Grassmannian structures on manifolds

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Abstract

Grassmannian structures on manifolds are introduced as subbundles of the second order framebundle. The structure group is the isotropy group of a Grassmannian. It is shown that such a structure is the prolongation of a subbundle of the first order framebundle. A canonical normal connection is constructed from a Cartan connection on the bundle and a Grassmannian curvature tensor for the structure is derived.

1 Introduction

The theory of Cartan connections has lead S. Kobayashi and T. Nagano, in 1963, to present a rigourous construction of projective connections [3]. Their construction, relating the work of Eisenhart, Veblen, Thomas a.o. to the work of E. Cartan, has a universal character which we intend to use in the construction of Grassmannian-like structures on manifolds. The principal aim is to generalise Grassmannians in a similar way. By doing so we very closely follow their construction of a Cartan connection on a principal bundle subjected to curvature conditions and the derivation of a normal connection on the manifold.

The action of the projective group $Pl(n_o)$ on a Grassmannian $G(l_o, n_o)$ of l_o-planes in $\mathbb{R}^{n_o}$ is induced from the natural action of $Gl(n_o)$ on $\mathbb{R}^{n_o}$. Let H be the isotropy group of this action at a fixed point e of $G(l_o, n_o)$. The generalisation will consist in the construction of a bundle P with structure group H and base manifold

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M of dimension $m_o = l_o k_o$ with $k_o = n_o - l_o$. The bundle P will be equipped with a Cartan connection with values in the Lie algebra of the projective group, which makes the bundle P completely parallelisable. We will show that such a connection exists and is unique if certain curvature conditions are imposed. The Cartan connection identifies the tangent space $T_x(M)$ for each $x \in M$ with the vectorspace $L(\mathbb{R}^{l_o}, \mathbb{R}^{k_o})$. Identifying $L(\mathbb{R}^{l_o}, \mathbb{R}^{k_o})$ with $V = \mathbb{R}^{m_o}$, the group H acts on V to the first order as $G_o = Gl(l_o) \times {}^{\tau}Gl(k_o)^{-1} / \exp tI_{n_o}$ properly embedded in $Gl(m_o)$. Let $\tilde{\boldsymbol{g}}^0$ denote the Lie algebra of this group, which is seen as a subspace of $V \otimes V^*$. We prove that if $k_o \geq 2$ and $l_o \geq 2$, the Lie algebra $\mathbf{h}$ of H, as subspace of $V \otimes V^*$, is the first prolongation of the Lie algebra $\tilde{\boldsymbol{g}}^0$. Moreover the second prolongation equals zero.

The action of H on V allows to define a homomorphism of P into the second order framebundle $F^2(M)$. The image, $Gr(k_o, l_o)(M)$, is called a Grassmannian structure on M. From the previous algebraic considerations it follows that a Grassmannian structure on a manifold is equivalent with a reduction of the framebundle $F^1(M)$ to a subbundle $B^{(k_o,l_o)}(M)$ with the structure group G_o. A Grassmannian connection from this point of view, is an equivalence class of symmetric affine connections, all of which are adapted to a subbundle of $F^1(M)$ with structure group G_o. The action of G_o in each fibre is defined by a local section $\sigma : x \in M \to F^1(M)(x)$ together with an identification of $T_x(M)$ with $M(k_o, l_o)$. This result explains in terms of G-structures the well known fact that the structure group of the tangent bundle on a Grassmannian, $G(l_o, n_o)$, reduces to $Gl(k_o) \times Gl(l_o)$ [6]. The consequences for the geometry and tensoralgebra are partly examined in the last paragraph, but will be studied in a future publication.

We remark that as a consequence of the algebraic structure the above defined structure is called Grassmannian if $k_o \geq 2$ and $l_o \geq 2$. Otherwise the structure is a projective structure. Hence the manifolds have dimension $m_o = k_o l_o$, with $k_o, l_o \geq 2$.

Let $(\bar{x}^\alpha)$, $\alpha = 1, \cdots, m_o$ be coordinates on $\mathbb{R}^{m_o}$, and (e^i_a), $a = 1, \cdots, k_o$; $i = 1, \cdots, l_o$, the natural basis on $M(k_o, l_o)$. (x^a_i) are the corresponding coordinates on $M(k_o, l_o)$. We will identify both spaces by $\alpha = (a-1)l_o + i$. Let $\sigma : \mathcal{U} \subset M \to F^1(M)$ be a local section and $\bar{\sigma}$ be the associated map identifying the tangent space $T_x(M)$ $(x \in \mathcal{U})$ with $M(k_o, l_o)$. An adapted local frame with respect to some coordinates $(\mathcal{U}, (x^\alpha))$ is given as $\bar{\sigma}^{-1}(x)(e^i_a) = E^{i\alpha}_a \frac{\partial}{\partial x^\alpha}(x)$. If ∇ and $\tilde{\nabla}$ are two adapted symmetric linear connections on $B^{(k_o,l_o)}(M)$, then there exists a map $\mu : \mathcal{U} \to M(l_o, k_o)$ such that for $X, Y \in \mathcal{X}(M)$:

$$\tilde{\nabla}_X Y = \nabla_X Y + \bar{\sigma}^{-1}[(\mu \,.\, \bar{\sigma}(X)) \,.\, \bar{\sigma}(Y) + (\mu \,.\, \bar{\sigma}(Y)) \,.\, \bar{\sigma}(X)].$$

Because $\mu \in M(l_o, k_o)$ and $\bar{\sigma}(X)(x) \in M(k_o, l_o)$, for $X \in \mathcal{X}(\mathcal{U})$, the term $(\mu \,.\, \bar{\sigma}(X)(x))$, as composition of matices, is an element of $M(l_o, l_o)$ which acts on $\bar{\sigma}(Y)(x)$ giving thus an element of $M(k_o, l_o)$.

Analogous to the projective case we will construct a canonical normal Grassmannian connection and calculate the expression of the coefficients with respect to an adapted frame. The curvature of the Grassmannian structure is given by the forms Ω^i_j, Ω^a_b, Ω^i_a, with respect to a Lie algebra decomposition of $\mathbf{h}$. We prove that if $l_o \geq 3$ or $k_o \geq 3$ the vanishing of Ω^i_j or Ω^a_b is necessary and sufficient for the local flatness of the bundle P. The two curvature forms Ω^i_j and Ω^a_b are basic forms on the quotient $\pi^2_1 : Gr(k_o, l_o)(M) \subset F^2(M) \to F^1(M)$ and hence determine the Grassmannian curvature tensor, whose local components are given by

$$K^\alpha_{\beta\gamma\sigma} = K^i_{j\gamma\sigma} F^a_{i\beta} E^{j\alpha}_a + K^b_{a\gamma\sigma} F^a_{i\beta} E^{i\alpha}_b,$$

with $\Omega^i_j = K^i_{j\alpha\beta} dx^\alpha \otimes dx^\beta$ and $\Omega^a_{b\alpha\beta} dx^\alpha \otimes dx^\beta$. $E^{j\alpha}_b$ is an adapted frame and $F^a_{i\beta}$ the corresponding coframe. It follows that the vanishing of the Grassmannian curvature tensor is a necessary and sufficient condition for the local flatness of the Grassmannian structure for any $l_o \geq 2$ and $k_o \geq 2$.

We assume all manifolds to be connected, paracompact and of class C^∞. All maps are of class C^∞ as well. $Gl(n_o)$ denotes the general linear group on $\mathbb{R}^{n_o}$ and $gl(n_o)$ its Lie algebra. We will use the summation convention over repeated indices. The indices take values as follows : $\alpha,\ \beta, \cdots = 1, \cdots, m_o = k_o l_o$; $a,\ b,\ c \cdots = 1, \cdots, k_o$; $i,\ j,\ k, \cdots = 1, \cdots, l_o$. Cross references are indicated by [(.)] while references to the bibliography by [.].

2 Grassmannians

A. Projective Group Actions

Let $G(l_o, n_o)$ be the Grassmannian of the l_o-dimensional subspaces in $\mathbb{R}^{n_o}$. Dim $G(l_o, n_o) = l_o k_o,\ n_o = l_o + k_o$. Let S be a k_o-dimensional subspace of $\mathbb{R}^{n_o}$. An associated big cell $\mathcal{U}(S)$ to S in $G(l_o, n_o)$ is determined by all transversal subspaces to S of dimension l_o in $\mathbb{R}^{n_o}$. One observes that

$$G(l_o, n_o) = \cup_I\, \mathcal{U}(S_I)$$

where I is any subset of length k_o of $\{1,\ 2,\ \cdots, n_o\}$ and S_I the subspace of dimension k_o spanned by the coordinates (x^I) in $\mathbb{R}^{n_o}$.

Let $(x^1 \cdots,\ x^{l_o},\ x^{l_o+1}, \cdots, x^{n_o})$ be the natural coordinates on $\mathbb{R}^{n_o}$. For simplicity we will choose a rearrangement of the coordinates such that S is given by the condition $x^1 = x^2 = \cdots = x^{l_o} = 0$.

Let $M(n_o, l_o)$ be the space of $(n_o \times l_o)$ matrices (n_o rows and l_o columns). Any element may be considered as l_o linearly independent vectors in $\mathbb{R}^{n_o}$. Hence each $y \in M(n_o, l_o)$ determines an l_o-plane in $\mathbb{R}^{n_o}$. We get a natural projection

$$\pi\ :\ M(n_o, l_o) \to \mathcal{U}(S), \tag{1}$$

which is a principal fibration over $\mathcal{U}(S)$ with structure group $Gl(l_o)$. Representing the coordinate system on $M(n_o, l_o)$ by a matrix Z, the big cell $\mathcal{U}(S)$ is coordinatised as follows. If $Z \in M(n_o, l_o)$, we will write

$$Z = \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix},$$

with Z_0 an $l_o \times l_o$ matrix and Z_1 an $k_o \times l_o$ matrix, $n_o = k_o + l_o$.

The coordinates are obtained by

$$\tilde{Z} = Z_1 . Z_0^{-1},$$

where we assumed Z_0 to be of maximal rank.
In terms of its elements we get

$$Z = \begin{pmatrix} z^i_j \\ z^a_j \end{pmatrix},$$

$i, j = 1, \cdots, l_o$ and $a = 1, \cdots, k_o$, to which we refer as the homogeneous coordinates. Denoting by w^i_j the inverse of z^i_j, we obtain

$$\tilde{Z} = (x^a_i) = (z^a_j w^j_i).$$

which are the local coordinates on the cell. In the sequel we will identify the cel with $M(k_o, l_o)$.

The action of the group $Gl(n_o)$ on $\mathrm{I\!R}^{n_o}$ induces a transitive action of $Pl(n_o)$ on $G(l_o, n_o)$. On a big cell the action of $Pl(n_o)$ is induced from the action of $Gl(n_o)$ on Z on the left. Let β be in $Gl(n_o)$. In matrix representation we write β as :

$$\beta = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix}, \tag{2}$$

with $\beta_{00} \in M(l_o,\, l_o)$, $\beta_{11} \in M(k_o,\, k_o)$, $\beta_{10} \in M(k_o,\, l_o)$, $\beta_{01} \in M(l_o,\, k_o)$.

The local action of an open neighbourhood of the identity in the subset of $Gl(n_o)$ defined by $\det \beta_{00} \neq 0$ on $M(k_o, l_o)$ is given in fractional form by

$$\phi_\beta : x \mapsto (\beta_{10} + \beta_{11}\, x)(\beta_{00} + \beta_{01}\, x)^{-1} \tag{3}$$

for $\beta \in Gl(n_o)$ as in [(2)] and $x \in M(k_o, l_o)$.

Because the elements of the center of $Gl(n_o)$ are in the kernel of ϕ_β this action induces an action of an open neighbourhood of the identity in $Pl(n_o)$.
In terms of the coordinates and using the notation

$$\beta_{00} = (\beta^i_j),\ \beta_{01} = (\beta^i_a),\ \beta_{10} = (\beta^a_i),\ \beta_{11} = (\beta^a_b) \ \text{ and } \ \beta_{00}^{-1} = (\gamma^i_j),$$

we find the Taylor expression

$$\begin{aligned} \bar{x}^a_l &= \beta^a_k \gamma^k_l + (\beta^a_b - \beta^a_k \gamma^k_j \beta^j_b) x^b_m \gamma^m_l \\ &\quad - \beta^a_c x^c_k \gamma^k_m \beta^m_b x^b_n \gamma^n_l + \beta^a_k \gamma^k_m \beta^m_c x^c_j \gamma^j_n \beta^n_e x^e_r \gamma^r_l + \cdots . \end{aligned} \tag{4}$$

Consequences :

(a) The orbit of the origin of the coordinates in $M(k_o, l_o)$, is locally given by $(0) \mapsto \beta_{10}\beta_{00}^{-1}$.
(b) The isotropy group H at $0 \in M(l_o, k_o)$ is the group

$$H : \{\beta = \begin{pmatrix} \beta_{00} & \beta_{01} \\ 0 & \beta_{11} \end{pmatrix} / \exp t.\mathrm{I}_{n_o}\}, \tag{5}$$

with $\beta_{00} \in Gl(l_o)$ and $\beta_{11} \in Gl(k_o)$. The subgroup H in Taylor form is given by

$$\bar{x}^a_j = \beta^a_b \gamma^m_j x^b_m - \frac{1}{2}[\beta^a_b \gamma^i_k \beta^k_c \gamma^l_j + \beta^a_c \gamma^l_k \beta^k_b \gamma^i_j] x^b_i x^c_l + \cdots . \tag{6}$$

B. The Maurer Cartan Equations

Let $(u^i_a, u^i_j, u^a_b, u^a_i)$, with $i, j = 1, \cdots, l_o$, $a, b = 1, \cdots, k_o$, be local coordinates at the identity on $Gl(n_o)$ according to the decomposition [(2)] and $(\bar{\omega}^i_a, \bar{\omega}^i_j, \bar{\omega}^a_b, \bar{\omega}^a_i)$ the left invariant forms coïnciding with $(du^i_a, du^i_j, du^a_b, du^a_i)$ at the identity. The Maurer Cartan equations are

$$\begin{array}{rrcl}
(1) & d\bar{\omega}^a_j & = & -\bar{\omega}^a_k \wedge \bar{\omega}^k_j - \bar{\omega}^a_b \wedge \bar{\omega}^b_j \\
(2) & d\bar{\omega}^i_j & = & -\bar{\omega}^i_k \wedge \bar{\omega}^k_j - \bar{\omega}^i_b \wedge \bar{\omega}^b_j \\
(3) & d\bar{\omega}^a_b & = & -\bar{\omega}^a_k \wedge \bar{\omega}^k_b - \bar{\omega}^a_c \wedge \bar{\omega}^c_b \\
(4) & d\bar{\omega}^i_a & = & -\bar{\omega}^i_k \wedge \bar{\omega}^k_a - \bar{\omega}^i_b \wedge \bar{\omega}^b_a .
\end{array}$$

Let $\bar{\omega}_1 = \bar{\omega}^i_i$ and $\bar{\omega}_2 = \bar{\omega}^a_a$. We define

$$\omega^i_j = \bar{\omega}^i_j - \frac{1}{l_o}\delta^i_j \, \bar{\omega}_1, \; \omega^a_b = \bar{\omega}^a_b - \frac{1}{k_o}\delta^a_b \, \bar{\omega}_2, \; \omega_* = \frac{1}{l_o}\bar{\omega}_1 - \frac{1}{k_o}\bar{\omega}_2 . \tag{7}$$

Passing to the quotient $Gl(n_o)/\exp t.\mathrm{I}_{n_o}$ we find the Maurer Cartan equations on $Pl(n_o)$.

Proposition 2.1 *The Maurer Cartan equations on $Pl(n_o)$ are*

$$\begin{array}{rrcl}
(1) & d\omega^a_j & = & -\omega^a_k \wedge \omega^k_j - \omega^a_b \wedge \omega^b_j - \omega^a_i \wedge \omega_* \\
(2) & d\omega^i_j & = & -\omega^i_k \wedge \omega^k_j - \omega^i_b \wedge \omega^b_j + \dfrac{1}{l}\, \delta^i_j \, \omega^k_c \wedge \omega^c_k \\
(3) & d\omega^a_b & = & -\omega^a_k \wedge \omega^k_b - \omega^a_c \wedge \omega^c_b + \dfrac{1}{k}\, \delta^a_b \, \omega^c_k \wedge \omega^k_c \\
(4) & d\omega^i_a & = & -\omega^i_k \wedge \omega^k_a - \omega^i_b \wedge \omega^b_a + \omega^i_a \wedge \omega_* \\
(5) & d\omega_* & = & \dfrac{k_o + l_o}{k_o l_o}\, \omega^a_i \wedge \omega^i_a .
\end{array} \tag{8}$$

Remark that $\omega^i_i = \omega^a_a = 0$.

The Lie algebra of $Pl(n_o)$, $\boldsymbol{g}$, in this representation is found by taking the tangent space at the identity, W, to the submanifold in $Gl(n_o)$ defined by $(\det \beta_{00})^k (\det \beta_{11})^l = 1$. The quotient of the algebra of left invariant vectorfields, originated from W, by the vectorfield $\exp t.\mathrm{I}_{n_o}$ determines the Lie algebra structure. The vectorspace for this Lie algebra is formed by the direct sum

$$\boldsymbol{g} = \boldsymbol{g}^{-1} \oplus \boldsymbol{g}^{0} \oplus \boldsymbol{g}^{1}, \tag{9}$$

where

$$\begin{array}{rcll} \boldsymbol{g}^{-1} & = & L(\mathrm{I\!R}^{l_o}, \mathrm{I\!R}^{k_o}) & \\ \boldsymbol{g}^{0} & = & \{(u,v) \in gl(l_o) \oplus gl(k_o)\,;\, k.\mathbf{T}r(u) + l.\mathbf{T}r(v) = 0\} & \\ \boldsymbol{g}^{1} & = & L(\mathrm{I\!R}^{k_o}, \mathrm{I\!R}^{l_o}). & (10) \end{array}$$

Let $x \in \boldsymbol{g}^{-1}$, ${}^{*}y \in \boldsymbol{g}^{1}$ and $(u,v) \in \boldsymbol{g}^{0}$, the induced brackets on this vector space are :

$$[u,x] = x.u\,;\quad [v,x] = v.x\,;\quad [u,\ {}^{*}y] = u.\ {}^{*}y\,;$$
$$[v,\ {}^{*}y] = {}^{*}y.v\,;\quad [x_1,x_2] = 0\,;\quad [\ {}^{*}y_1,\ {}^{*}y_2] = 0\,;$$

$$\begin{array}{rcll} [u_1+v_1, u_2+v_2] & = & [u_1,u_2] + [v_1,v_2]\,; & (11) \\ [x, {}^{*}y] & = & x^{*}y - {}^{*}y.x - (l_o - k_o)\dfrac{\mathbf{T}r(x.{}^{*}y)}{2k_o l_o}.\mathbf{I}\mathrm{d}_{n_o}. & \end{array}$$

$\mathbf{I}\mathrm{d}_{n_o}$ denotes the identity on $\mathrm{I\!R}^{l_o} \oplus \mathrm{I\!R}^{k_o}$.

C. Representations and prolongation

We will use the following identifications :

$$\begin{array}{ccccc} M(k_o,l_o) & \stackrel{\kappa}{=} & \mathrm{I\!R}^{k_o\times l_o} & \stackrel{\varsigma}{=} & \mathrm{I\!R}^{m_o} \\ x_i^a & \stackrel{\kappa}{=} & x^{a\,i} & \stackrel{\varsigma}{=} & x^{\alpha} \end{array} \qquad (12)$$

where $\mathrm{I\!R}^{k_o\times l_o}$ stands for $\underbrace{\mathrm{I\!R}^{l_o}\times\cdots\times\mathrm{I\!R}^{l_o}}_{k_o\ times}$; $\alpha = (a-1)l_o + i$, $m_o = k_o l_o$; $\alpha = 1,\cdots,m_o$; $a = 1,\cdots,k_o$ and $i = 1,\cdots,l_o$.

We introduce the following two subgroups.

(1) The subgroup G_o of $Gl(l_o)\times Gl(k_o)$:

$$G_o = \{(A,B) \in Gl(l_o)\times Gl(k_o)\,|\,(\det(A))^{k_o}.(\det(B))^{l_o} = 1\}. \qquad (13)$$

Let (A,B) and (A',B') be elements in G_o. Then from $(\det(A))^{k_o}(\det(B))^{l_o} = 1$ and $(\det(A'))^{k_o}(\det(B'))^{l_o} = 1$ it follows that $(\det(AA'))^{k_o}(\det(BB'))^{l_o} = 1$. We also remark that G_o is isomorphic to the subgroup defined by $\beta_{01} = \beta_{10} = 0$ in $Gl(l_o+k_o)/\exp t.\mathrm{I}_{n_o}$. There indeed always exists an α such that

$$(\det\alpha A)^{k_o}.(\det\alpha B)^{l_o} = \alpha^{k+l}(\det(A))^{k_o}.(\det(B))^{l_o} = 1.$$

(2) The subgroup $\tilde{G}_o$ of $Gl(m_o)$ defined by

$$\{A^{\alpha}_{\beta}\,\delta^{(a-1)l_o+i}_{\alpha}\,\delta^{\beta}_{(b-1)l_o+j} = A^i_j A^a_b\,|\,(A^i_j)\in Gl(l_o),\,(A^a_b)\in Gl(k_o)\}. \qquad (14)$$

Multiplication in the group yields

$$A^{\alpha}_{\gamma}A^{\gamma}_{\beta}\,\delta^{(a-1)l_o+i}_{\alpha}\,\delta^{\beta}_{(b-1)l_o+j} = A^i_k A^k_j A^a_c B^c_b.$$

We will intoduce the following notations

$$A^\alpha_\beta x^\beta = \tilde{x}^\alpha \overset{\varsigma}{\leftrightarrow} A^i_j A^a_b x^{bj} = \tilde{x}^{ai} \overset{\kappa}{\leftrightarrow} A^a_b x^b_j\ {}^\tau\!A^j_i = \tilde{x}^a_i, \tag{15}$$

which we will use throughout this paper. We also will use κ for $\kappa\, o\, \varsigma$.

Let (A_1, B_1) and (A_2, B_2) be in G_o. We then have

$$(A_1.A_2, B_1.B_2) \mapsto (\ {}^\tau(A_1.A_2)^{-1}, B_1.B_2) = (\ {}^\tau(A_1)^{-1}.\ {}^\tau(A_2)^{-1}, B_1.B_2),$$

which proves the following proposition.

Proposition 2.2 *The morphism*

$$\begin{array}{rcl} \tau : G_o & \rightarrow & \tilde{G}_o \\ (A, B) & \mapsto & (\ {}^\tau\!A^{-1}, B) \end{array} \tag{16}$$

is a group isomorphism sending left invariant vectorfields into left invariant vectorfields.

Proposition 2.3 *The Lie algebra, $\tilde{\boldsymbol{g}}_o$ of $\tilde{G}_o$ is given by the subalgebra of the $(m_o \times m_o)$ matrices which are defined by*

$$z^\alpha_\beta \overset{\kappa}{=} \tilde{u}^j_i \delta^a_b + \tilde{u}^a_b \delta^j_i \tag{17}$$

with $\alpha = (a-1)l_o + i,\ \beta = (b-1)l_o + j,\ (\tilde{u}^i_j) \in gl(l_o),\ (\tilde{u}^a_b) \in gl(k_o)$.

It is a direct consequence of proposition [(2.2)] that this Lie algebra, $\tilde{\boldsymbol{g}}^0$, is isomorphic to $\boldsymbol{g}^0$. The isomorphism is induced from ${}^\tau u = -(\tilde{u}^i_j),\ v = (\tilde{u}^a_b)$.

Let V be the real vectorspace isomorphic to $\mathbb{R}^{m_o}$. The algebra $\tilde{\boldsymbol{g}}^0$ is a subalgebra of $V \otimes V^*$. The first prologation $\tilde{\boldsymbol{g}}^{(1)}$ is defined as the vectorspace $V^* \otimes \tilde{\boldsymbol{g}}^0 \cap S^2(V^*) \otimes V$ and the k^{th} prolongation likewise as the vectorspace [1] [10]

$$\tilde{\boldsymbol{g}}^{(k)} = \underbrace{V^* \otimes \cdots \otimes V^*}_{k\ times} \otimes \tilde{\boldsymbol{g}}^0 \cap S^{k+1}(V^*) \otimes V.$$

A subspace of $V^* \otimes V$ is called of finite type if $\tilde{\boldsymbol{g}}^{(k)} = 0$ for some (and hence all larger) k and otherwise of infinite type. We refer to [10] [1] [8] for the details.

We then have the following theorem.

Theorem 2.1 *The algebra $V \oplus \tilde{\boldsymbol{g}}^0$ is of infinite type if k_o or l_o equals 1. If k_o and l_o are both different from 1 the algebra is of finite type. Moreover in this case $\tilde{\boldsymbol{g}}^{(2)} = 0$ and the algebra $V \oplus \tilde{\boldsymbol{g}}^0 \oplus \tilde{\boldsymbol{g}}^{(1)}$ is isomorphic to the algebra $\boldsymbol{g}^{-1} \oplus \boldsymbol{g}^0 \oplus \boldsymbol{g}^1$.*

In order to prove the theorem we will make use of the representation of $\tilde{\boldsymbol{g}}^0$ into the linear polynomial vectorfields on V. Let (x^α) be the coordinates on V. Define the subalgebra $\boldsymbol{g}^0$ as the set of vectorfields

$$\tilde{u}^\alpha_\beta x^\beta \frac{\partial}{\partial x^\alpha} \quad \text{with } \tilde{u}^\alpha_\beta \overset{\kappa}{=} \tilde{u}^j_i \delta^a_b + \tilde{v}^a_b \delta^j_i. \tag{18}$$

If $k_o = 1$ or $l_o = 1$, the algebra $\boldsymbol{g}^{-1} \oplus \boldsymbol{g}^0 \oplus \boldsymbol{g}^1$ is the algebra of projective transformations on $\mathbb{R}^{m_o}$ [11]. Hence $\boldsymbol{g}^0 = \tilde{\boldsymbol{g}}^0 = gl(m_o)$, from which it follows that the algebra $V \oplus \tilde{\boldsymbol{g}}^0$ is of infinite type.

We assume from now on k_o and l_o to be different from 1. The second prolongation $\tilde{\boldsymbol{g}}^{(2)}$ is zero as a consequence of a classification theorem by Matsushima [7] [8] or by a direct calculation from $\tilde{\boldsymbol{g}}^{(1)}$ once this is derived.

Before proving the theorem we will prove the following lemmas.

Lemma 2.1 *Any second order vectorfield* $X \overset{\kappa}{=} T^{ilc}_{adk} x^a_i x^d_l \frac{\partial}{\partial x^c_k}$*, such that*

$$[\![\, u^b_l \frac{\partial}{\partial x^b_l}, X]\!] \in \tilde{\boldsymbol{g}}^0$$

is of the form

$$T^{ilc}_{adk} = u^i_a \delta^l_k \delta^c_d + u^l_d \delta^i_k \delta^c_a.$$

Proof

For any $u^b_j \frac{\partial}{\partial x^b_j} \in V$ the bracket with any homogeneous second order vectorfield $T^{ijc}_{abk} x^a_i x^b_j \frac{\partial}{\partial x^c_k}$ taking values in $\tilde{\boldsymbol{g}}^0$ satisfies the equation

$$[\![\, u^b_j \frac{\partial}{\partial x^b_j}, T^{ijc}_{abk} x^a_i x^b_j \frac{\partial}{\partial x^c_k}]\!] = [A^l_k \delta^c_d + B^c_d \delta^l_k] x^d_l \frac{\partial}{\partial x^c_k},$$

for some constants A^l_k and B^c_d.
This equation becomes

$$2u^a_i T^{ilc}_{adk} = A^l_k \delta^c_d + B^c_d \delta^l_k.$$

Which together with the symmetry $T^{ilc}_{adk} = T^{lic}_{dak}$ proves the lemma.

□

Call W be the vector space of the second order vectorfields of the form $X \overset{\kappa}{=} T^{ilc}_{adk} x^a_i x^d_l \frac{\partial}{\partial x^c_k}$. Let $X \in V$, $Y \in \tilde{\boldsymbol{g}}^0$ and $Z \in W$. Because the set of all formal vectorfields on V is a Lie algebra, we can consider the Jacobi identity

$$[\![\, [\![\, X, Y]\!] \, , Z]\!] + [\![\, [\![\, Y, Z]\!] \, , X]\!] + [\![\, [\![\, Z, X]\!] \, , Y]\!] = 0.$$

Lemma 2.2 *Let* $Y \in \tilde{\boldsymbol{g}}^0$ *and* $Z \in W$*. Then :*

$$[\![\, Y, Z]\!] \in \tilde{\boldsymbol{g}}^0.$$

Proof

Because $[\![\, X, Y]\!] \in V$ the first term takes values in $\tilde{\boldsymbol{g}}^0$. The thirth term also takes values in $\tilde{\boldsymbol{g}}^0$ by the construction of W. Hence the second term $[\![\, [\![\, Y, Z]\!] \, , X]\!]$ takes values in $\tilde{\boldsymbol{g}}^0$. But this imples that $[\![\, Y, Z]\!]$ takes values in W by the former lemma.

□

As a consequence of both lemmas we are able to write the algebra $V \oplus \tilde{\boldsymbol{g}}^0 \oplus \tilde{\boldsymbol{g}}^{(1)}$ as the vectorspace $\mathcal{L}$ spanned by the vectorfields

$$(\tilde{u}_i^a \frac{\partial}{\partial x_i^a}, (\tilde{u}_i^j \delta_b^a + \tilde{u}_b^a \delta_i^j) x_j^b \frac{\partial}{\partial x_i^a}, (\tilde{u}_b^k \delta_c^a \delta_i^j + \tilde{u}_c^j \delta_b^a \delta_i^k) x_j^b x_k^c \frac{\partial}{\partial x_i^a})$$

$$= (\tilde{u}_i^a \frac{\partial}{\partial x_i^a}, \tilde{u}_i^j x_j^a \frac{\partial}{\partial x_i^a} + \tilde{u}_b^a x_j^b \frac{\partial}{\partial x_j^a}, \tilde{u}_c^k x_k^a x_i^c \frac{\partial}{\partial x_i^a}). \tag{19}$$

We find the following proposition.

Proposition 2.4 *Both Lie algebras $\mathcal{L}$ and $\boldsymbol{g}$ are isomorphic. The isomorphism*

$$\tau : \boldsymbol{g} = \boldsymbol{g}^{-1} \oplus \boldsymbol{g}^0 \oplus \boldsymbol{g}^1 \to \mathcal{L}$$

is induced from

$$\tau(u_i^a) = \tilde{u}_i^a,\ \tau(u_j^i) = -\tilde{u}_j^i,\ \tau(u_b^a) = \tilde{u}_b^a,\ \tau(u_a^i) = \tilde{u}_a^i, \tag{20}$$

with $(u_i^a,\, u_j^i,\, u_b^a,\, u_a^i) \in \boldsymbol{g}$.

This proposition together with both lemmas proves the theorem.

3 The Cartan connections

A. The structure equations

Let P be a principal bundle, of dimension $n_o^2 - 1$ ($n_o = k_o + l_o$), over M with fibre group H, the isotropy group [(5)]. We then have dim $P/H = k_o l_o$. The right action of H on P is denoted as R_a, for $a \in H$, while ad stands for the adjoint representation of H on the Lie algebra $\boldsymbol{g} = \mathbf{pl}(\mathbf{n_o})$. Every $A \in \boldsymbol{h}$ induces in a natural manner a vectorfield $A^\star$, called fundamental vectorfield, on P as a consequence of the action of H on P. The vectorfield $A^\star$ obviously is a vertical vectorfield on P.

A Cartan connection on P is a 1-form ω on P, with values in the Lie algebra $\boldsymbol{g}$, such that :

$$\begin{array}{ll} (1) & \omega(A^\star) = A,\ \forall\, A \in \mathbf{h} \\ (2) & R_a^\star\, \omega = ad(a^{-1})\, \omega,\ a \in H \\ (3) & \omega(X) \neq 0\,,\ \forall X \in \mathcal{X}(P) \text{ with } X \neq 0. \end{array} \tag{21}$$

The form ω defines for each $x \in P$ an isomorphism of $T_x P$ with g. Hence the space P is globally parallelisable.

In terms of the natural basis in matrix representation of $\mathbf{pl}(\mathbf{n_o})$ as given in [(10)] and [(11)], we write the connection form ω as $(\omega_i^a,\, \omega_j^i,\, \omega_b^a,\, \omega_*,\, \omega_a^i)$, with $\omega_i^i = \omega_a^a = 0$. As basis for the subalgebra $\mathbf{h} = \mathbf{sl}(\mathbf{l_o}) \oplus \mathbf{sl}(\mathbf{k_o}) \oplus \mathbb{R} \oplus L(\mathbb{R}^{k_o}, \mathbb{R}^{l_o})$ we choose $(e_j^i,\, e_b^a,\, e_*,\, e_i^a)$.

The structure equations of Cartan on P are now defined as

$$\begin{array}{llll} (1) & d\omega^a_j & = & -\omega^a_k \wedge \omega^k_j - \omega^a_b \wedge \omega^b_j - \omega^a_j \wedge \omega_* + \Omega^a_j \\ (2) & d\omega^i_j & = & -\omega^i_k \wedge \omega^k_j - \omega^i_b \wedge \omega^b_j + \frac{1}{l}\,\delta^i_j\,\omega^k_c \wedge \omega^c_k + \Omega^i_j \\ (3) & d\omega^a_b & = & -\omega^a_k \wedge \omega^k_b - \omega^a_c \wedge \omega^c_b + \frac{1}{k}\,\delta^a_b\,\omega^c_k \wedge \omega^k_c + \Omega^a_b \\ (4) & d\omega^i_a & = & -\omega^i_k \wedge \omega^k_a - \omega^i_b \wedge \omega^b_a + \omega^i_a \wedge \omega_* + \Omega^i_a \\ (5) & d\omega_* & = & \frac{k_o + l_o}{k_o l_o}\,\omega^a_i \wedge \omega^i_a + \Omega_*, \end{array} \tag{22}$$

with $\omega^i_i = \omega^a_a = \Omega^i_i = \Omega^a_a = 0$.

In analogy with the projective case described by Kobayashi and Nagano, the form Ω^a_i is called the torsion form while ($\Omega^i_j,\ \Omega^a_b,\ \Omega^i_a,\ \Omega_*$) are called the curvature forms of the connection. The connection form satisfies the following conditions : $\omega^a_i(A^\star) = 0$, $\omega^i_j(A^\star) = A^i_j$, $\omega^a_b(A^\star) = A^a_b$, $\omega_*(A^\star) = A_*$ for $A = (A^i_j,\ A^a_b,\ A^i_a,\ A_*) \in \mathbf{h}$. Furthermore if $X \in \mathcal{X}$ such that $\omega^a_i(X) = 0$, then X is vertical.

Proposition 3.1 *The torsion and the curvature forms are basic forms on the bundle P. Hence we define :*

$$\Omega^a_i = K^{a\,bc}_{i\,jk}\,\omega^j_b \wedge \omega^k_c,\ \ \Omega^i_j = K^{i\,bc}_{j\,lk}\,\omega^l_b \wedge \omega^k_c,$$

$$\Omega^a_b = K^{a\,dc}_{b\,jk}\,\omega^j_d \wedge \omega^k_c,\ \ \Omega^i_a = K^{i\,bc}_{a\,jk}\,\omega^j_b \wedge \omega^k_c,\ \ \Omega_* = K^{bc}_{*,jk}\,\omega^j_b \wedge \omega^k_c \tag{23}$$

Proof

Let $F_x,\ x \in M$, be the fibre above x. The restriction of ω^a_i to F_x is identically zero and the forms $\omega^i_j,\ \omega^a_b,\ \omega^i_a,\ \omega_*$ are linearly independent on F_x as a consequence of [(21 (1)(3))]. Because the form ω sends the fundamental vectorfields A^* which are tangent to F_x into the left invariant vectorfields A on the group H, the forms $\omega^i_j,\ \omega^a_b,\ \omega^i_a,\ \omega_*$ satisfy the equations of Maurer cartan on H. The combination of these equations and equations [(22)] implies the vanishing of the curvature forms when restricted to F_x.

□

From now on we assume the torsion Ω^a_i to be zero.

Proposition 3.2 *Let P be a principal fibre bundle over M with structure group H and ($\omega^i_b,\ \omega^i_j,\ \omega^a_b,\ \omega^a_j$) a Cartan connection on P satisfying the structure equations [(22)]. The curvature forms possess the following properties :*

$$\begin{array}{llll} (1) & 0 & = & \omega^a_k \wedge \Omega^k_j + \Omega^a_b \wedge \omega^b_j + \omega^a_j \wedge \Omega_* \\ (2) & 0 & = & d\Omega_* - \omega^a_i \wedge \Omega^i_a \end{array} \tag{24}$$

Proof

These equations are obtained by taking the exterior differential of equations [(22,(1) and (5))].

□

B. The normal Cartan connection

The first equation of the structure equations of Cartan with $\Omega_i^a = 0$ is called the torsion zero equation and does not contain the form ω_a^i, while the other equations define the curvature forms. A natural question then arises, namely : let $(\omega_i^a,\, \omega_j^i,\, \omega_b^a, \omega_*)$ be given a priori on P which satisfy the torsion equation, does there then exists a ω_a^i such that ω is a Cartan connection on P and if so is there a canonical one.

Theorem 3.1 *Let the bundle P be given as defined and $(\omega_i^a,\, \omega_b^a,\, \omega_j^i,\, \omega_*)$ be 1-forms satisfying :*
(1) $\omega_i^a(A^*) = 0$, $\omega_b^a(A^*) = A_b^a$, $\omega_j^i(A^*) = A_j^i$, $\omega_*(A^*) = A_*$

$$\forall A = (A_j^i,\, A_b^a,\, A_a^i,\, A_*) \in \mathbf{h}$$

(2) $(R_a)^*(\omega_i^a,\, \omega_j^i,\, \omega_b^a, \omega_*) = ad(a^{-1})(\omega_i^a,\, \omega_j^i,\, \omega_b^a, \omega_*)$, $\forall a \in H$
(3) If $X \in \mathcal{X}(P)$ such that $\omega_i^a(X) = 0$, then X is vertical.

(4) $d\omega_i^a = -\omega_b^a \wedge \omega_i^b - \omega_j^a \wedge \omega_i^j - \omega_i^a \wedge \omega_*$.

If $l_o \neq 1$ and $k_o \neq 1$ then there exists an unique Cartan connection ω on P

$$\omega = (\omega_i^a,\, \omega_j^i,\, \omega_b^a, \omega_*,\, \omega_a^i),$$

such that :

$$\Omega_* = 0 \quad \text{and} \quad K_{lab}^{ilk} = K_{adb}^{dki}. \tag{25}$$

Proof

The existence of a Cartan connection satisfying the given conditions follows from a classical construction using the partition of unity. Because the manifold is supposed to be paracompact there exists a locally finite cover $\{\mathcal{U}_\alpha\}$ of M such that $P(\mathcal{U}_\alpha)$ is trivial, for each α. Let $\{(f_\alpha, \mathcal{U}_\alpha)\}$ then be a subordinate partition of unity. If for each α the form ω_α is a Cartan connection on $P(\mathcal{U}_\alpha)$ with prescribed $(\omega_i^a,\, \omega_b^a,\, \omega_j^i,\, \omega_*)$, then $\sum_\alpha (f_\alpha\, o\, \pi)\omega_\alpha$ is a Cartan connection in P (π being the bundle projection $P \to M$).

Hence the problem is reduced to a local problem for a trivial P. Let $\sigma : \mathcal{U} \subset M \to P$ be a local section we define the 1-form ω_a^i over σ as $\omega_a^i(X) = 0$ for all tangent vectors to σ and $\omega_a^i(A^*) = A_a^i$ for $A \in \mathbf{h}$. Now any vectorfield Y on P can be written uniquely as $Y = R_a(X) + V$, where X is tangent to σ and $a \in H$ and V is tangent to the fibre. Hence the condition

$$\omega(Y)(p.a) = ad(a^{-1})(\omega(X))(p) + A, \quad p = \sigma(x),\ x \in M$$

with ${}^*A$ the unique fundamental vectorfield corresponding to A, such that ${}^*A(p.a) = V(p.a)$, determines $\omega_a^i(Y)$.

We will prove the existence of a Cartan connection satisfying the required conditions [(25)] by means of a set of propositions.

□

Proposition 3.3 *Let ω be a Cartan connection on P. Then there exists a Cartan connection satifying the condition $\Omega_* = 0$. Two Cartan connections satisfying this same condition are related by $\bar{\omega}^i_a = \omega^i_a - A^{ik}_{ab}\omega^b_k$, with $A^{ik}_{ab} = A^{ki}_{ba}$.*

Proof

Using conditions [(21,(1) (3))] the unkown form can be written as

$$\bar{\omega}^i_a = \omega^i_a - A^{ik}_{ab}\omega^b_k.$$

Equation [(22, (5))] then yields

$$0 = \frac{k_o + l_o}{k_o l_o}\omega^a_i \wedge A^{ik}_{ab}\omega^b_k + \Omega_* - \bar{\Omega}_*.$$

If $\Omega_* \neq 0$ choose A^{ik}_{ab} such that

$$0 = \frac{k_o + l_o}{k_o l_o}\omega^a_i \wedge A^{ik}_{ab}\omega^b_k + \Omega_*$$

or

$$A^{ik}_{ab} - A^{ki}_{ba} = -\frac{2k_o l_o}{k_o + l_o} K^{ik}_{*\,ab}.$$

As follows directly from this equation two Cartan connections satisfying the curvature condition $\Omega_* = 0$ are related by $\bar{\omega}^i_a = \omega^i_a - A^{ik}_{ab}\omega^b_k$, with $A^{ik}_{ab} - A^{ki}_{ba} = 0$.

□

Proposition 3.4 *Let ω be a Cartan connection on P satisfying condition $\Omega_* = 0$. Then the Bianchi identities [(24)] become*

$$(1)\quad K^{klm}_{jcb}\,\delta^a_d + K^{alm}_{dcb}\,\delta^k_j + K^{mkl}_{jdc}\,\delta^a_b + K^{akl}_{bdc}\,\delta^m_j + K^{lmk}_{jbd}\,\delta^a_c + K^{amk}_{abd}\,\delta^l_j = 0$$

$$(2)\quad K^{ikl}_{acb} + K^{lik}_{bac} + K^{kli}_{cba} = 0. \tag{26}$$

Consequences : From equation [(26,(1))] we find by contraction of the indices k_o & j and a & d

$$K^{mkl}_{kbc} + K^{dml}_{bdc} - K^{lkm}_{kcb} - K^{dlm}_{cdb} = 0 \tag{27}$$

and by contraction of k_o & j and a & c the expression

$$kK^{mjl}_{jdc} - lK^{aml}_{dac} - K^{ljm}_{jdc} + K^{aml}_{cad} = 0. \tag{28}$$

Lemma 3.1 *The expression*

$$K^{mkl}_{kbc} - K^{dlm}_{cdb}$$

is symmetric in the pair $((m,b),(l,c))$.

Proposition 3.5 *Let ω be a Cartan connection on P satisfying $\Omega_* = 0$. Then there exists a unique Cartan connection satisfying the curvature conditions [(25)].*

Proof

It is sufficient to consider the class of Cartan connections determined by the condition $\Omega_* = 0$. Two such connections are related by

$$\bar{\omega}^i_a = \omega^i_a - A^{ik}_{ab}\omega^b_k,$$

with $A^{ik}_{ab} = A^{ki}_{ba}$ [(3.3)].
Equation [(22, (2))] then gives

$$\Omega^i_j - \bar{\Omega}^i_j - A^{il}_{ab}\,\omega^b_l \wedge \omega^a_j = 0$$

or

$$\left[K^{ilk}_{jba} - \bar{K}^{ilk}_{jba} - A^{il}_{ab}\,\delta^k_j\right]\omega^b_l \wedge \omega^a_k = 0,$$

which yields

$$K^{ilk}_{jba} - \bar{K}^{ilk}_{jba} - \frac{1}{2}\left(\delta^k_j A^{il}_{ab} - \delta^l_j A^{ik}_{ba}\right) = 0.$$

Summation on the indices l and j gives :

$$K^{ilk}_{lba} - \bar{K}^{ilk}_{lba} - \frac{1}{2}\left(A^{ik}_{ab} - l\,A^{ik}_{ba}\right) = 0. \tag{29}$$

From [(22),(3)] we derive in a similar way the following equation :

$$\frac{1}{2}\left(\delta^d_a\,A^{lk}_{bc} - \delta^d_c\,A^{kl}_{ba}\right) + K^{dkl}_{bca} - \bar{K}^{dkl}_{bca} = 0.$$

Contraction on d and c yields

$$\frac{1}{2}\left(A^{ik}_{ab} - kA^{ki}_{ab}\right) + K^{dki}_{adb} - \bar{K}^{dki}_{adb} = 0. \tag{30}$$

From the lemma [(3.1)] we know that the expression

$$K^{ilk}_{lba} - K^{dki}_{adb}$$

is symmetric in the pair $((i,b),(k,a))$. If

$$K^{ilk}_{lba} - K^{dki}_{adb} \neq 0$$

we define A^{ik}_{ab} such that

$$K^{ilk}_{lba} - K^{dki}_{adb} = A^{ik}_{ab} - \frac{1}{2}(l_o + k_o)A^{ik}_{ba}. \tag{31}$$

Or

$$A^{ki}_{ab} = \frac{4}{4-(l_o+k_o)^2}\left[\Delta^{ki}_{ba} + \frac{1}{2}(l_o+k_o)\Delta^{ik}_{ba}\right], \tag{32}$$

with

$$\Delta^{ik}_{ba} = K^{ilk}_{lba} - K^{dki}_{adb}. \tag{33}$$

Substitution of [(31)] in the sum of equations [(29)] and [(30)] gives

$$\bar{K}^{ilk}_{lba} - \bar{K}^{dki}_{adb} = 0.$$

This proves the theorem for $l_o \neq 1$ and $k_o \neq 1$. If l_o or k_o equals one we refer to the projective case treated by Kobayashi S., Nagano T. [3]. The uniqueness follows from the same considerations.

□

Definition 3.1 *The unique Cartan connection ω on P satisfying the curvature conditions [(25)], will be called the normal Grassmannian connection on P.*

Proposition 3.6 *Let ω be a normal Cartan connection on the bundle P. The following curvature equations are identities :*

$$\begin{aligned} k_o K^{mjl}_{jdc} &= l_o K^{aml}_{dac} \\ k_o K^{mjl}_{jdc} &= l_o K^{ljm}_{jcd} \\ k_o K^{alm}_{cad} &= l_o K^{aml}_{dac} \end{aligned} \tag{34}$$

Proof

These relations follow from the conditions [(25)] and the identities [(28)].

□

Proposition 3.7 *Let P and ω be as above. If $\Omega^i_j = 0$ and $\Omega^a_b = 0$, then it follows that $\Omega^i_a = 0$.*

Proof

If $k_o \neq 1$ and $l_o \neq 1$ then the manifold M has dimension larger than 3. The proposition follows from differentiation of equations [(22, (2)(3))] :

$$d\Omega^i_j - \Omega^i_k \wedge \omega^k_l + \omega^i_k \wedge \Omega^k_j + \frac{1}{l_o}\delta^i_j\,\Omega^k_c \wedge \omega^c_k - \Omega^i_b \wedge \omega^b_j = 0 \tag{35}$$

and

$$d\Omega^a_b - \Omega^a_c \wedge \omega^c_b + \omega^a_c \wedge \Omega^c_b - \frac{1}{k_o}\delta^a_b \omega^c_k \wedge \Omega^k_c + \omega^a_k \wedge \Omega^k_b = 0. \tag{36}$$

From equation [(35)] one finds

$$\frac{1}{l_o}\Omega^k_c \wedge \omega^c_k \wedge \omega^a_j - \Omega^i_b \wedge \omega^b_j \wedge \omega^a_i = 0.$$

While from equation [(36)] one has

$$\frac{1}{k_o}\Omega^k_c \wedge \omega^c_k \wedge \omega^a_j - \Omega^k_b \wedge \omega^a_k \wedge \omega^b_j = 0.$$

Combining the two equations gives

$$(k_o + l_o)\Omega^k_b \wedge \omega^a_k \wedge \omega^b_j = 0,$$

which substituted in equation [(35)] gives

$$\Omega^i_b \wedge \omega^b_j = 0$$

and in equation [(36)]

$$\omega^a_k \wedge \Omega^k_b = 0.$$

Or in terms of the components we find the two equations :

$$K^{ikl}_{bcd}\delta^j_m + K^{ijk}_{dbc}\delta^l_m + K^{ilj}_{cdb}\delta^k_m = 0.$$

and

$$K^{ikl}_{bcd}\delta^a_e + K^{kli}_{bde}\delta^a_c + K^{lik}_{bec}\delta^a_d = 0.$$

In case $l_o \geq 3$, let l be different from k and j. We find by taking $m = l$ that $K^{ijk}_{dbc} = 0$. In case $k_o \geq 3$, let c be different from e and d. One finds the same result by taking $a = c$.
The special case $k_o = 2$ and $l_o = 2$ is easily proven by consideration of the different cases $k = j = l$, $k = l \neq j$, $e = c = d$ and $e = c \neq d$.

□

Proposition 3.8 *Let P with $k_o \geq 3$, $l_o \geq 3$ and ω be as above. Then*

$$\Omega^i_j = 0 \quad \textit{iff} \quad \Omega^a_b = 0.$$

Proof

From the Bianchi identities [(24,(1))] we find with $\Omega^i_j = 0$

$$\Omega^a_b \wedge \omega^b_j = 0.$$

In terms of the components this equation is

$$K^{alm}_{dcb}\delta^k_j + K^{akl}_{bdc}\delta^m_j + K^{amk}_{cbd}\delta^l_j = 0.$$

Let m be different from k and l. Taking j equal to m yields $K^{akl}_{bdc} = 0$.
Conversally, the condition $\Omega^a_b = 0$ implies $\Omega^i_j = 0$ by an analogous argument using the Bianchi equations [(24,(1))].

□

This proves the following theorem.

Theorem 3.2 *Let P with $k_o \geq 3$, $l_o \geq 3$ and ω as above. The bundle P is locally flat iff $\Omega^i_j = 0$ or $\Omega^a_b = 0$.*

Local flatness of P means vanishing of the structure functions [2].

4 The Ehresmann connection

A. Second order frames

Let M be a manifold of dimension m_o and f a diffeomorphism of an neighborhood of 0 in $\mathbb{R}^{m_o}$ onto an open neighborhood of M. If f is a local diffeomorphism then the r-jet $j_0^r(f)$ is an r-frame at $x = f(0)$. The set of r-frames of M will be denoted by $F^r(M)$, while the set of r-frames at $f(0)$ forms a group $G^r(m_o)$ with multiplication defined by the composition of jets :

$$j_0^r(g_1) \,.\, j_0^r(g_2) = j_0^r(g_1 \, o \, g_2).$$

The group $G^r(m_o)$ acts on $F^r(M)$ on the right :

$$j_0^r(f) \,.\, j_0^r(g) = j_0^r(f \, o \, g). \tag{37}$$

The Lie algebra of $G^r(m_o)$ will be denoted by $g^r(m_o)$. We mainly will be interested in the bundle of 2-frames on M. Let (x^α) be some local coordinates on M and $\bar{x}^\alpha$ the natural coordinates on $\mathbb{R}^{m_o}$. A 2-frame u then is given by $u = j_0^2(f)$.
From

$$f(\bar{x}) = x^\alpha e_\alpha + u^\alpha_\beta \bar{x}^\beta e_\alpha + \frac{1}{2} u^\alpha_{\beta\gamma} \bar{x}^\beta \bar{x}^\gamma e_\alpha, \tag{38}$$

we get a set of local coordinates $(x^\alpha,\ u^\alpha_\beta,\ u^\alpha_{\beta\gamma})$ on $F^2(M)$.

In a similar way we may use $(s^\alpha_\beta,\ s^\alpha_{\beta\gamma})$ as coordinates on $G^2(m_o)$. The multiplication in $G^2(m_o)$ is given by

$$(\bar{s}^\alpha_\beta,\ \bar{s}^\alpha_{\beta\gamma}).(s^\alpha_\beta,\ s^\alpha_{\beta\gamma}) = (\bar{s}^\alpha_\sigma s^\sigma_\beta,\ \bar{s}^\alpha_\sigma s^\sigma_{\beta\gamma} + \bar{s}^\alpha_{\sigma\rho} s^\sigma_\beta s^\rho_\gamma), \tag{39}$$

while the action of $G^2(m_o)$ on $F^2(M)$ is given by

$$(x^\alpha,\ u^\alpha_\beta,\ u^\alpha_{\beta\gamma}).(s^\alpha_\beta,\ s^\alpha_{\beta\gamma}) = (x^\alpha,\ u^\alpha_\sigma s^\sigma_\beta,\ u^\alpha_\sigma s^\sigma_{\beta\gamma} + u^\alpha_{\sigma\rho} s^\sigma_\beta s^\rho_\gamma). \tag{40}$$

Let

$$(e_\alpha = \frac{\partial}{\partial \bar{x}^\alpha},\ e^\alpha_\beta = \frac{\partial}{\partial \bar{x}^\alpha} \otimes d\bar{x}^\beta)$$

be a basis for the Lie algebra of affine transformations on $\mathbb{R}^{m_o}$. The canonical one form θ on $F^2(M)$, which we write as

$$\theta = \theta^\alpha e_\alpha + \theta^\alpha_\beta e^\beta_\alpha,$$

is given in local coordinates by (with v^α_β is the inverse matrix of u^α_β) [4] :

$$\theta^\alpha = v^\alpha_\beta dx^\beta, \tag{41}$$

$$\theta^\alpha_\beta = v^\alpha_\gamma du^\gamma_\beta - v^\alpha_\gamma u^\gamma_{\rho\beta} v^\rho_\sigma dx^\sigma. \tag{42}$$

Because the group $G^2(m_o)$ acts on $F^2(M)$ on the right, with each $A \in g^2(m_o)$ corresponds a fundamental vectorfield $A^\star \in \mathcal{X}(F^2(M)$. Let $\pi^2_1 : g^2(m_o) \to g^1(m_o)$, we have the following proposition [3] :

Proposition 4.1
(1) $\theta(A^\star) = \pi_1^2(A)$ *for* $A \in g^2(m_o)$
(2) $R_a^\star \theta = ad(a^{-1})\theta,\ a \in G^2(m_o)$.

The canonical form satisfies the structure equation [4] :

$$d\theta^\alpha = -\,\theta^\alpha_\beta \wedge \theta^\beta. \tag{43}$$

B. The Grassmannian bundle $Gr(k_o, l_o)(M)$

We will now define a subbundle of $F^2(M)$ which is isomorphic with the bundle P. In this section we use the identification $\mathbb{R}^{k_o\times l_o} \stackrel{\varsigma}{=} \mathbb{R}^{m_o}$.

Proposition 4.2 *The embedding* $H \to G^2(m_o)$, $m_o = k_o l_o$, *defined by*

$$\left(\beta^i_j, \beta^a_b, \beta^k_c\right) \mapsto \begin{cases} s^\alpha_\beta & \stackrel{\varsigma}{=}\ \alpha^i_j \beta^a_b \\ s^\alpha_{\beta\gamma} & \stackrel{\varsigma}{=}\ -\left[\beta^a_b \alpha^l_j \gamma_{lc} \alpha^i_k + \beta^a_c \alpha^l_k \gamma_{lb} \alpha^i_j\right], \end{cases} \tag{44}$$

with $\alpha = (a-1)l_o + j$, $\beta = (b-1)l_o + j$, $\gamma = (c-1)l_o + k$ *and* $\alpha^i_j =\ {}^\tau\beta^{-1}{}^i_j$, $\gamma_{kc} = \beta^k_c$, *is a group morphism. Let* $\tilde{H}$ *designate image of the embedding in* $G^2(m_o)$.

Proof

The multiplication in H yields

$$(\hat\beta^i_j, \hat\beta^i_c, \hat\beta^a_b).(\beta^j_k, \beta^j_c, \beta^b_c) = (\hat\beta^i_j \beta^j_k, \hat\beta^i_j \beta^j_b + \hat\beta^i_c \beta^c_b, \hat\beta^a_b \beta^b_c). \tag{45}$$

Let

$$s^\alpha_\beta = \alpha^i_j \beta^a_b, \quad s^\alpha_{\beta\gamma} = -\left[\beta^a_b \alpha^l_j \gamma_{lc} \alpha^i_k + \beta^a_c \alpha^l_k \gamma_{lb} \alpha^i_j\right]$$

and

$$\hat s^\alpha_\beta = \hat\alpha^i_j \hat\beta^a_b, \quad \hat s^\alpha_{\beta\gamma} = -\left[\hat\beta^a_b \hat\alpha^l_j \hat\gamma_{lc} \hat\alpha^i_k + \hat\beta^a_c \alpha^l_k \hat\gamma_{lb} \hat\alpha^i_j\right].$$

We find for the multiplication

$$(\bar s^\alpha_\beta,\ \bar s^\alpha_{\beta\gamma}).(s^\alpha_\beta,\ s^\alpha_{\beta\gamma}) = (\hat\alpha^i_j \hat\beta^a_b \alpha^j_k \beta^b_c,$$

$$-\hat\alpha^i_j \hat\beta^a_b \left[\beta^b_d \alpha^l_m \gamma_{lc} \alpha^i_k + \beta^b_c \alpha^l_k \gamma_{ld} \alpha^j_m\right] - \left[\hat\beta^a_b \hat\alpha^l_j \hat\gamma_{le} \hat\alpha^i_m + \hat\beta^a_e \alpha^l_m \hat\gamma_{lb} \hat\alpha^i_j\right] \alpha^j_m \beta^b_d \alpha^m_k \beta^e_c),$$

which proves the group morphism.

□

Definition 4.1 *A Grassmannian structure,* $Gr(k_o, l_o)(M)$, *on a manifold* M *is a subbundle of* $F^2(M)$ *with structure group* $\tilde{H}$.

Proposition [(4.2)] together with some classical results in bundle theory [9] proves the following theorem.

Theorem 4.1 *Let P be a H-bundle over M. Then there exists a $Gr(k_o,l_o)(M)$, subbundle of $F^2(M)$, which is isomorphic to P.*

Definition 4.2 *A (k_o,l_o)-structure on a manifold, $B^{(k_o,l_o)}(M)$, is a subbundle of $F^1(M)$ with structure group G_o.*

Theorem 4.2 *Each Grassmannian structure, $Gr(k_o,l_o)(M)$, on M is the prolongation of a (k_o,l_o)-structure. Moreover this structure has vanishing second prolongation.*

Proof

Let $B^{(k_o,l_o)}(M)$ be any subbundle of $F^1(M)$ with structure group G_o. The first prolongation of $B^{(k_o,l_o)}(M)$ is a subbundle of $F^2(M)$ with structure group the semi direct product of G_o and the group of automorphisms of $V \simeq \mathbb{R}^{m_o}$ generated by the Lie algebra $\tilde{\boldsymbol{g}}^{(1)}$ [10]. Hence the first prolongation is a $Gr(k_o,l_o)(M)$.

Let $Gr(k_o,l_o)(M)$ be given and $\pi_1^2 : F^2(M) \to F^1(M)$ the bundle projection. Then $\pi_1^2(Gr(k_o,l_o)(M)$ is a bundle $B^{(k_o,l_o)}(M)$ whose prolongation coïncides with $Gr(k_o,l_o)(M)$ by the isomorphism of the structure groups. The second prolongation of a $B^{(k_o,l_o)}(M)$ vanishes identically [(2.1)].

□

We refer to S. Sternberg [10] for a detailled exposition of the relationship between connections on G structures and prolongations. In particular the set of adapted symmetric connections is parametrised by the first prolongation of the Lie algebra $\tilde{\boldsymbol{g}}^{(1)}$. To make this clear we first need the following lemma on symmetric affine connections.

Lemma 4.1 *Let $\Gamma : M \to F^2(M)/Gl(m_o)$ be an affine symmetric connection. Then there exists a canonical homomorphism $\tilde{\Gamma} : F^1(M) \to F^2(M)$ canonically associated with Γ.*

Proof

For a proof we refer to [5]. In local coordinates the map Γ is given by

$$\tilde{\Gamma} : \bar{x}^\alpha = x^\alpha \ ; \quad \bar{u}^\alpha_\beta = u^\alpha_\beta \ ; \quad \bar{u}^\alpha_{\beta\gamma} = -u^\sigma_\beta \Gamma^\alpha_{\sigma\rho} u^\rho_\gamma. \tag{46}$$

□

Remark that

$$\tilde{\Gamma}^* \theta^\alpha_\beta = v^\alpha_\gamma (du^\gamma_\beta + \Gamma^\gamma_{\rho\sigma} u^\sigma_\beta dx^\rho). \tag{47}$$

Let $B^{(k_o,l_o)}(M)$ be a (k_o,l_o) structure on M. An adapted affine symmetric connection on $B^{(k_o,l_o)}(M)$ is a map $\Gamma : M \to F^2(M)/Gl(m_o)$ such that $\tilde{\Gamma}^*\theta^\alpha_\beta$ restricted to $B^{(k_o,l_o)}(M)$ is a connection form with values in $\boldsymbol{g}^0$. Let $\Phi(B^{(k_o,l_o)})(M)$ be the set of adapted affine symmetric connection and denote the set of associated homomorphisms by $\tilde{\Phi}(B^{(k_o,l_o)})(M)$.

In order to prove the next proposition we need some local expressions. Let $(\bar{x}^{ai} \overset{\varsigma}{=} \bar{x}^{\alpha})$ be the coordinates on $\mathbb{R}^{m_o} \simeq \mathbb{R}^{k_o \times l_o}$. The Lie algebra of the second order formal vector fields $\mathcal{L}$ on this space as given in [(19)] has the following basis

$$e_{ai} = \frac{\partial}{\partial \bar{x}^{ai}},\ e^i_j \delta^a_b + e^a_b \delta^i_j = \delta^a_b \bar{x}^{ci} \frac{\partial}{\partial \bar{x}^{cj}} + \delta^i_j \bar{x}^{ak} \frac{\partial}{\partial \bar{x}^{bk}},\ e^{ai} = \bar{x}^{aj} \bar{x}^{ci} \frac{\partial}{\partial \bar{x}^{cj}}. \tag{48}$$

In terms of local coordinates on M and taking the identification ς directly into account, a 2-frame is given by

$$f(\bar{x}) = \left[x^{\alpha} + u^{\alpha}_{bj} \bar{x}^{bj} + u^{\alpha}_{bjck} \bar{x}^{bj} \bar{x}^{ck} \right] e_{\alpha}. \tag{49}$$

Let σ be a local section of $F^1(M)$, then σ is given by the functions

$$\sigma : (x) \mapsto E^{\alpha}_{bj}(x) = \sigma^* u^{\alpha}_{bj}. \tag{50}$$

The fundamental form along σ becomes

$$\bar{\theta}^{ai} = \sigma^* \theta^{ai} = F^{ai}_{\beta}(x) dx^{\beta}, \tag{51}$$

while the connection form with respect to a given $\tilde{\Gamma} \in \Phi(B^{(k_o,l_o)}(M)$ is

$$\bar{\theta}^{ai}_{bj} = \sigma^* \theta^{ai}_{bj} = F^{ai}_{\alpha} dE^{\alpha}_{bj} + F^{ai}_{\alpha} \Gamma^{\alpha}_{\rho\sigma} E^{\sigma}_{bj} dx^{\rho}. \tag{52}$$

The form $\bar{\theta}^{ai}_{bj}$ satisfies the structure equation [(43)]

$$d\bar{\theta}^{ai} = -\bar{\theta}^{ai}_{bj} \wedge \bar{\theta}^{bj}.$$

Let $\hat{\theta}^{ai}_{bj}$ be a second connection form with respect to a different morphism belonging to $\Phi(B^{(k_o,l_o)}(M)$. This form satisfies the same equation [(43)]. Hence we find

$$0 = (\bar{\theta}^{ai}_{bj} - \hat{\theta}^{ai}_{bj}) \wedge \hat{\theta}^{bj}. \tag{53}$$

The difference $(\bar{\theta}^{ai}_{bj} - \hat{\theta}^{ai}_{bj})$ defines a morphism $V \to \boldsymbol{g}^0 \subset V \otimes V^*$ at each $x \in M$, satisfying [(53)] and hence defines an element in $\boldsymbol{g}^{(1)}$. This implies that at $x \in M$:

$$\bar{\theta}^{ai}_{bj} - \hat{\theta}^{ai}_{bj} = u_{bk} \delta^a_c \delta^i_j + u_{cj} \delta^a_b \delta^i_k, \tag{54}$$

with $u^i_a \in M(l_o, k_o)$.

Proposition 4.3 *Any two adapted affine symmetric connections on $B^{(k_o,l_o)}(M)$ are locally related by :*

$$\Gamma'^{\gamma}_{\alpha\sigma} - \Gamma^{\gamma}_{\alpha\sigma} = 2 u_{bk} E^{\gamma}_{cj} F^{ck}_{(\alpha} F^{bj}_{\sigma)}. \tag{55}$$

with u_{bk} an element of $M(l_o, k_o)$.

Proof

We know that any connection form on $B^{(k_o,l_o)}(M)$ takes values in $\boldsymbol{g}^0$. Hence

$$\theta^{ai}_{bj\alpha} = \theta^a_{b\alpha}\delta^i_j + \theta^i_{j\alpha}\delta^a_b.$$

We find along the section σ :

$$\bar{\theta}^a_{b\alpha}E^\alpha_{ck}\delta^i_j + \bar{\theta}^i_{j\alpha}E^\alpha_{ck}\delta^a_b = F^{ai}_\gamma(\frac{\partial}{\partial x^\alpha}E^\gamma_{bj})E^\alpha_{ck} + F^{ai}_\gamma\Gamma^\gamma_{\alpha\sigma}E^\alpha_{ck}E^\sigma_{bj}.$$

From the theorem [(2.1)] and equation [(54)] it follows that for any two of such connection forms there exists an element u_{bk} such that

$$u_{bk}\delta^a_c\delta^i_j + u_{cj}\delta^a_b\delta^i_k = F^{ai}_\gamma(\Gamma'^\gamma_{\alpha\sigma} - \Gamma^\gamma_{\alpha\sigma})E^\alpha_{ck}E^\sigma_{bj}.$$

Hence

$$E^\gamma_{ai}\left[u_{bk}\delta^a_c\delta^i_j + u_{cj}\delta^a_b\delta^i_k\right]F^{ck}_\alpha F^{bj}_\sigma = \Gamma'^\gamma_{\alpha\sigma} - \Gamma^\gamma_{\alpha\sigma}.$$

$\square$

Because the first prolongation $\tilde{\boldsymbol{g}}^{(1)}$ can be identified with $M(l_o,k_o)$ this describes the parametrisation of the set of adapted connections. This allows us to formulate the following theorem.

Theorem 4.3 *Let $B^{(k_o,l_o)}(M)$ be a (k_o,l_o)-structure on M. The set*

$$\{\tilde{\Gamma}(B^{(k,l)}(M)) \mid \tilde{\Gamma} \in \tilde{\Phi}(B^{(k,l)})(M)\} \tag{56}$$

forms a Grassmannian structure on M.

Consequences :

(1) Each $Gr(k_o,l_o)(M)$ is locally determined by a section

$$\tilde{\Gamma}\, o\, \sigma : M \to F^2(M)$$

where $\tilde{\Gamma} \in \tilde{\Phi}(B^{(k_o,l_o)})(M)$ and σ a section $M \to B^{(k_o,l_o)}(M)$.
(2) The set of $Gr(k_o,l_o)(M)$ bundles is given by $F^2(M)/H$. Each local section $\tilde{\Gamma}\, o\, \sigma$ determines locally an element of $F^2(M)/H$.
(3) Each $Gr(k_o,l_o)(M)$ is equivalent with a $B^{(k_o,l_o)}(M)$ together with its set of adapted connections.

As alternative formulation of former theorem we have :

Theorem 4.4 *Each $Gr(k_o,l_o)(M)$ is locally uniquely defined by a section $\sigma : M \to F^1(M)$ and an identification $\mathbb{R}^{m_o} \stackrel{\leq}{=} \mathbb{R}^{k_o l_o}$.*

C. The normal Grassmannian connection coefficients

We will now investigate the coefficients of a normal Grassmannian connection in terms of an adapted frame and give an expression of the normal Grassmannian curvature tensor. Let $Gr(k_o, l_o)(M)$ be a Grassmannian structure defined as a subbundle of $F^2(M)$. Let θ^α, θ^α_β be the fundamental and the connection form on $Gr(k_o, l_o)(M)$. Because of the identification $\mathbb{R}^{m_o} \stackrel{\varsigma}{=} \mathbb{R}^{k_o \times l_o}$ we write these forms as $(\theta^{ai}, \theta^i_j, \theta^a_b)$ with $k_o\theta^i_i - l_o\theta^a_a = 0$ in order to fix their uniqueness in the decomposition. We then define on $Gr(k_o, l_o)(M)$

$$\omega^a_i = \theta^{ai}\,;\; \omega^i_j = -\,{}^\tau\theta^i_j + \frac{1}{l_o}\,{}^\tau\theta^k_k\delta^i_j\,;\; \omega^a_b = \theta^a_b - \frac{1}{k_o}\theta^c_c\delta^a_b\,;\; \omega_* = -\frac{1}{l_o}\theta^i_i - \frac{1}{k_o}\theta^a_a. \tag{57}$$

As a consequence of theorem [(3.1)] there exists a unique normal connection form $\omega = (\omega^a_i, \omega^i_j, \omega^a_b, \omega_*, \omega^i_a)$ on $Gr(k_o, l_o)(M)$.

Theorem 4.5 *Let M be a manifold equipped with a (k_o, l_o) structure $B^{(k_o,l_o)}(M)$ and $\mathcal{U} \subset M$ an open subset carrying an adapted coframe $F^a_{i\alpha}dx^\alpha$. Let further $Gr(k_o, l_o)(M)$ be the Grassmannian structure on M determined by $B^{(k_o,l_o)}(M)$ and*

$$\omega = (\omega^a_i, \omega^i_j, \omega^a_b, \omega_*, \omega^i_a)$$

the normal Cartan connection.

Then there exists a unique local section $\nu : \mathcal{U} \to Gr(k_o, l_o)(M)$ determined by the conditions

$$\nu^*\omega^a_{i\alpha}dx^\alpha = F^a_{i\alpha}dx^\alpha, \quad \nu^*\omega_* = 0. \tag{58}$$

Proof

Any section ν may be decomposed into a section σ of $B^{(k_o,l_o)}(M)$ and a section $\vartheta : B^{(k_o,l_o)}(M) \to Gr(k_o, l_o)(M)$. The requirement $\nu^*\omega^a_{i\alpha}dx^\alpha = F^a_{i\alpha}dx^\alpha$ implies $\sigma^*\omega^a_{i\alpha}dx^\alpha = F^a_{i\alpha}dx^\alpha$, which determines the section σ. Let $\tilde{\Gamma}$ be a morphism $F^1(M) \to F^2(M)$ defined by an adapted symmetric connection. Using proposition [(4.3)] and expression [(48)] the map ϑ can be written as

$$u^\alpha_{\beta\gamma} = -u^\sigma_\beta[\Gamma^\alpha_{\sigma\rho} + 2u_{bk}u^\alpha_{cj}v^{ck}_{(\sigma}v^{bj}_{\rho)}]u^\rho_\gamma,$$

with u_{bk} a function on $\mathcal{U}$. Or also

$$u^\alpha_{bj\,ai} = -E^\sigma_{bj}[\Gamma^\alpha_{\sigma\rho} + 2u_{bk}E^\alpha_{cj}F^{ck}_{(\sigma}F^{bj}_{\rho)}]E^\rho_{ai},$$

with E^α_{ai} the local frame dual to the coframe F^{ai}_α.

We remark that $\theta^\alpha_\alpha = -\frac{1}{k_o l_o}\omega_*$. The calculation of $(\vartheta\, o\, \sigma)^*\theta^\alpha_\alpha = 0$ yields, with the use of expression [(42)], the equation

$$F^{ai}_\gamma dE^\gamma_{ai} + \Gamma^\gamma_{\rho\gamma}dx^\rho + 2u_{bk}E^\beta_{cj}F^{ck}_{(\rho}F^{bj}_{\beta)}dx^\rho = 0$$

or

$$u_{ck}F^{ck}_\rho dx^\rho = -\frac{1}{2}[F^{ai}_\gamma dE^\gamma_{ai} + \Gamma^\gamma_{\rho\gamma}dx^\rho].$$

The unicity follows from the same calculations. Any two morphisms of $B^{(k_o,l_o)}(M)$ into $F^2(M)$ indeed are, as a consequence of proposition [(4.3)], defined by affine connections on $B^{(k_o,l_o)}(M)$ which are related by

$$\Gamma'^{\gamma}_{\alpha\sigma} - \Gamma^{\gamma}_{\alpha\sigma} = 2u_{bk}E^{\gamma}_{cj}F^{ck}_{(\alpha}F^{bj}_{\sigma)}.$$

A simple substitution then yields the unicity.

□

The theorem allows us to introduce the normal Grassmannian connection coefficients. We set

$$\sigma^*\omega^i_j = \Pi^i_{j\alpha}dx^\alpha, \quad \sigma^*\omega^a_b = \Pi^a_{b\alpha}dx^\alpha, \quad \sigma^*\omega^i_a = \Pi^i_{a\alpha}dx^\alpha. \tag{59}$$

Dual to the coframe $F^a_{i\alpha}dx^\alpha$ we define the frame $E^{i\alpha}_a\frac{\partial}{\partial x^\alpha}$ by the conditions

$$F^a_{i\alpha}E^{j\alpha}_b = \delta^a_b\delta^j_i. \tag{60}$$

From equation [(22)(5)] we find

$$\sigma^*\omega^a_i \wedge \sigma^*\omega^i_a = 0$$

or

$$F^a_{i\alpha}\Pi^i_{a\beta} - F^a_{i\beta}\Pi^i_{a\alpha} = 0.$$

Define $\Pi^i_{a\beta}E^{k\beta}_c = \Pi^{ik}_{ac}$. The former equation becomes

$$\Pi^{kl}_{cd} - \Pi^{lk}_{dc} = 0. \tag{61}$$

Let

$$L^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{62}$$

and

$$L^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \tag{63}$$

The equations [(22)(2) and (3)] become

$$\begin{aligned} K^i_{j\alpha\beta} &= L^i_{j\alpha\beta} + \frac{1}{2}(\Pi^i_{b\alpha}F^b_{j\beta} - \Pi^i_{b\beta}F^b_{j\alpha}) \\ K^a_{b\alpha\beta} &= L^a_{b\alpha\beta} + \frac{1}{2}(\Pi^a_{i\alpha}F^i_{b\beta} - \Pi^a_{i\beta}F^i_{b\alpha}). \end{aligned} \tag{64}$$

Using the notations

$$L^i_{l\alpha\beta}E^{k\alpha}_c E^{j\beta}_b = L^{ikj}_{lcb} \tag{65}$$

and

$$L^a_{d\alpha\beta}E^{k\alpha}_c E^{j\beta}_b = L^{akj}_{dcb}, \tag{66}$$

we find

$$\begin{aligned} K^{ikl}_{jcd} &= L^{ikl}_{jcd} + \frac{1}{2}(\Pi^{ik}_{bc}\delta^b_d\delta^l_j - \Pi^{il}_{bd}\delta^b_c\delta^k_j) \\ K^{akl}_{bcd} &= L^{akl}_{bcd} + \frac{1}{2}(\Pi^{ml}_{bd}\delta^a_c\delta^k_m - \Pi^{mk}_{bc}\delta^a_d\delta^l_m). \end{aligned} \tag{67}$$

From the condition

$$K^{ilk}_{lba} - K^{dki}_{adb} = 0$$

we obtain

$$L^{ilk}_{lba} - L^{dki}_{adb} - \frac{k_o + l_o}{2}\Pi^{ki}_{ab} + \Pi^{ik}_{ab} = 0.$$

This gives

$$\Pi^{ki}_{ab} = \frac{2}{(k_o+l_o)^2 - 4}\left[(k_o + l_o)(L^{ilk}_{lba} - L^{dki}_{adb}) + 2(L^{kli}_{lba} - L^{dik}_{adb})\right]. \tag{68}$$

Let M be equipped with an adapted symmetric affine connection on $B^{(k_o,l_o)}(M)$. We define the coefficients $(\gamma^{jk}_{lc}, \gamma^{dk}_{bc})$ by

$$\nabla_{E^i_a} E^j_b = \gamma^{di}_{ba} E^j_d + \gamma^{ji}_{la} E^l_b,$$

together with $k_o \gamma^{i\,k}_{i\,c} - l_o \gamma^{d\,k}_{d\,c} = 0$.
A Grassmannian related covariant derivation is defined as

$$\begin{aligned} \tilde{\nabla}_{E^i_a} E^j_b &= \left[\left(\gamma^{di}_{ba} + u^i_b \delta^d_a\right)\delta^j_l \right. \\ &+ \left.\left(\gamma^{ji}_{la} + u^j_a \delta^i_l\right)\delta^d_b\right] E^l_d. \end{aligned} \tag{69}$$

Or

$$\tilde{\nabla}_{E^i_a} E^j_b = \nabla_{E^i_a} E^j_b + u^i_b E^j_a + u^j_a E^i_b. \tag{70}$$

Using this expression we find

Proposition 4.4 *Let $X, Y \in \mathcal{X}(M)$, ∇ and $\tilde{\nabla}$ be two adapted connections on the bundle $B^{(k_o,l_o)}(M)$. Let further $\sigma : \mathcal{U} \to B^{(k,l)}(M)$ be a local section and $\bar{\sigma}(x)$ the corresponding identification of the tangent space $T_x(M)$ at $x \in \mathcal{U}$ with $M(k_o, l_o)$. Then there exists a map $\mu : \mathcal{U} \to M(l_o, k_o)$ such that*

$$\tilde{\nabla}_X Y = \nabla_X Y + \bar{\sigma}^{-1}[(\mu \,.\, \bar{\sigma}(X))\,.\,\bar{\sigma}(Y) + (\mu\,.\,\bar{\sigma}(Y))\,.\,\bar{\sigma}(X)]. \tag{71}$$

Because $\mu \in M(l_o, k_o)$ and $\bar{\sigma} \in M(k_o, l_o)$ the composition $(\mu\,.\,\bar{\sigma}(X)(x))$ is an element of $M(l_o, l_o)$ which acts on $\bar{\sigma}(Y)(x)$ by composition, giving thus an element of $M(k_o, l_o)$.

Remark We can define the $(2, 1)$-tensorfield

$$\tilde{\mu} = \bar{\sigma}^{-1}.\mu.\bar{\sigma}.$$

The Grassmannian relationship of two symmetric affine adapted connections is then given by

$$\hat{\nabla}_X Y = \nabla_X Y + \tilde{\mu}(X)(Y) + \tilde{\mu}(Y)(X).$$

We define the splitting of the coefficients γ into the trace free parts and the trace part as $(\bar{\gamma}^{ak}_{bc}, \bar{\gamma}^{ik}_{jc}, \bar{\gamma}^{k}_{*\,c})$, with $(\bar{\gamma}^{ak}_{ac} = \bar{\gamma}^{ik}_{ic} = 0)$. A Grassmannian related covariant derivation is then given by

$$\begin{aligned}
\tilde{\nabla}_{E_a^i} E_b^j &= \left[\left(\bar{\gamma}_{ba}^{di} + u_b^i \delta_a^d - \frac{1}{k_o} u_a^i \delta_b^d \right) \delta_l^j \right. \\
&+ \left(\bar{\gamma}_{la}^{ji} + u_a^j \delta_l^i - \frac{1}{l_o} u_a^i \delta_l^j \right) \delta_b^d \\
&+ \left. \left(\bar{\gamma}_{*\,a}^{i} + \frac{k_o + l_o}{k_o l_o} u_a^i \right) \delta_b^d \delta_l^j \right] E_d^l \qquad (72)
\end{aligned}$$

with

$$\bar{\gamma}_{*\,a}^{i} = \frac{1}{k_o l_o} (k_o \gamma_{ja}^{ji} + l_o \gamma_{ca}^{ci}). \qquad (73)$$

The normal Cartan connection is defined by the requirement

$$u_a^i = -\frac{1}{k_o + l_o} (k_o \gamma_{ja}^{ji} + l_o \gamma_{ca}^{ci})$$

and the coefficients of this connection are

$$\begin{aligned}
\Pi_{la}^{ji} &= -\bar{\gamma}_{la}^{ji} - u_a^j \delta_i^j + \frac{1}{l_o} u_a^i \delta_l^j \\
\Pi_{ba}^{di} &= \bar{\gamma}_{ba}^{di} + u_b^i \delta_a^d - \frac{1}{k_o} u_a^i \delta_b^d. \qquad (74)
\end{aligned}$$

We now are able to investigate the Grassmannian curvature tensor. Because the bundle $Gr(k_o, l_o)(M)$ is a subbundle of $F^2(M)$ the restriction of the homomorphism $\pi_1^2 : F^2(M) \to F^1(M)$ to $Gr(k_o, l_o)(M)$ is the homomorphism :

$$\eta : Gr(k_o, l_o)(M) \to B^{(k_o, l_o)}(M). \qquad (75)$$

The fibres of η are isomorphic to the kernel $\mathcal{M}^*$ of the homomorphism $H \to G_o$. The following theorem proves that the curvature forms Ω_j^i and Ω_b^a are defined on the bundle $B^{(k,l)}(M) \subset F^1(M)$.

Proposition 4.5 *Let $Gr(k_o, l_o)(M)$ be a Grassmannian structure equipped with a normal Grassmannian connection. Then the curvature forms Ω_j^i and Ω_b^a satisfy the following conditions. Let A^* be a fundamental vectorfield with $A \in \boldsymbol{g}^1$.*
Then

(1)

$$\mathcal{L}_{A^*}(\Omega_j^i) = \mathcal{L}_{A^*}(\Omega_b^a) = 0. \qquad (76)$$

(2) The tensor

$$K_{\beta\gamma\sigma}^{\alpha} = K_{j\gamma\sigma}^{i} F_{i\beta}^{a} E_a^{j\alpha} + K_{a\gamma\sigma}^{b} F_{i\beta}^{a} E_b^{i\alpha} \qquad (77)$$

is a $(1,3)$-tensorfield on M, which we call the Grassmannian curvature tensor.

Proof

The relations (1) are a direct consequence of the equations [(36)], while (2) is a consequence of the fact that $B^{(k_o,l_o)}(M)$ is a subbundle of $F^1(M)$ together with proposition [(3.1)]. Writing the curvature forms as $\Omega^i_j = K^i_{j\alpha\beta} dx^\alpha \otimes dx^\beta$ and $\Omega^a_{b\alpha\beta} dx^\alpha \otimes dx^\beta$, $E^{j\alpha}_b$, the Grassmannian curvature tensorfield is defined as

$$K^\alpha_{\beta\gamma\sigma} = \left[K^i_{j\gamma\sigma} \delta^b_a + K^b_{a\gamma\sigma} \delta^i_j \right] F^a_{i\beta} E^{j\alpha}_b$$

which is equivalent with [(77)].

$\square$

We call a Grassmannian structure on M locally flat if the structure has vanishing structure constants, which means that the structure is locally isomorphic with a flat structure [2]. The flat structure here means the structure of a Grassmannian. As a consequence of proposition [(3.7)] and because the dimension of the manifold admitting a Grassmannian structure is larger than 3, we have

Theorem 4.6 *A Grassmannian structure on M is locally flat iff the Grassmannian curvature equals zero.*

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