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The Partition Problem for Equifacetal Simplices

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Abstract. Associated with any equifacetal d-simplex, which necessarily has a vertex transitive isometry group, there is a well-defined partition of d that counts the number of edges of each possible length incident at a given vertex. The partition problem asks for a characterization of those partitions that arise from equifacetal simplices. The partition problem is resolved by proving that a partition of the number d arises this way if and only if the number of odd entries in the partition is at most $\iota(d+1)$, the maximum number of involutions in a finite group of order $d+1$. When n is even the number $\iota(n)$ is shown to be $n/2 + n_2/2 - 1$, where n_2 denotes the 2-part of n. Those extremal equifacetal d-simplices for which the number of odd entries of the associated partition is exactly $\iota(d+1)$ are characterized.

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1. Introduction

An equifacetal d-simplex is a geometric simplex of dimension d in Euclidean space such that all of its codimension one faces are congruent to one another. The starting point for the author's earlier investigation of equifacetal simplices was the observation that the isometry group of such a simplex is transitive on vertices [2]. This isometry group also acts on the edges of the simplex and the set of edges of the simplex of a given length is a union of orbits for this action.

One may also consider edge-colorings of the complete graph K_{d+1} on $d+1$ vertices, with the property that its colored-graph automorphism group is transitive on vertices. Such colored graphs arise from the 1-skeleta of equifacetal simplices, with edges colored according to their lengths. Moreover, every such edge-coloring of K_{d+1} arises from an equifacetal simplex [2].

Let $\mathcal{S}_n$ denote the symmetric group on n symbols. Given any transitive group $G < \mathcal{S}_n$ (where $n = d+1$) we obtain a vertex-transitive edge-coloring of K_n simply by assigning a color to each orbit of edges. Moreover, one obtains a related equifacetal d-simplex by choosing distinct edge lengths to correspond to each color. Provided the edge lengths are all close enough to 1, say, then there is an equifacetal geometric simplex realizing those lengths by [2].

Associated to an equifacetal d-simplex or to a vertex-transitive edge coloring of K_{d+1} there is a partition $\pi = [d_1, d_2, \ldots, d_k]$ of d, defined as follows: let $c_1, c_2, \ldots, c_k$ be the distinct lengths of edges or colors of edges. Fix a vertex v; then d_i denotes the number of edges incident at v and labelled c_i. By vertex-transitivity, the partition is well-defined, independent of the choice of vertex.

The study in [2] led to the problem of characterizing the partitions that arise from equifacetal d-simplices.

In [2] this partition problem was solved when d is even, in which case it turned out that a partition arises if and only if each entry d_i in the partition is even.

The case when d is odd remained more mysterious. It was shown that a partition with at most one odd entry could be realized. It was shown that more odd entries could be realized in certain cases. But it was also shown that if the partition $[1, 1, \ldots, 1]$ arises, then $d+1$ is a power of 2.

It thus remained to understand the general situation when d is odd.

Below we summarize the main results to be proved in subsequent sections.

For present purposes, we call the number of odd entries in a partition π the *oddness* of π and denote it $\mathcal{O}(\pi)$. We denote by $\iota(n)$ the maximum number of involutions in any finite group of order n.

Theorem 1.1. *If d is odd and π is a partition of d with oddness $\mathcal{O}(\pi) \leq \iota(d+1)$, then π is the partition associated with an equifacetal d-simplex.*

The short proof is given in Section 3. It thus becomes important to compute the number $\iota(n)$.

Theorem 1.2. *For any even positive integer n, the maximum involution number is given by*

$$\iota(n) = n/2 + n_2/2 - 1$$

where n_2 denotes the 2-part of n.

The proof is given in Section 4. This section is the heart of the paper. We also explicitly determine in Section 4 all "2-maximal" groups of order n containing $\iota(n)$ involutions, as follows.

Theorem 1.3. *A group of even order n is 2-maximal if and only if it has a Sylow 2-subgroup of exponent 2 and can be expressed as a semidirect product extension of an abelian normal subgroup of order $n/2$ by a cyclic group of order 2 acting by inversion on the normal subgroup.*

The main aim of this paper is the following result saying that indeed the maximum involution number gives the upper bound on the oddness of the partition associated with an equifacetal d-simplex when d is odd.

Theorem 1.4. *If d is odd and π is the partition associated with an equifacetal d-simplex, then $\mathcal{O}(\pi) \leq \iota(d+1)$.*

The proof is completed in Section 5. Theorem 1.3 provides the main step in the proof.

We are also able to characterize the extreme equifacetal d-simplices that realize the maximum oddness. Recall that a permutation group is said to be *regular* or to act *regularly* if no element besides the identity of the group fixes any point.

Theorem 1.5. *If the partition π associated with an equifacetal d-simplex S, d odd, has maximal oddness $\mathcal{O}(\pi) = \iota(d+1)$, then the isometry group G of S acts regularly and transitively on the vertices of S.*

It follows that such a group is characterized as in Theorem 1.3 above. The proof is given in Section 6.

1.1. A comment on computation

The approach to the partition problem, as represented by the discussion above, began with a few tedious by-hand constructions in low dimensions, some of which were given in [2]. A proposed solution took shape after extensive computer searches, with the software system GAP [3]. Utilizing the library of transitive permutation groups, developed by A. Hulpke [4] and included in the GAP distribution, the answer to the partition problem was verified up through degree 30. Even that was not quite enough to get the general answer straight. Although nothing presented here depends on computer calculations, the results were motivated and informed by such calculations.

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2. Edge-colorings and their groups

As discussed above, an equifacetal d-simplex determines a vertex-transitive edge-coloring of the complete graph K_{d+1}, and any such vertex-transitive edge-coloring arises from an equifacetal simplex. Moreover, every vertex-transitive edge-coloring χ of K_{d+1} has an associated group $G(\chi)$, the maximal (transitive) subgroup of $\mathcal{S}_{d+1}$ that preserves the coloring of the edges.

On the other hand, any transitive group $G < \mathcal{S}_{d+1}$ gives rise to an edge-coloring $\chi(G)$ where each orbit of edges is colored by a different color.

We say that a vertex-transitive edge-coloring χ' of K_{d+1} *refines* an edge-coloring χ if the set of edges of a given χ-color is a union of the sets of edges of given χ'-colors.

In general it is possible that $\chi(G(\chi)) \neq \chi$. The most one can say in general is that $\chi(G(\chi))$ refines χ. This would happen if the automorphism group of χ does not act transitively on the set of edges of a given color. Similarly, in general, one always has $G(\chi(G)) \supseteq G$, but not necessarily equality. One sees that $G(\chi(G))$ defines a kind of closure operator on the set of transitive groups.

Lemma 2.1. *If χ is a vertex-transitive edge-coloring of K_{d+1} with partition π, and χ' is a second vertex-transitive edge-coloring with partition π', such that χ' refines χ, then $\mathcal{O}(\pi') \geq \mathcal{O}(\pi)$.* □

As a consequence, it suffices to solve the partition problem for edge-colorings that admit no nontrivial refinement.

Lemma 2.2. *If χ is a vertex-transitive edge-coloring of K_{d+1} that admits no refinement, then its automorphism group $G(\chi)$ acts transitively on its set of edges of any given color.* □

These lemmas (whose elementary proofs may safely be omitted) reduce the study of the partition problem, as represented by the results summarized in Section 1, to the study of edge-colorings and their partitions arising from transitive permutation groups. We may focus on the transitive group, rather than the equifacetal simplex or vertex-transitive edge-coloring. In particular, in the proofs of Theorems 1.4 and 1.5 we may and shall assume the partitions in question arise directly from a transitive group action.

3. Realization of partitions

Let G be a finite group of order $n = d + 1$, viewed as a permutation group acting on itself by left translations. We may identify the vertices of the complete graph K_{d+1} with the elements of G itself and then examine the induced edge-coloring $\chi = \chi(G)$. Such an action is *regular* in the sense that the only group element that fixes a point is the identity. In a partition, an entry of the form $a(k)$ stands for a sequence of k's of length a.

Lemma 3.1. *The partition π associated with a regular transitive group $G < \mathcal{S}_n$ has the form $[a(2), b(1)]$, where $n = 2a + b$ and b is the number of elements of order 2 in G.*

Proof. In the regular case edges $[e, g_1]$ and $[e, g]$ lie in the same orbit under G if and only if $gg_1 = e$. It follows that the valence of the orbit of $[e, g]$ is 1 if and only if g has order 2, so that $g_1 = g$. Otherwise the only other edge incident at e in the same orbit as $[e, g]$ is $[e, g^{-1}]$ and the orbit graph containing $[e, g]$ has valence 2. □

Theorem 3.2. *If d is odd and π is a partition of d such that $\mathcal{O}(\pi) \leq \iota(d+1)$, then there is an equifacetal d-simplex with partition π.*

Proof. It suffices to realize a partition of the form $[a(1), b(2)]$ where $a = \iota(d+1)$ by a corresponding vertex-transitive edge-coloring of K_{d+1}. For the given partition can then be realized by appropriately recoloring some different colors to be the same. Let G be a finite group of order $d+1$ with exactly $\iota(d+1)$ involutions. Then by the preceding lemma, the associated colored graph $\chi(G)$ has the required properties. □

This completes the proof of Theorem 1.1, as stated in the introduction. To make further progress we need to determine the actual value of the bound $\iota(d+1)$.

4. Groups with maximal number of involutions

We determine the maximum number of involutions in a group of even order n as stated in Theorem 1.2 and classify the groups of order n that contain the maximal number of involutions, as stated in Theorem 1.3. This will give the proof of the oddness bound stated in Theorem 1.4 in the special case of an equifacetal simplex with regular transitive automorphism group.

4.1. Statement of results

Let $\mathcal{I}(G)$ denote the set of involutions, or elements of order 2, in the finite group G, and let $\iota(G)$ denote the cardinality of $\mathcal{I}(G)$.

Theorem 4.1. (Main inequality) *If G is a finite group of even order , then $\iota(G) \leq |G|/2 + |G|_2/2 - 1$.*

Theorem 4.2. (Classification) *A group G of even order satisfies $\iota(G) = |G|/2 + |G|_2/2 - 1$ if and only if its Sylow 2-subgroup is elementary abelian of exponent 2 and G is a semidirect product extension of an abelian group K by an involution inverting its elements.*

The simplest examples of such "2-maximal" groups include the dihedral groups D_{2h} and D_{4h}, having orders $2h$ and $4h$, and their products with an abelian group of exponent 2. [Thanks to Michael Aschbacher for the inclusive statement above that captures some examples I had previously neglected.]

4.2. Toward a proof of the main inequality

In the end we will prove the main inequality by induction on the order of the group. But our initial effort will be to describe some of the properties of groups such that at least half the elements are involutions.

Let G be a finite group of even order $n = 2^r h$, h odd, $r \geq 1$. Let $S < G$ be a Sylow 2-subgroup, of order $|S| = 2^r$. Let $S_1, \ldots, S_k$ be the set of all distinct Sylow 2-subgroups, all of order 2^r, and all conjugate to one another. By the Sylow theorems, $k|h$. Note that $|G|/2 + |G|_2/2 - 1 = 2^{r-1}(h+1) - 1$.

Let $N = N_G(S)$ denote the normalizer of S in G. Then $[G : N] = k$ and $[N : S] = h/k$, and both k and n/k are odd.

We approach the main inequality through several special cases of independent interest, in which often stronger inequalities hold true.

Proposition 4.3. *If G is a finite group of even order $2^r h$, h odd, with Sylow 2-subgroup S and $k = [G : N_G(S)]$ satisfies $1 \leq k < h$, then $\iota(G) \leq (2^r - 1)k \leq 2^{r-1}(h+1) - 1$.*

Proof. Since k is a divisor of the odd number h and $k < h$, then $k \leq h/3$. Thus

$$\begin{aligned} \iota(G) &= \iota(S_1 \cup \cdots \cup S_k) \\ &\leq (2^r - 1)k \\ &= 2^r k - k \\ &= 2^{r-1}(2k) - k \\ &\leq 2^{r-1}(h+1) - 1. \end{aligned}$$ □

Corollary 4.4. *If G is a finite group with Sylow 2-subgroup S, and $\iota(G) \geq |G|/2$, then $N_G(S) = S$.* □

The following special case of the main result will be important at a certain point in the main proof.

Proposition 4.5. *Let $G = HS$ be a finite group of order $2^r h$, h odd, with Sylow 2-subgroup S and normal 2-complement H. Then $\iota(G) \leq 2^{r-1}(h+1) - 1$.*

Proof. If $x \in S$ and $y \in H$, then yx has order ≤ 2 if and only if $1 = yxyx = yy^x x^2$ if and only if $y^x = y^{-1}$ and $x^2 = 1$.

$$\begin{aligned} \iota(G) &= \sum_{x \in \mathcal{I}(S)} \left|\{y \in H : y^x = y^{-1}\}\right| \\ &= \sum_{y \in H} \left|\{x \in \mathcal{I}(S) : y^x = y^{-1}\}\right| \\ &\leq \sum_{y \in H} \left|\{x \in S^{\#} : y^x = y^{-1}\}\right|. \end{aligned}$$

(Here $S^\#$ denotes the set of nontrivial elements of S.) Now $y = 1$ contributes at most $2^r - 1$. If $y \neq 1$, note that y has odd order, so that $y \neq y^{-1}$. The only possible x's in S that contribute are in the isotropy subgroup $S_{\{y,y^{-1}\}}$ consisting of all $x \in S$ conjugating y to $y^{\pm 1}$. Moreover, the x's that conjugate y to y^{-1} form a coset of the smaller subgroup S_y of elements of S that fix y. Such a coset contributes at most half the elements of S, namely 2^{r-1}. It follows that

$$\iota(G) \leq 2^r - 1 + 2^{r-1}(h-1) = 2^{r-1}(h+1) - 1$$

as required. □

Lemma 4.6. *If G admits a surjection $G \to C_k$ to a cyclic group of order k, $k > 2$, or a non-split surjection $G \to C_2$, then $\iota(G) < |G|/2$.*

Proof. If k is odd, then all of $\mathcal{I}(G)$ lies in the kernel, which has size $|G|/k \leq |G|/3 < |G|/2$. If $k = 2$ and the surjection is non-split, then, again, all the involutions lie in the kernel, of size $|G|/2$. Since the identity element does not lie in $\mathcal{I}(G)$, it follows that $\iota(G) < |G|/2$. If $k > 2$ is even, then all of $\mathcal{I}(G)$ lies in the kernel of the further projection to $C_{k/2}$, which has size $2|G|/k \leq 2|G|/4 = |G|/2$. Since the identity element does not lie in $\mathcal{I}(G)$, we are done. □

Corollary 4.7. *If $\iota(G) \geq |G|/2$, then the abelianization $A = G/[G,G]$ has exponent 2, and the projection of G onto any C_2 is a split surjection.* □

We will see below that the abelianization is indeed nontrivial in the context of Corollary 4.7. Counting involutions in a slightly more sophisticated way we have the following general statement.

Proposition 4.8. *If the finite group G has a normal subgroup H, then*

$$\iota(G) \leq \iota(G/H)|H| + \iota(H).$$

Proof. The natural projection $G \to G/H$, denoted by $x \to \bar{x}$, induces a function $\mathcal{I}(G) - \mathcal{I}(H) \to \mathcal{I}(G/H)$. Let y_i be a list of image values in $\mathcal{I}(G/H)$. For each index i choose a representative involution $x_i \in \mathcal{I}(G)$. Define a function $\mathcal{I}(G) - \mathcal{I}(H) \to H$ by $x \to xx_i^{-1}$ if $\bar{x} = \bar{x}_i$. This defines an injective function $\mathcal{I}(G) - \mathcal{I}(H) \to \mathcal{I}(G/H) \times H$. The result follows. □

If the subgroup $H \subset G$ is *central*, then we can do a little better.

Proposition 4.9. *If the finite group G has a central subgroup H, then*

$$\iota(G) \leq \iota(G/H)(\iota(H) + 1) + \iota(H).$$

Proof. The proof proceeds as before. But this time the rule $x \to xx_i^{-1}$ (where $\bar{x} = \bar{x}_i$) can be checked to define a function $\mathcal{I}(G) - \mathcal{I}(H) \to \mathcal{I}(H) \cup \{e\}$. Thus we obtain an injective function $\mathcal{I}(G) - \mathcal{I}(H) \to \mathcal{I}(G/H) \times (\mathcal{I}(H) \cup \{e\})$. The result follows. □

If the extension above is actually a direct product, then the inequality becomes an equality, i.e., writing Q in place of G/H, $\iota(Q \times H) = (\iota(Q) + 1)(\iota(H) + 1) - 1$.

Proposition 4.10. *Let G be a finite group of even order and k denote the number of conjugacy classes in G. Then $\iota(G)^2 - \iota(G) \leq (k-1)|G|$.*

Proof. Follow the proof of Aschbacher [1], (45.3), up to inequality (b), which is our desired result. Note that in that proof our ι is m and and our k is his $k+1$, since his k only counts nonidentity conjugacy classes. Note also that the hypothesis that $Z(G) = 1$ is not invoked until the paragraph following formula (b). □

Proposition 4.11. *Let G be a finite group and k denote the number of conjugacy classes in G. Then*

$$k \leq \frac{|G| + 3\,|G/[G,G]|}{4}.$$

Proof. The irreducible complex representations of G are in one-to-one correspondence with the conjugacy classes of elements of G. It is a consequence of the orthogonality relations for characters of representations over $\mathbb{C}$ that $|G|$ is expressed as the sum of the squares of the dimensions of the irreducible representations. The irreducible representations of dimension 1 are in one-to-one correspondence with the elements of the abelianization $G/[G,G]$. So we can write

$$|G| = |G/[G,G]| + \sum_{j=1}^{c} n_j^2 \geq |G/[G,G]| + 4c$$

where the sum is over the conjugacy classes in $[G,G]$. Each term in the sum is at least $2^2 = 4$. Thus

$$c \leq \frac{|G| - |G/[G,G]|}{4}$$

and, adding back in as many as $|G/[G,G]|$ conjugacy classes, the total of *all* conjugacy classes is

$$k \leq \frac{|G| + 3\,|G/[G,G]|}{4}.$$ □

We have the following extension of result (45.3) of Aschbacher [1].

Proposition 4.12. *Let G be a finite group of even order, with trivial center $Z(G)$ and such that $\iota(G) \geq |G|/2$. Then G has a normal subgroup of index 2, expressing G as a semidirect product extension of a normal subgroup K of order $|G|/2$ by a cyclic group of order 2.*

Proof. Let $|G| = 2^r h$, h odd. Since $Z(G)$ is trivial, G is not a 2-group, and so $h \geq 3$. If $h = 3$, we can assume $r \geq 2$, since both groups of order 6 have a subgroup of index 2.

Let $s =$ the minimum index of a proper subgroup of G. Applying Aschbacher [1], (45.3), we have

$$s \leq 2\left(\frac{|G|}{\iota(G)}\right)^2 \leq 2\left(\frac{2^r h}{2^{r-1} h}\right)^2 = 8.$$

Thus we have $2 \leq s \leq 8$. We now look more closely at the end of the proof of (45.3). We set $m = \iota(G)$ and $n = |G|/\iota(G)$, as in that proof.

$$\begin{aligned} s &\leq \frac{n(n - m^{-1})}{1 - m^{-1}} \\ &= \frac{\frac{|G|}{\iota(G)}\left(\frac{|G|}{\iota(G)} - \frac{1}{\iota(G)}\right)}{1 - \frac{1}{\iota(G)}} \\ &= \frac{|G|(|G| - 1)}{\iota(G)(\iota(G) - 1)} \\ &\leq \frac{2^r h(2^r h - 1)}{2^{r-1}h(2^{r-1}h - 1)} \\ &= 2\frac{2^r h - 1}{2^{r-1}h - 1} \\ &= 4\left(\frac{h - 1/2^r}{h - 1/2^{r-1}}\right) \\ &< 24/5 \end{aligned}$$

At the end we used the observation that the last fraction above involving h is less than $6/5$ provided when $h = 3$ we have $r \geq 2$, as discussed above.

Thus $2 \leq s \leq 4$.

First suppose $s = 2$. Let K be a (normal) subgroup of index 2. The extension

$$1 \to K \to G \to C_2 \to 1$$

is a semidirect product by Lemma 0.8.

Now we need to rule out $s = 3$.

If $s = 3$ and there is a normal subgroup K with $G/K \cong C_3$, then $\iota(G) = \iota(K) \leq |G|/2$, by Lemma 4.6.

Otherwise, if $s = 3$ then there is a subgroup K of G of index 3 that is not normal. Left translation on cosets then induces a surjective homomorphism $\rho : G \to \mathcal{S}_3$. The preimage of the subgroup of order 3 then provides a subgroup of G of index 2, as required.

Finally we rule out $s = 4$. A subgroup of index 4 yields a transitive representation in $\mathcal{S}_4$. But each such image group has a subgroup of index 2 or 3 and we are reduced to the cases already considered. (If a subgroup of $\mathcal{S}_4$ is not contained in $\mathcal{A}_4$, then intersecting with $\mathcal{A}_4$ yields a subgroup of index 2. But $\mathcal{A}_4$ has a normal Sylow 2-subgroup of index 3. So a subgroup of $\mathcal{A}_4$ not contained in its Sylow 2-subgroup contains a (normal) subgroup of index 3. Finally the Sylow 2-subgroup of $\mathcal{A}_4$ is $C_2 \times C_2$ and yields subgroup of index 2 for any nontrivial subgroup.) □

When a group is presented as a semidirect product extension of a subgroup by a cyclic group of order 2, as above, we can give a useful formula for the the number of involutions.

Proposition 4.13. *Suppose $G = KC$, a semidirect product of a subgroup K of index 2 by a cyclic group C of order 2 generated by x. Then*

$$\iota(G) = \iota(K) + \#\{y \in K : y^x = y^{-1}\}.$$

Proof. The elements of order 2, but not in K, have the form yx, $y \in K$. We compute $(yx)^2 = yxyx = yy^x$, which equals e if and only if $y^x = y^{-1}$. □

4.3. The proof of the main inequality

The overall structure of the proof will be by induction on the order of the group. But the inductive hypothesis will only be used in case the group has nontrivial center.

4.3.1. The case of nontrivial center

First we isolate the inductive part of the proof of the main inequality, handling the case of nontrivial center.

Proposition 4.14. *Let G be a finite group of even order with nontrivial center $Z = Z(G)$, such that $N_G(S) = S$ where S is a Sylow 2-subgroup. Suppose the main inequality is true for groups of order $< |G|$. Then $\iota(G) \leq 2^{r-1}(h+1) - 1$.*

Proof. Note that in this case the center $Z \subset N_G(S) \subset S$. In particular, $|Z| = 2^s$ for some s, $1 \leq s < r$. (If $s = r$, then S would be normal in G.)

Then by induction

$$\iota(G/Z) \leq 2^{r-s-1}(h+1) - 1$$

and so

$$\begin{aligned}\iota(G) \leq \iota(G) &\leq \iota(G/Z)(\iota(Z)+1) + \iota(Z) \text{ by Proposition 4.9}\\ &\leq (2^{r-s-1}(h+1)) - 1)(2^s) + (2^s - 1)\\ &= 2^{r-1}(h+1) - 1.\end{aligned}$$ □

Continuing with the non-inductive part of the proof of the main inequality, we may now assume that G has trivial center.

4.3.2. The case of trivial center

Thus we are reduced to giving a proof of the main inequality in the case of trivial center. In this part we do not invoke induction.

Proposition 4.15. *Let G be a finite group of even order, with trivial center. Then $\iota(G) \leq |G|/2 + |G|_2/2 - 1$.*

Proof. Let $|G| = 2^r h$, h odd, so that the required inequality becomes $\iota(G) \le 2^{r-1}(h+1) - 1$. We may assume the odd factor $h \ge 3$ since the claimed result is trivial for $h = 1$. (Of course, any 2-group has nontrivial center.) We may also assume that $\iota(G) \ge |G|/2$. Thus we know that the commutator quotient $G/[G,G] \approx C_2^\alpha$, $1 \le \alpha \le r$, by Corollary 4.7. Moreover, each element of order 2 in the quotient is the image of an element of order 2 in G.

First consider the extreme case when $\alpha = r$. In this case the commutator subgroup has odd order h and contains no involutions at all. It also follows that the surjection $G \to A = G/[G,G]$ is split by any choice of Sylow 2-subgroup in G. Therefore Proposition 4.5 implies that $\iota(G) \le 2^{r-1}(h+1) - 1$, as required. Henceforth we assume $1 \le \alpha \le r-1$.

When $1 \le \alpha \le r-1$ we bring together Propositions 4.10 and 4.11 to complete the proof as follows.

We have

$$\begin{aligned} \iota(G)^2 - \iota(G) &\le \left(\frac{|G| + 3\,|G/[G,G]|}{4} - 1 \right) |G| \\ &= \left(\frac{|G|}{2} \right)^2 + (3 \times 2^{\alpha-2} - 1)|G|. \end{aligned}$$

Thus, completing the square, we have

$$\iota^2 - \iota \le \left(\frac{|G|}{2} + 3 \cdot 2^{\alpha-2} - 1 \right)^2 - (3 \cdot 2^{\alpha-2} - 1)^2 \le \left(\frac{|G|}{2} + 3 \cdot 2^{\alpha-2} - 1 \right)^2 \quad (1)$$

and

$$(\iota - 1/2)^2 \le \left(\frac{|G|}{2} + 3 \cdot 2^{\alpha-2} - 1 \right)^2 + 1/4.$$

We warn the reader that we will eventually have to return and account for the term $(3 \cdot 2^{\alpha-2} - 1)^2$ ignored in this inequality. Therefore

$$\begin{aligned} \iota &\le \sqrt{\left(\frac{|G|}{2} + 3 \cdot 2^{\alpha-2} - 1 \right)^2 + 1/4} + 1/2 \\ &\le \left(\frac{|G|}{2} + 3 \cdot 2^{\alpha-2} - 1 \right) + 1/2 + 1/2 \\ &= \frac{|G|}{2} + 3 \cdot 2^{\alpha-2} \\ &= 2^{r-1}h + 3 \cdot 2^{\alpha-2} \\ &\le 2^{r-1}h + 2^\alpha - 2^{\alpha-2} \\ &\le 2^{r-1}h + 2^{r-1} - 2^{\alpha-2} \quad \text{(since by assumption } \alpha \le r-1) \\ &\le 2^{r-1}h + 2^{r-1} - 1 \quad \text{(provided } \alpha \ge 2). \end{aligned}$$

It remains to look more closely at the case $\alpha = 1$, $r > 1$. From above we have

$$\begin{aligned}\iota &\le \left(\frac{|G|}{2} + 3\cdot 2^{\alpha-2} - 1\right) + 1/2 + 1/2\\ &= \left(\frac{|G|}{2} + 1/2\right) + 1\\ &= 2^{r-1}h + 3/2\\ &< 2^{r-1}h + 2^{r-1} - 1 \quad (\text{for } r \ge 3).\end{aligned}$$

Finally we need to examine the case where $r = 2$ and $\alpha = 1$. For this we have to go back further in the analysis and add back in the quantity $(3\cdot 2^{\alpha-2} - 1)^2$ we dropped out before in order to simplify things.

We return to the inequality (1), in which we substitute $r = 2$ and $\alpha = 1$.

$$\iota^2 - \iota \le \left(\frac{|G|}{2} + 3\cdot 2^{\alpha-2} - 1\right)^2 - (3\cdot 2^{\alpha-2} - 1)^2 = (2h + 1/2)^2 - (1/2)^2.$$

Completing the square, we have

$$\iota^2 - \iota + 1/4 \le 4h^2 + 2h + 1/4$$

or

$$(\iota - 1/2)^2 \le (2h + 1/2)^2,$$

so

$$\iota - 1/2 \le 2h + 1/2$$

or

$$\iota \le 2h + 1 = 2(h + 1) - 1.$$

The proof is complete. □

This also completes the proof of Theorem 1.2 of the introduction and the main inequality, Theorem 4.1.

4.4. The proof of the classification theorem

Let G be a group such that $\iota(G) = |G|/2 + |G|_2/2 - 1$. Call such a group 2-*maximal*. Again we let $|G| = 2^r h$, so that $\iota(G) = 2^{r-1}(h + 1) - 1$.

First of all we know that $N_G(S) = S$ for any Sylow 2-subgroup, by Corollary 4.4.

4.4.1. The abelian case

If G is abelian, then $\iota(G) = |G|/2 + |G|_2/2 - 1 = |G_2| - 1$. It follows that G is an elementary abelian 2-group, and we are done. Henceforth assume that G is nonabelian.

4.4.2. The case of nontrivial center

Now suppose the center $Z = Z(G) \neq \{e\}$, and G is nonabelian. Then G/Z is nontrivial, too. Moreover, the center normalizes the Sylow 2-subgroup S, so lies in S, and so has order a power of 2. We note that G/Z also has even order. For if $|G/Z|$ were odd, then $\iota(G) = \iota(Z) \leq |Z|/2+|Z_2|/2-1 < |G|/2+|G_2|/2-1 = \iota(G)$, again a contradiction.

Now one can see that both Z and G/Z must also be 2-maximal, as follows. Examining the proof of Proposition 4.14, note that the inequalities there must be equalities in the case when G is 2-maximal. From that we see that $\iota(Z) = 2^s - 1$ and $\iota(G/Z) = 2^{r-s-1}(h+1) - 1$, which means Z and G/Z are 2-maximal.

Therefore induction can be applied to G/Z (and to Z). Thus Z is an abelian group of exponent 2 and there is a normal subgroup $H \subset G$ of index 2, containing the center Z, and an element $x \in G$ not in H such that $x^2 \in Z$. Moreover, G/Z has Sylow 2-subgroup of exponent 2 and H/Z is abelian and x acts by inversion on H/Z. Moreover, we can assume $x^2 = e$ in G itself by Corollary 4.7.

To see that G itself has standard form, we still need to show that a Sylow 2-subgroup $S \subset G$ is of exponent 2 and we need to see that $G/H \approx C_2$ acts by inversion on all of H.

Now

$$\begin{aligned}\iota(H) &\leq \iota(H/Z)(\iota(Z)+1) + \iota(Z)\\ &= (2^{r-s-1} - 1)(2^s) + 2^s - 1\\ &= 2^{r-1} - 1.\end{aligned}$$

Next

$$\iota(G) = \iota(H) + \left|\{w \in H : w^x = x^{-1}\}\right|.$$

Thus

$$2^{r-1}(h+1) - 1 \leq 2^{r-1} - 1 + \#\{w \in H : w^x = x^{-1}\}.$$

Whence

$$\left|\{w \in H : w^x = x^{-1}\}\right| \geq 2^{r-1}h.$$

That is, x inverts all of H. It follows that H is abelian.

Now

$$\iota(G) = \iota(H) + |H|,$$

so that

$$2^{r-1}(h+1) - 1 = \iota(H) + 2^{r-1}h,$$

from which it follows that

$$\iota(H) = 2^{r-1} - 1.$$

From this it follows that the Sylow 2-subgroup H_2 has exponent 2. Thus we see that a Sylow subgroup for G is a central, split extension of C_2^{r-1} by C_2, hence isomorphic to C_2^r, as required. This completes the case when the center is nontrivial

4.4.3. The case of trivial center

In what follows we may therefore assume that $Z(G) = \{e\}$. Then the various inequalities we developed for groups with $\iota(G) \geq |G|/2$ in Proposition 4.15 must collapse to equalities. But in most cases this cannot happen because of the strict inequality in inequality (1) that arose when the term $-(3 \cdot 2^{\alpha-2} - 1)^2$ was dropped out.

Going through the proof of Proposition 4.15, we are thus left to look more closely at the case $\alpha = r$ and the case $\alpha = 1, r = 2$.

• The case $\alpha = r$. The analysis in this case refers back to Proposition 4.5, dealing with the case of a semidirect product $G = HS$, with normal 2-complement H. But an equality here requires that all nontrivial elements of S act by inversion on H. This forces H to be abelian, and S to be cyclic of order 2, acting on H by inversion. So G clearly has the required form.

• The case $\alpha = 1, r = 2$. We have a group G of order $|G| = 4h$, h odd, with $\iota(G) = 2(h+1)-1 = 2h+1$. We may assume that $Z(G) = \{1\}$, and $G/[G, G] \approx C_2$ or $C_2 \times C_2$. Also we may assume that if $S < G$ is a Sylow 2-subgroup, then $N_G(S) = S$. We may also assume that there is an involution $t \in G$ representing the generator of $G/K = C_2$.

It follows from the Burnside Normal p-Complement Theorem that S has a normal 2-complement, a normal subgroup H of order h and that $G = HS$, a semidirect product. (See [1], (39.1).) The precise hypothesis is that the Sylow subgroup be contained in the center of its normalizer.)

It suffices to prove that one of the three involutions of S acts trivially on H and that another one (actually both of the others) acts by inversion on H.

We look again at the argument of Proposition 4.5. Note that a given nontrivial element of H can be inverted by at most two of the three involutions of S, since any involution is the product of the other two, while all three involutions trivially invert the identity element $e \in H$.

$$\begin{aligned}
\iota(G) &= \sum_{x \in \mathcal{I}(S)} \left|\{y \in H : y^x = y^{-1}\}\right| \\
&= \sum_{y \in H} \left|\{x \in \mathcal{I}(S) : y^x = y^{-1}\}\right| \\
&\leq 2(h-1) + 3 \\
&= 2h + 1.
\end{aligned}$$

The assumption that $\iota(G) = 2h + 1$ therefore implies that each nontrivial element of H is inverted by exactly two of the three involutions of S and fixed by the remaining one.

Note that if an involution t inverts one nontrivial element $y \in H$, then it must invert them all. Otherwise there is a second nontrivial element $x \in H$ fixed by t. But then xy is neither fixed nor inverted by t.

From this it follows easily that H can be expressed in standard form of a semidirect product of an abelian group by an involution acting by inversion. □

5. The oddness bound in the case of a general transitive permutation group

Here we complete the proof of Theorem 1.4 as stated in the introduction. We are reduced to showing that the desired inequality holds for the orbit-edge-coloring associated with a transitive permutation group. The preceding section handles the case when the permutation group acts regularly. The focus here is on the case of a nontrivial point stabilizer.

First we set up the necessary, somewhat elaborate, notation for the general situation.

In the case of a (weakly) primitive transitive group we will prove the stronger statement that $\mathcal{O}(\pi) \leq d/3$. Then we complete the proof of the main Theorem 1.4, in general, by showing that the desired inequality holds for (weakly) imprimitive actions.

5.1. The general set up

Here $G < \mathcal{S}_{d+1}$ is a transitive subgroup with nontrivial isotropy subgroup $H < G$ of a point. On the one hand, the bigger the group, the fewer the orbits of edges. On the other hand the analysis of which edges lie in the same orbit is more complicated.

We identify the vertices of K_{d+1} with the cosets G/H. When are two edges in the same edge-orbit? We answer this by describing the stabilizer group $\mathrm{Stab}([g_1H, g_2H])$ of a general edge $[g_1H, g_2H]$. First consider the subgroup of all $g \in G$ fixing both end points. This amounts to the conditions that $gg_1H = g_1H$ and $gg_2H = g_2H$. This yields the condition $g \in g_1Hg_1^{-1} \cap g_2Hg_2^{-1}$. This at least contains the identity element and is a subgroup.

The full stabilizer group is an index 1 or 2 extension allowing for the possibility of $g \in G$ such that $gg_1H = g_2H$ and $gg_2H = g_1H$. This yields the condition $g \in g_2Hg_1^{-1} \cap g_1Hg_2^{-1}$.

We conclude that

$$\mathrm{Stab}([g_1H, g_2H]) = \left(g_1Hg_1^{-1} \cap g_2Hg_2^{-1}\right) \cup \left(g_2Hg_1^{-1} \cap g_1Hg_2^{-1}\right).$$

Note that $\mathrm{Stab}([g_1H, g_2H])$ contains

$$\mathrm{Stab}(g_1H) \cap \mathrm{Stab}(g_2H) = g_1Hg_1^{-1} \cap g_2Hg_2^{-1}$$

as a subgroup of index 1 or 2, as one sees by noting that the product of any two elements in the second part of $\mathrm{Stab}([g_1H, g_2H])$ lies in the first part. Moreover, the product of an element of the first part with an element of the second part lies in the second part.

Any edge is in the same orbit as an edge of the form $[H, gH]$. In the latter case the formulas above specialize to say that

$$\mathrm{Stab}([H, gH]) = \left(H \cap gHg^{-1}\right) \cup \left(gH \cap Hg^{-1}\right).$$

The number of edges in the orbit of edge $[H, gH]$ is then

$$E_g = \left|G/\left[\left(H \cap gHg^{-1}\right) \cup \left(gH \cap Hg^{-1}\right)\right]\right|.$$

Standard elementary counting of edge-ends shows that $v_g(d+1) = 2E_g$ where v_g is the common valence of all vertices in the edge-orbit graph. Thus

$$v_g = \frac{2E_g}{d+1} = \frac{2\,|G/\left[(H \cap gHg^{-1}) \cup (gH \cap Hg^{-1})\right]|}{|G/H|}.$$

Our goal is to determine for which cosets $gH \neq H$ we have v_g odd.

We distinguish four cases for further analysis.

1. $g \in N(H),\ gH \cap Hg^{-1} = \emptyset$.
2. $g \in N(H),\ gH \cap Hg^{-1} \neq \emptyset$ (whence $Hg^{-1} = g^{-1}H = gH$).
3. $g \notin N(H),\ gH \cap Hg^{-1} = \emptyset$.
4. $g \notin N(H),\ gH \cap Hg^{-1} \neq \emptyset$.

In all cases $|G\ /\,(H \cap gHg^{-1})| = |G/H| \times |H\ /\,(H \cap gHg^{-1})| = ns$. Also, $s = 1$ if and only if $g \in N(H)$ (cases (1) and (2)). Then

1. $v_g = 2n/n = 2$, even.
2. $v_g = 2n/2/n = 1$, odd.
3. $v_g = 2ns/n = 2s \geq 4$, even.
4. $v_g = 2ns/2/n = s \geq 2$, potentially odd.

For future reference we note that the condition in (2) is equivalent to gH representing an element of order 2 in $N(H)/H$. The condition in (1) is equivalent to gH representing an element of order greater than 2 in $N(H)/H$. It is less clear how to understand the conditions in (3) and (4).

5.2. The partition problem in the primitive case

A transitive permutation group G is said to be *primitive* if no nontrivial block decomposition is preserved. This corresponds to having no subgroup strictly between the permutation group G and its point stabilizer H. So we may assume in particular that $N(H) = H$. (If $N(H) = G$ then H would act trivially; but we assume our actions are faithful.) For lack of a better name for a transitive permutation group such that $H = N(H)$, we will call it *weakly primitive.*

We can give a stronger bound on the oddness in the primitive case.

Proposition 5.1. *If d is odd and π is the partition of d associated with a (weakly) primitive transitive group of degree $n = d+1$ that is not regularly transitive, then Oddness$(\pi) \leq d/3$.*

Proof. A weakly primitive transitive group satisfies $N(H) = H$. It follows that the valence of an edge orbit containing $[e, g]$ is s or $2s$, where $s = |H/H \cap gHg^{-1}|$ and that $s \geq 2$. The odd valences then are at least 3 and there are at most $d/3$ of them. □

5.3. The general partition problem

Consider an arbitrary transitive permutation group $G < \mathcal{S}_n$ of degree $n = 2^r h$, h odd. We assume that $r \geq 1$, so that n is even.

Let H denote the stabilizer of one point, and $N = N(H)$ be the normalizer of H in G. Set $[N : H] = p$ and $[G : N] = q$, where $n = pq$.

If $p = 1$, then $N = H$ and we have seen that Oddness $\leq (n-1)/3$, and $(n-1)/3 \leq 2^{r-1}(h+1) - 1$, since $r \leq 1$.

If $q = 1$, then H is normal in G, and since the action is faithful, we conclude that $H = \{e\}$ and that G must act regularly. From earlier work we have Oddness $\leq n - h$, as required.

Henceforth assume both $p \geq 2$ and $q \geq 2$.

According to the analysis of elements of order 2 in a group, there are up to $p/2 + p_2/2 - 1$ terms equal to 1 in the partition associated to G. That leaves $d - p/2 - p_2/2 + 1$ possible entries that could be partitioned into odd terms of size at least 3. That is we have at most

$$\frac{pq - p/2 - p_2/2}{3}$$

additional odd terms of size at least 3 in the partition. In particular,

$$\mathcal{O}(G) \leq p/2 + p_2/2 - 1 + \frac{pq - p/2 - p_2/2}{3} = \frac{pq + p + p_2 - 3}{3}.$$

We have to determine when

$$\frac{pq + p + p_2 - 3}{3} \leq 2^{r-1}(h+1) - 1.$$

Now letting $h = pq/2^r$, our required inequality is equivalent to

$$\frac{pq + p + p_2 - 3}{3} \leq pq/2 + 2^{r-1} - 1$$

or

$$\frac{pq + p + p_2 - 3}{3} \leq \frac{pq + 2^r - 2}{2},$$

which is equivalent to

$$2pq + 2p + 2p_2 - 6 \leq 3pq + 2^r \cdot 3 - 6$$

or

$$2p + 2p_2 \leq pq + 2^r \cdot 3,$$

which always holds, since $2 \leq q$ and $p_2 \leq 2^r$. □

6. Extremal groups

Define a transitive permutation group $G < \mathcal{S}_n$ of even degree n to be *extremal* if its associated partition has maximal oddness: $\mathcal{O}(\pi) = n/2 + n_2/2 - 1 = 2^{r-1}(h+1) - 1$, where $n = 2^r h$, h odd and $r \geq 1$. We will see after the fact that this is equivalent to the condition that the associated partition of $d = n-1$ has the form $[a(1), b(2)]$, where $a = n/2 + n_2/2 - 1$ and $b = (n/2 - n_2/2)/2$.

Theorem 6.1. *An extremal, regularly transitive group G of even degree is isomorphic to one of the following groups: a group with Sylow 2-subgroup of exponent 2 that can be expressed as a split extension of dihedral type of an abelian group by a cyclic group of order 2 (acting by inversion).*

Proof. This is just a re-interpretation of the characterization of 2-maximal groups given in Theorem 4.2. □

In fact an extremal permutation group must act regularly, as we now show.

Theorem 6.2. *Any extremal transitive permutation group of even degree acts regularly, and hence is one of the groups described for the extremal, regularly transitive case.*

Proof. Assume $G < \mathcal{S}_n$ is a transitive group with point stabilizer $H < G$ nontrivial. We proceed to show that such a group cannot be extremal.

If $H = N(H)$ then we proved that $\mathcal{O}(G) \leq d/3 = (n-1)/3$. But if $(n-1)/3 = n/2 + n_2/2 - 1$, then $4 = n + 3n_2$, which cannot happen since $n_2 \geq 2$.

Now set $[N : H] = p$ and $[G : N] = q$, where $n = pq$. According to the preceding observation we may assume $p > 1$.

If $q = 1$, then H is normal in G, and since the action is faithful, we conclude that $H = \{e\}$ and that G must act regularly.

Henceforth assume both $p \geq 2$ and $q \geq 2$. We aim to show G cannot be extremal under these assumptions.

We examine the earlier analysis, assuming that all inequalities are equalities. One concludes that $2p + 2p_2 = pq + 2^r \cdot 3$. But this cannot happen, since $2 \leq q$ and $2p_2 \leq 2^{r+1} < 2^r \cdot 3$. □

Corollary 6.3. *A transitive permutation group of even degree n, is extremal if and only if its associated partition of $d = n - 1$ has the form $[a(1), b(2)]$, where $a = n/2 + n_2/2 - 1$.*

Proof. Since the group is extremal, we know that it acts regularly. This implies that its associated partition has the form $[a(1), b(2)]$, where a is the number of involutions in the group. Extremality implies the number of involutions is $n/2 + n_2/2 - 1$. □

Note that it follows that, for example, the isometry group of an equifacetal 5-simplex with partition $[1, 1, 3]$ is the same as that of an equifacetal 5-simplex with partition $[1, 1, 1, 2]$. The simplex in the case of the former partition is extremal, but then the partition associated with its full isometry group is the latter one. Moreover, every vertex-transitive coloring of K_6 with partition type $[1,1,3]$ admits a refinement of partition type $[1,1,1,2]$.

Remark 1. [Groups of odd degree] The main objects of study in this paper have been permutation groups of even degree n. Is there is an analogue of the preceding result that an extremal group acts regularly when $n = d+1$ is odd instead of even? In this case the terms in the associated partition of d are necessarily all even. One might define a transitive permutation group of odd degree to be *extremal* if its associated partition has the form $[(d/2)2]$. However, such an extremal transitive permutation group of odd degree need not act regularly. For example, both the cyclic group C_n of order n and the dihedral group D_{2n} of order $2n$ are naturally viewed as permutation groups in $\mathcal{S}_n$. But both lead to the same coloring and partition $[2,2,\ldots,2]$ when $n \geq 5$ and odd. The former is regular and the latter is not. Are there other, substantially different, examples?

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