined by

$$\rho(X_1^{a_1} \cdots X_n^{a_n} Y^b Z) = X_1^{a_1} \cdots X_n^{a_n} Z.$$

Assume ι splits $\rho_{(P,H,w)}$. Since the Newton polytope of a product is a Minkowski sum of the Newton polytopes of the factors, we get: the polynomials ψ and η_i, mentioned in the proof of (a), that correspond to ι, satisfy the conditions: $\psi \in k[Y]$ and $\eta_i \in k[Y] + X_1k[Y] + \cdots + X_nk[Y]$. Is is also clear that upon evaluation at $Y = 1$ we get $\psi(1), \eta_i(X_1, \ldots, X_n, 1) \in k^*$, $i \in [0,n]$. Therefore, after the normalizations $\psi \mapsto \psi^{-1}(1)\psi$, $\eta_i \mapsto \eta_i(X_1, \ldots X_n, 1)^{-1}\eta_i$ we conclude that ι is obtained by a polytope change applied to $A \star B^{(c)}$ as above (with respect to $a = \psi$, $B_0 = \eta_i Z$, $i \in [0,n]$).

For a lattice fibration (P,H,W) of higher codimension similar arguments show that ι is obtained by a polytope change applied to $A \star B^{(c)}$, where

B is a splitting of a projection of the type $\rho_{(P',H',W')}$ such that the base polytope $P' \cap H'$ is a unit simplex and

A is a homomorphism defined by a single polynomial whose Newton polytope is parallel to W'.

We skip the details for splittings of face projections and only remark that similar arguments based on Newton polytopes imply the following. All such splittings are obtained by polytope changes applied to $A \star B^{(c)}$ where B is a splitting of a face projection onto a polynomial ring and A is again defined by a single polynomial. □

Lemma 4.5. *Assume we are given an integral affine mapping* $\alpha : c\Delta_n \to \mathbb{R}_+^d$ *for some natural numbers* c, n *and* d. *Then there exists an element* $v \in \mathbb{Z}_+^d$ *and a integral affine mapping* $\beta : \Delta_n \to \mathbb{R}_+^d$ *such that* $\alpha = v + c\beta$.

Proof. Assume $\alpha(cx_i) = (a_{i1}, \ldots, a_{id})$, $i \in [0,n]$ (the x_i as above). Consider the vector

$$v = \bigl(\min\{a_{i1}\}_{i=0}^n, \ldots, \min\{a_{id}\}_{i=0}^n\bigr).$$

It suffices to show that all the vectors $\alpha(cx_i) - v$ are cth multiples of integral vectors. But for any index $l \in [1,d]$ the lth component of either $\alpha(cx_i) - v$ or $\alpha(cx_j) - v$ for some $j \ne i$ is zero. In the first case there is nothing to prove and in the second case the desired divisibility follows from the fact that $\alpha(cx_i) - \alpha(cx_j) = (\alpha(cx_i) - v) - (\alpha(cx_j) - v)$ is a cth multiple of an integral vector (because α is integral affine). □

Remark 4.6. Theorems 3.2, 3.4 and 4.3 provide a possibility for computing the $\Gamma_k(P)$-variety of idempotent endomorphisms $\mathrm{Idemp}(k[P])$ for a polygon P and an algebraically closed field k ($\Gamma_k(P)$ acts by conjugation): the orbits of codimension 1 idempotent endomorphisms are naturally associated to the segmental fibration structures and edges of P, and all codimension 2 idempotent endomorphisms factor through $k[t]$.

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