Bialgebra structures on a real semisimple Lie algebra.

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Abstract

We describe some results on the classification of bialgebra structures on a real semisimple Lie algebra. We first describe the possible Manin algebras (i.e. big algebra in the Manin triple) for such a bialgebra structure. We then determine all the bialgebra structures on a real semisimple Lie algebra for the nonzero standard modified Yang-Baxter equation. Finally we consider the case of a real simple Lie algebra the complexification of which is not simple and we give some partial results about the bialgebra structures for any nonzero modified Yang-Baxter equation.

1 Definitions and notations.

Our work is a continuation of a paper from M. Cahen, S. Gutt and J. Rawnsley [1]; we use the same notations as theirs which we now recall.

Definition 1. (cf[3]) A *Lie bialgebra* $(\mathfrak{g}, p)$ is a Lie algebra $\mathfrak{g}$ with a 1-cocycle $p : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ (relative to the adjoint action) such that $p^* : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ $(\xi, \eta) \to [\xi, \eta]$ *with*

$$\langle [\xi, \eta], X\rangle = \langle \xi \ \wedge \eta, p(X)\rangle$$

is a Lie bracket on $\mathfrak{g}^*$. One also denotes the bialgebra by $(\mathfrak{g}, \mathfrak{g}^*)$.
A Lie bialgebra $(\mathfrak{g}, p)$ is said to be *exact* if the 1-cocycle p is a coboundary, $p = \partial Q$, for $Q \in \Lambda^2\mathfrak{g}$.

This means that $\partial Q_X = [X, Q]$ and then the condition for $(\mathfrak{g}, \partial Q)$ to be a Lie bialgebra is that the bracket $[Q,Q]$ be invariant under the adjoint action in $\Lambda^3\mathfrak{g}$.

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Definition 2. (cf [3]) A *Manin triple* consists of three Lie algebras $(\mathfrak{L}, \mathfrak{g}_1, \mathfrak{g}_2)$ and a nondegenerate invariant symmetric bilinear form $\ll, \gg$ on $\mathfrak{L}$ such that

1)$\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras of $\mathfrak{L}$;

2)$\mathfrak{L} = \mathfrak{g}_1 + \mathfrak{g}_2$ as vector spaces;

3)$\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isotropic for $\ll\ ,\ \gg$.

We shall call the Lie algebra $\mathfrak{L}$ the associated *Manin algebra.*

Proposition 1. (cf [3]) *There is a bijective correspondance between Lie bialgebras and Manin triples.*

Notation. Let $\mathfrak{g}$ be a Lie algebra of dimension n. Consider any vector space $\mathfrak{L}$ of dimension $2n$ with a nondegenerate symmetric bilinear form $\ll\ ,\ \gg$ and a skewsymmetric bilinear map $[\ ,\ ]:\mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ such that

i) $\mathfrak{L}$ contains $\mathfrak{g}$;

ii) the bracket restricted to $\mathfrak{g} \times \mathfrak{g}$ is the Lie bracket of $\mathfrak{g}$;

iii) $\mathfrak{g}$ is isotropic;

iv) $\ll [X,Y], Z \gg + \ll Y, [X,Z] \gg = 0,\ \forall X, Y, Z \in \mathfrak{L}$.

Then, choosing an isotropic subspace supplementary to $\mathfrak{g}$ in $\mathfrak{L}$ and identifying it with $\mathfrak{g}^*$ *via* $\ll\ ,\ \gg$, $\mathfrak{L} = \mathfrak{g} + \mathfrak{g}^*$ as vector spaces and one has :

1) $\ll (X, \alpha), (Y, \nu) \gg = \langle \alpha, Y \rangle + \langle \nu, X \rangle$;

2) $[(X, \alpha), (Y, \nu)] = ([X,Y] + C_1(\alpha, Y) - C_1(\nu, X) + \bar{S}(\alpha, \nu), ad^*_X \nu - ad^*_Y \alpha + T(\alpha, \nu))$.

The invariance condition reads:

3) $\mathrm{S}(\alpha, \nu, \gamma) = \langle \gamma, \bar{S}(\alpha, \nu) \rangle$ is totally skewsymmetric;

4) $\langle T(\alpha, \nu), Z \rangle = \langle \alpha, C_1(\nu, Z) \rangle$.

We denote by $\mathfrak{L}_{S,p}$ where $p = {}^tT : \mathfrak{g} \to \Lambda^2\mathfrak{g}$, the space $\mathfrak{L} = \mathfrak{g} + \mathfrak{g}^*$ with $\ll\ ,\ \gg$ and $[\ ,\ ]$ defined by 1 and 2 with the conditions 3 and 4.

Definition 3. (cf [4]) A *Manin pair* is a pair of Lie algebras $(\mathfrak{L}, \mathfrak{g})$ and a nondegenerate symmetric bilinear form $\ll\ ,\ \gg$ on $\mathfrak{L}$ such that the conditions i),ii),iii),iv) are satisfied.

So if $(\mathfrak{L}, \mathfrak{g})$ is a Manin pair, then a choice of an isotropic subspace in $\mathfrak{L}$ supplementary to $\mathfrak{g}$ identifies $\mathfrak{L}$ with a Lie algebra $\mathfrak{L}_{S,p}$

Remark that the bracket defined on $\mathfrak{L}$ is a Lie bracket (i.e. satisfies Jacobi's identity) if and only if :

5) $\partial p = 0$ where $p = {}^tT : \mathfrak{g} \to \Lambda^2\mathfrak{g}$;

6) $[\mathrm{X,S}](\alpha, \nu, \gamma) + \langle \Sigma_{\alpha\nu\gamma} T(T(\alpha, \nu), \gamma)), X \rangle = 0$ where Σ denotes the sum over cyclic permutations;

7) $\Sigma_{\alpha\eta\gamma}(S(T(\alpha, \eta), \gamma, \nu) + S(T(\alpha, \nu), \eta, \gamma)) = 0$.

Definition 4. A map $\phi : \mathfrak{L}_{S,p} \to \mathfrak{L}_{S',p'}$ which is linear, maps $\mathfrak{g}$ to $\mathfrak{g}$, preserves $\ll\ ,\ \gg$ and is such that $\phi[(X, \alpha), (Y, \nu)]_{S,p} = [\phi(X, \alpha), \phi(Y, \nu)]_{S',p'}$ is called an isomorphism of Manin pair.

Remark that it is of the form $\phi(X, \alpha) = (A(X + \hat{Q}(\alpha)),\ {}^tA^{-1}(\alpha))$ where

i) $A : \mathfrak{g} \to \mathfrak{g}$ is Lie automorphism of $\mathfrak{g}$ and $\hat{Q} : \mathfrak{g}^* \to \mathfrak{g}$ is induced by an element

$Q \in \Lambda^2\mathfrak{g}$ through $\langle \nu, \hat{Q}(\alpha)\rangle = Q(\alpha, \nu)$ such that

ii) $A^{-1} \cdot p' - p = -\partial Q$;

iii) $(A^{-1} \cdot S' - S)(\alpha, \nu, \gamma) = \Sigma_{\alpha\nu\gamma}(Q(T(\alpha, \nu), \gamma) + \langle \alpha, [\hat{Q}(\nu), \hat{Q}(\gamma)]\rangle)$

$$= \ 1/2[Q, Q](\alpha, \nu, \gamma) + \Sigma_{\alpha\nu\gamma}(T(\alpha, \nu), \gamma)$$

where $(A \cdot p')_X(\alpha, \nu) = p'_{A^{-1}(X)}({}^tA\alpha, {}^tA\nu)$ and $(A \cdot S)(\alpha, \nu, \gamma) = S({}^tA\alpha, {}^tA\nu, {}^tA\gamma)$. We then say that $\mathfrak{L}_{S,p}$ and $\mathfrak{L}_{S',p'}$ are isomorphic under ϕ.

Remark 1. A Manin pair $(\mathfrak{L}, \mathfrak{g})$ yields a Manin triple $(\mathfrak{L}, \mathfrak{g}, \mathfrak{g}^*)$ if and only if there is an isotropic subspace supplementary to $\mathfrak{g}$ in $\mathfrak{L}$ which is a subalgebra of $\mathfrak{L}$. Hence a bialgebra structure on $\mathfrak{g}$ yields as its corresponding Manin algebra an algebra $\mathfrak{L}_{S,p'}$ which is isomorphic to a Lie algebra $\mathfrak{L}_{0,p}$ and vice versa.

Definition 5. (cf [1]) We shall say that two *Manin algebras* $\mathfrak{L}$ and $\mathfrak{L}'$ are isomorphic if there exists a map $\phi : \mathfrak{L} \to \mathfrak{L}'$ which

- is an isomorphism of Lie algebras ,
- maps $\mathfrak{g}$ to $\mathfrak{g}$,
- is a homothetic transformation from $\mathfrak{L}$ to $\mathfrak{L}'$, i.e. $\ll \phi(X), \phi(Y) \gg' = s \ll X, Y \gg \ \forall X, Y \in \mathfrak{L}$ for some nonzero real s .

Lemma 1. (cf [1]) *Two Lie bialgebra structures on a given Lie algebra $\mathfrak{g}$, $(\mathfrak{g}, p)$ and $(\mathfrak{g}, p')$ yield isomorphic Manin algebras if and only if there are $Q \in \Lambda^2\mathfrak{g}$, A an automorphism of $\mathfrak{g}$ and s a nonzero real number such that*

$$\begin{cases} p' = sA(p - \partial Q); \\ 1/2[Q, Q](\alpha, \nu, \gamma) + \Sigma_{\alpha,\nu,\gamma} Q({}^tp(\alpha, \nu), \gamma) = 0. \end{cases}$$

In particular, two exact Lie bialgebra structures on $\mathfrak{g}, (\mathfrak{g}, \partial Q)$ and $(\mathfrak{g}, \partial Q')$ yield isomorphic Manin algebras if and only if $[Q, Q] = s^2 A[Q', Q']$ for some automorphism A of $\mathfrak{g}$ and some $s \neq 0 \in \mathbb{R}$.

Lemma 2. *If $\mathfrak{g}$ is a (real or complex) semisimple Lie algebra and β its Killing form, the linear map $\rho : \ (S^2\mathfrak{g}^*)^{inv} \to (\Lambda^3\mathfrak{g})^{inv}$ defined by $\beta^{(3)}\langle \rho B, X \wedge Y \wedge Z) = B([X, Y], Z)$ for $X, Y, Z \in \mathfrak{g}$ is a linear isomorphism.*

Hence any bialgebra structure one $\mathfrak{g}$ is defined by a $Q \in \Lambda^2\mathfrak{g}$ such that $[Q, Q] = \rho B$ where $B \in (S^2\mathfrak{g}^*)^{inv}$ is of the form $B(X, Y) = \beta(MX, Y)$.

Suppose $\mathfrak{g}$ has a nondegenerate invariant symmetric bilinear form β. Then Q determines a linear map $\widetilde{Q} : \mathfrak{g} \to \mathfrak{g}$ defined by $\langle \alpha, \widetilde{Q}(X)\rangle = \beta(\hat{Q}(\alpha), X)$
or equivalentely $\beta(\widetilde{Q}(Y), X) = \beta^{(2)}(Q, X \wedge Y) = Q(\hat{\beta}(X), \hat{\beta}(Y))$
where $\hat{\beta} : \mathfrak{g} \to \mathfrak{g}^*$ is such that $\langle \hat{\beta}(X), Y\rangle = \beta(X, Y)$.

Remark 2. If $\mathfrak{g}_0$ is a simple real Lie algebra such that $\mathfrak{g} = \mathfrak{g}_0{}^{\mathbb{C}}$ is simple then $(\Lambda^3\mathfrak{g}_0)^{inv}$ is 1- dimensional. Hence any exact Lie bialgebra structure on $\mathfrak{g}_0$ is of the form $(\mathfrak{g}_0, p)$ where $p = \partial Q$ with $Q \in \Lambda^2\mathfrak{g}_0$ and $[Q, Q] = \lambda\Omega$ such that

$$\beta^{(3)}(X \wedge Y \wedge Z, \Omega) = \beta(X, [Y, Z]).$$

Corollary 1. *When we look at all the Lie bialgebra structures on a simple real Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g} = \mathfrak{g}_0{}^{\mathbb{C}}$ is simple there are only three cases up to isomorphism:*

- $\lambda = 0$;
- $\lambda > 0$;
- $\lambda < 0$.

2 The Manin algebras associated to a real semisimple Lie algebra

Aim: we determine the Manin algebra $\mathfrak{L}$ which is associated to the Lie bialgebra structure $(\mathfrak{g}_0, p)$ where $\mathfrak{g}_0$ is a semisimple real Lie algebra, p satisfies $p = \partial Q$ and $[Q,Q] = \rho B$ where $B \in (S^2\mathfrak{g}^*)^{inv}$, hence $B(X,Y) = \beta(MX,Y)$ where $M \circ adX = adX \circ M \quad \forall X \in \mathfrak{g}_0$.

Theorem 1. *Up to isomorphism the Manin algebra $\mathfrak{L}$ associated to a real semisimple Lie algebra $\mathfrak{g}_0$ which can be writen $\mathfrak{g}_0 = \underset{1\le k\le p}{\oplus} \mathfrak{I}_k$ where $\mathfrak{I}_k$ are simple ideals of $\mathfrak{g}_0$, is of the form $\mathfrak{L} = \underset{1\le k\le p}{\oplus} \mathfrak{L}_k$ where $\mathfrak{L}_k$ is one of the following:*

$$\begin{cases} \mathfrak{L}_k = Lie(T^*(I_k)) \text{ where } I_k \text{ is the Lie group associated to the Lie algebra } \mathfrak{I}_k; \\ \mathfrak{L}_k = (\mathfrak{I}_k)^{\mathbb{C}}; \\ \mathfrak{L}_k = \mathfrak{I}_k \oplus \mathfrak{I}_k. \end{cases}$$

The rest of this paragraph is devoted to the proof of the theorem 1.

The Manin algebra we consider is given by $\mathfrak{L}_{0,\partial Q}$ which is isomorphic to $\mathfrak{L}_{-1/2[Q,Q],0} = \mathfrak{L}$.
So $\mathfrak{L} = \mathfrak{g}_0 + \mathfrak{g}_0{}^*$ as vector spaces
with the duality $\ll (X,\alpha),(Y,\nu) \gg_M = \langle \alpha, Y\rangle + \langle \nu, X\rangle$
and the bracket :

$$[(X,\alpha),(Y,\nu)]_M = ([X,Y] + \bar{S}(\alpha,\nu), ad^*_X\nu - ad^*_Y\alpha) \tag{1}$$

where $S = -1/2[Q,Q] = -1/2\rho B$ so that $\beta^{(3)}\langle S, X\wedge Y\wedge Z) = -1/2\beta(M[X,Y],Z)$ and $M\circ adX = adX\circ M\ \forall X \in \mathfrak{g}_0$. So we have $M[X_k,Y_k] = [X_k, MY_k]$ for $1 \le k \le p$, this implies $M(\mathfrak{I}_k) \subset \mathfrak{I}_k$
Thus we write $M(\sum_{1\le k\le p} X_k) = \sum_{1\le k\le p} M_k(X_k)$

Proposition 1. *Suppose $\mathfrak{g}$ is a real semisimple Lie algebra and S=0 then $\mathfrak{L} = Lie(T^*G)$.*

proof: We first identify T^*G with $G\times\mathfrak{g}^*$: to $\alpha \in T^*G$ we associate the couple $(g,\tilde{\alpha})$ such that $\tilde{\alpha} = L^*{}_g\alpha_g$.
We define the product on T^*G by $\alpha_g\cdot\nu_{g'} = (R^*{}_{g'^{-1}}\alpha_g + L^*{}_{g^{-1}}\nu_{g'})$, so it is given on $G\times\mathfrak{g}^*$ by $(g,\tilde{\alpha})\cdot(g',\tilde{\nu}) = (gg', Ad^*_{g'^{-1}}\tilde{\alpha} + \tilde{\nu})$.
The bracket on $Lie(T^*G)$ reads: $ad(X,\tilde{\alpha})\cdot(Y,\tilde{\nu}) = ([X,Y], -ad^*_Y\tilde{\alpha} + ad^*_X\tilde{\nu})$ which is the bracket (1) when $S \equiv 0$. □

Remark that in what follows the hypothesis that $\mathfrak{g}_0$ is semisimple is too strong, it's sufficient that $\mathfrak{g}_0$ possesses a nondegenerate invariant bilinear form.
We identify $\mathfrak{g}_0 + \mathfrak{g}_0^*$ with $\mathfrak{g}_0 + \mathfrak{g}_0$ by $\Psi: \ \mathfrak{g}_0 + \mathfrak{g}_0^* \to \mathfrak{g}_0 + \mathfrak{g}_0 \ (X, \alpha) \mapsto (X, \hat{\beta}^{-1}(\alpha))$
Hence the duality is given by

$$\ll (X,A),(Y,B) \gg_M = \langle \hat{\beta}(A), Y \rangle + \langle \hat{\beta}(B), X \rangle = \beta(A,Y) + \beta(B,X)$$

And from (1) the bracket is given by:

(2)

$$\begin{cases} [(X,A),(Y,B)]_M & = ([X,Y] + \bar{S}(\hat{\beta}(A), \hat{\beta}(B)), \hat{\beta}^{-1}(ad_X^* \hat{\beta}(B)) - \hat{\beta}^{-1}(ad_Y^* \hat{\beta}(A))) \\ & = ([X,Y] - 1/2M[A,B], \ [X,B] - [Y,A]) \end{cases}$$

As $\mathfrak{g}_0 = \bigoplus_{1 \le k \le p} \mathfrak{I}_k$ any $X \in \mathfrak{g}_0$ is of the form $X = \sum_{1 \le k \le p} X_k$ so the bracket reads:

$$[(X,A),(Y,B)]_M = \sum_{1 \le k \le p} ([X_k, Y_k] - 1/2M_k[A_k, B_k], \ [X_k, B_k] - [Y_k, A_k])$$

Thus we only have to study the Manin algebra $\mathfrak{L}$ associated to a real simple Lie algebra $\mathfrak{g}_0$. In this case ${\mathfrak{g}_0}^{\mathbb{C}}$ is either simple or not simple. If it is not simple then $M = a + bJ$ where $a, b \in \mathbb{R}$ and J is such that $J^2 = -Id, \ J \circ adX = adX \circ J \ \forall X \in \mathfrak{g}_0$; if it is simple $M = \lambda Id$ for $\lambda \in \mathbb{R}$.

Proposition 2. *Assume that there exists $N: \mathfrak{g}_0 \ \rightarrow \ \mathfrak{g}_0$ such that*

1) *N is a linear isomorphism;*
2) *$N^2 = 1/2M$;*
3) *$N \circ adX = adX \circ N \ \forall X \in \mathfrak{g}_0$.*

Then

$$\mathfrak{L} \approx {\mathfrak{g}_0}^{\mathbb{C}} \ \textit{as Lie algebras}, \ \mathfrak{g}_0 \approx \{(X,0) \in {\mathfrak{g}_0}^{\mathbb{C}}\}$$

and the duality is given by $\ll (X,A),(Y,B) \gg_M = \beta(X, N^{-1}B) + \beta(N^{-1}A, Y)$.

proof: the isomorphism is given by:

$$\Psi : {\mathfrak{g}_0}^{\mathbb{C}} \ \rightarrow \ \mathfrak{L} = \mathfrak{g}_0 + \mathfrak{g}_0 \ (X,A) \ \mapsto \ (X, N^{-1}A)$$

Recall that ${\mathfrak{g}_0}^{\mathbb{C}} = \{(X,Y) \ | \ X, Y \in \mathfrak{g}_0\}$ with the bracket $[\ ,\ ]_{\mathbb{C}}$ given by:

$$[(X,A),(Y,B)]_{\mathbb{C}} = ([X,Y] - [A,B], [X,B] - [Y,A]). \qquad \square$$

Corollary 1. *If $M = \lambda Id$ with $\lambda > 0$ we obtain the Manin pair $({\mathfrak{g}_0}^{\mathbb{C}}, \mathfrak{g}_0)$ where the duality is given by:*

$$\ll (X,A),(Y,B) \gg = \beta(A,Y) + \beta(B,X).$$

Corollary 2. *If ${\mathfrak{g}_0}^{\mathbb{C}}$ is not simple then $M = a + bJ$ with $a^2 + b^2 \neq 0$ if $M \neq 0$, thus N exits and is given by N=c+dJ with $(c+id)^2 = a+ib$ in $\mathbb{C}$. Then the Manin algebra associated to a Lie bialgebra structure with $M \neq 0$ on a real simple Lie algebra $\mathfrak{g}_0$ such that ${\mathfrak{g}_0}^{\mathbb{C}}$ is not simple, is $\mathfrak{L} \approx {\mathfrak{g}_0}^{\mathbb{C}}$.*

Remark 1. If ${\mathfrak{g}_0}^{\mathbb{C}}$ is simple and $M = \lambda Id$ with $\lambda < 0$ there is no possible N.

Proposition 3. *If ${\mathfrak{g}_0}^{\mathbb{C}}$ is simple and $M = \lambda Id$ with $\lambda < 0$ then $\mathfrak{L} \approx \mathfrak{g}_0 \oplus \mathfrak{g}_0$, the direct sum of two copies of the Lie algebra $\mathfrak{g}_0$, $\mathfrak{g}_0 \approx \Delta\mathfrak{g}_0 = \{(X, X) \mid X \in \mathfrak{g}_0\}$ and the duality is given by:*

$$\ll (X, Y), (X', Y') \gg = 1/2\beta(X, X') - 1/2\beta(Y, Y').$$

proof: we can suppose that $\lambda = -1$ because when ${\mathfrak{g}_0}^{\mathbb{C}}$ is simple the structures are isomorphic when multiplied by a positive constant, then from (2) the bracket is given by:

$$[(X, Y), (X', Y')] = ([X, X'] + [Y, Y'], [X, Y'] + [Y, X']).$$

The isomorphism is $\phi : \mathfrak{L} = \mathfrak{g}_0 + \mathfrak{g}_0 \ \rightarrow \ \mathfrak{g}_0 \oplus \mathfrak{g}_0 = \mathfrak{L}' \ (X, Y) \ \mapsto \ (X + Y, X - Y)$
The duality is given by:
$\ll (X, Y), X', Y') \gg' = \ll \phi^{-1}(X, Y), \phi^{-1}(X', Y') \gg_{\mathfrak{L}} = \frac{1}{2}(\beta(X, X') - \beta(Y, Y'))$ □

Remark 2. We have ${\mathfrak{g}_0}^{\mathbb{C}} \approx \mathfrak{g}_0 \oplus \mathfrak{g}_0$ if and only if there exits such a J.
The isomorphism is $\phi : \mathfrak{g}_0 \oplus \mathfrak{g}_0 \ \rightarrow \ {\mathfrak{g}_0}^{\mathbb{C}} \ (X, Y) \ \mapsto \ (\frac{X+iJX}{2}, \frac{Y+iJY}{2})$
Remark that if ${\mathfrak{g}_0}^{\mathbb{C}} \approx \mathfrak{g}_0 \oplus \mathfrak{g}_0$ then ${\mathfrak{g}_0}^{\mathbb{C}}$ is not simple.

Theorem 3. *The Manin algebra $\mathfrak{L}$ which is associated to the Lie bialgebra structure $(\mathfrak{g}_0, p)$ where $\mathfrak{g}_0$ is a simple real Lie algebra, p satisfies $p = \partial Q$ and $[Q, Q] = \rho B$ where $B \in (S^2\mathfrak{g}^*)^{inv}$, hence $B(X, Y) = \beta(MX, Y)$ where $M \circ adX = adX \circ M \ \ \forall X \in \mathfrak{g}_0$, is*

- $\mathfrak{L} = Lie(T^*G)$ *if $B = 0$, where G is the Lie group associated to $\mathfrak{g}_0$;*
- $\mathfrak{L} = {\mathfrak{g}_0}^{\mathbb{C}}$ *if ${\mathfrak{g}_0}^{\mathbb{C}}$ is not simple;*
- $\mathfrak{L} = {\mathfrak{g}_0}^{\mathbb{C}}$ *if ${\mathfrak{g}_0}^{\mathbb{C}}$ is simple and $M = \lambda Id$ with $\lambda \in {\mathbb{R}^*}_+$;*
- $\mathfrak{L} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$ *if ${\mathfrak{g}_0}^{\mathbb{C}}$ is simple and $M = \lambda Id$ with $\lambda \in {\mathbb{R}^*}_-$.*

3 Solutions of the nonzero standard modified Yang-Baxter's equation.

Aim: we want to find all the solutions of the modified Yang-Baxter equation of the form:

$$\left\{ \begin{array}{l} [\tilde{Q}X, \tilde{Q}Y] - \tilde{Q}[\tilde{Q}X, Y] - \tilde{Q}[X, \tilde{Q}Y] = \lambda[X, Y] \\ \beta(\tilde{Q}X, Y) = -\beta(X, \tilde{Q}Y) \end{array} \right. \tag{3}$$

for $X, Y \in \mathfrak{g}_0$ when $\mathfrak{g}_0$ is a real semisimple Lie algebra, and for nonzero λ.

In this case the algebra $\mathfrak{g} = {\mathfrak{g}_0}^{\mathbb{C}}$ is semisimple; remark that this equation has been studied for complex semisimple Lie algebra by A. Belavin and V. Drinfeld [2]; we use their methods and results.

A. The case $\lambda > 0$.

a. Existence of a solution.

Proposition 1. *There always exists a solution $\widetilde{Q} \in End(\mathfrak{g}_0)$, it is related to the existence of a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0$ contains a maximal torus of $\mathfrak{k}$ where $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}_0$.*

proof: to obtain this result we work on $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$, we extend $\widetilde{Q}$ $\mathbb{C}$-linearly to $\mathfrak{g}$. The equation satisfied by $\widetilde{Q}$ on $\mathfrak{g}$ is:

$$\begin{cases} [\widetilde{Q}X, \widetilde{Q}Y] - \widetilde{Q}[\widetilde{Q}X, Y] - \widetilde{Q}[X, \widetilde{Q}Y] = \lambda[X, Y] \\ \beta(\widetilde{Q}X, Y) = -\beta(X, \widetilde{Q}Y) \end{cases} \tag{4}$$

For any complex number μ let $\mathfrak{g}_\mu$ denote the generalized eigenspace given by:

$$\mathfrak{g}_\mu = \{X \in \mathfrak{g} \mid (\widetilde{Q} - \mu)^k X = 0 \; for \; some \; positive \; integer \; k\}.$$

Let $a^2 = -\lambda$, then a is purely imaginary; $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ are subalgebras of $\mathfrak{g}$ which are isotropic with respect to β and $\mathfrak{g}_{-a} = \overline{\mathfrak{g}_a}$.
Besides $\mathfrak{g}' = \sum_{\mu \neq \pm a} \mathfrak{g}_\mu$ is a subalgebra and $\overline{\mathfrak{g}'} = \mathfrak{g}'$.
From ([2] and [1] p. 8) we know that there exist two Borel subalgebras $\mathfrak{b}_\pm$ of $\mathfrak{g}$ such that $\mathfrak{g}_a + \mathfrak{g}' \subset \mathfrak{b}_+$ and $\mathfrak{g}_{-a} + \mathfrak{g}' \subset \mathfrak{b}_-$, moreover they satisfy $\overline{\mathfrak{b}_+} = \mathfrak{b}_-$ and $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\overline{\mathfrak{h}} = \mathfrak{h}$ thus $\mathfrak{h} = {\mathfrak{h}_0}^{\mathbb{C}}$ where $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$.

Let $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}_0$ and let $\mathfrak{h}_0 = \mathfrak{t} + \mathfrak{a}$ be the corresponding decomposition of $\mathfrak{h}_0$ i.e. $\mathfrak{t} \subset \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$.
Denote by Δ^+ the set of roots of $(\mathfrak{g}, \mathfrak{h})$ such that the corresponding eigenspaces are in $\mathfrak{b}_+$.
Denote by α' the restriction of $\alpha \in \Delta$ to $\mathfrak{h}_0$.
Then $\overline{\mathfrak{g}^\alpha} = \mathfrak{g}^{-\widetilde{\alpha}}$ where $-\widetilde{\alpha}' = \overline{\alpha}'$ and $\widetilde{\alpha} \in \Delta^+$ if $\alpha \in \Delta^+$.
Remark that $\mathfrak{t}' = \mathfrak{t} + i\mathfrak{a}$ is a maximal torus of $\mathfrak{k} + i\mathfrak{p}$ which is a compact subalgebra of $\mathfrak{g}$.
Hence we have $\alpha \in \Delta^+$ if and only if $\exists X \in i\mathfrak{t}'$ such that $\alpha(X) > 0$.
But $(\alpha \in \Delta^+) \Rightarrow (\overline{\alpha'} \in \Delta^-)$ thus we obtain: $\begin{cases} i\alpha'(Z) + \alpha'(Y) > 0 \\ -i\alpha'(Z) + \alpha'(Y) < 0 \end{cases}$
So there must be a $Z \in \mathfrak{t}$ such that $\alpha(Z) \neq 0 \; \forall \alpha \in \Delta$.

Reciprocally suppose there exists $X \in \mathfrak{t}$ such that $\alpha(X) \neq 0 \; \forall \alpha \in \Delta$ then let $\Delta^+ = \{\alpha \in \Delta \mid i\alpha(X) > 0\}$, $b_+ = (\mathfrak{h}_0)^{\mathbb{C}} + \sum\limits_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ and define $\widetilde{Q}$ as follow:

$$\widetilde{Q}(X) = \begin{cases} aX \text{ if } X \in \sum\limits_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \\ 0 \text{ if } X \in \mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}} \\ -aX \text{ if } X \in \overline{\sum\limits_{\alpha \in \Delta^+} \mathfrak{g}_\alpha} \end{cases} \tag{5}$$

Such a $\widetilde{Q}$ satisfies (4) and when restricted to $\mathfrak{g}_0$ it satisfies (3).

So there exists a solution of (3) if and only if there exits $X \in \mathfrak{t}$ *such that* $\alpha(X) \neq 0\ \forall \alpha \in \Delta^+$. *And in this case a solution of (3) is given by (5). Remark that the fact that there exits* $X \in \mathfrak{t}$ *such that* $\alpha(X) \neq 0\ \forall \alpha \in \Delta$ *is equivalent to the fact that* $\mathfrak{t}$ *is a maximal torus of* $\mathfrak{k}$. *Hence we can always find a solution of (3) for* $\lambda > 0$.

□

b. Research of all the solutions.

We work on $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$, we extend $\widetilde{Q}$ $\mathbb{C}$-lineairely to $\mathfrak{g}$.
The equation satisfied by $\widetilde{Q}$ on $\mathfrak{g}$ is (4)

But we impose futhermore that $\widetilde{Q} \in End(\mathfrak{g}_0)$ i.e. $\overline{\widetilde{Q}(X)} = \widetilde{Q}(\overline{X})\ \forall X \in \mathfrak{g}$.

We have from Belavin-Drinfeld [2] :

Theorem 1. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra and let* $Q \in \Lambda^2\mathfrak{g}$ *satisfying*

$$\beta^{(3)}([Q,Q], X \wedge Y \wedge Z) = \beta(\frac{\lambda}{2}[X,Y], Z).$$

Then, there exists a Cartan subalgebra $\mathfrak{h}$ *of* $\mathfrak{g}$, *a system of positive roots* Δ^+ *of* $(\mathfrak{g}, \mathfrak{h})$, *two subsets* Γ_+ *and* Γ_- *of the set* Φ *of simple roots corresponding to* Δ^+ *and a map* $\tau : \Gamma_+ \to \Gamma_-$ *satisfying*

(1) $< \tau(\alpha), \tau(\nu) >=< \alpha, \nu >,\ \forall \alpha, \nu \in \Gamma_+;$

(2) $\forall \alpha \in \Gamma_+$, *there exists a positive integer* k *such that* $\tau^l(\alpha) \in \Gamma_+,\ \forall l < k$ *and* $\tau^k(\alpha) \notin \Gamma_+$ *such that, for a choice of Weyl bases* E_α *in* $\mathfrak{g}^\alpha$ *with* $\beta(E_\alpha, E_{-\alpha}) = 1$:

$$Q = Q_0 + a(\sum_{\alpha \in \Delta^+} E_{-\alpha} \wedge E_\alpha + 2 \sum_{\alpha \in \hat{\Gamma}_+, \alpha < \nu} E_{-\nu} \wedge E_\alpha)$$

where $a^2 = -\lambda$ *and* $Q_0 \in \Lambda^2\mathfrak{h}$ *is determined by* $Q(\alpha, \nu),\ \forall \alpha, \nu \in \Phi$ *and those must verify:*

(3) $Q(\tau(\alpha), \nu) = Q(\alpha, \nu) - a(< \alpha, \nu > + < \tau(\alpha), \nu >),\ \forall \alpha \in \Gamma_+,\ \forall \nu \in \Phi.$

Where $\hat{\Gamma}_+$ is the set of the positive roots which can be written as integer combinations of the simple roots in Γ_+.
Where $< \alpha, \nu >= \beta(H_\alpha, H_\nu)$.
Where the notation $\nu > \alpha$ for $\alpha \in \hat{\Gamma}_+$ means that there exists an integer $k \geq 1$ such that $\tau^k(\alpha) = \nu$.

Lemma 1. *As we work on* $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ *with* $\lambda > 0$ *we have:*

1) $\mathfrak{h} = {\mathfrak{h}_0}^{\mathbb{C}}$;
2) $\overline{\mathfrak{g}^\alpha} = \mathfrak{g}^{-\widetilde{\alpha}}$ *where* $-\widetilde{\alpha}_{|\mathfrak{h}_0} = \overline{\alpha}_{|\mathfrak{h}_0}$ *thus* $\overline{E_\alpha} = \lambda_\alpha E_{-\widetilde{\alpha}}$;
3) $\Gamma_- = \{\widetilde{\alpha}$ *when* $\alpha \in \Gamma_+\}$;
4) $\overline{H_\alpha} = -H_{\widetilde{\alpha}}$ *for* $\alpha \in \Delta$;
5) $\widetilde{Q}(E_{-\alpha}) = -aE_{-\alpha} - 2a \sum_{\nu < \alpha} E_{-\nu}$ *where* $\sum_{\nu < \alpha} E_{-\nu} = 0$ *if* $\nu \notin \hat{\Gamma}_+$

proof:
∗) from the paragraphe 3.a we already have 1 and 2.
∗) for 3: if $c_+ = Im(\widetilde{Q} + a)$ then $c_- = Im(\widetilde{Q} - a) = \overline{c_+}$

and $\sum_{\alpha\in\hat{\Gamma}_+}(\mathfrak{g}^{\alpha}+\mathfrak{g}^{-\alpha}+[\mathfrak{g}^{\alpha},\mathfrak{g}^{-\alpha}])$ is the Levy factor of c_+ so $\overline{\hat{\Gamma}_+}=\hat{\Gamma}_-$ i.e. $\hat{\Gamma}_-=\{\widetilde{\alpha}$ when $\alpha\in\Gamma_+\}$

$*$) for 4: we use $[E_\alpha,E_{-\alpha}]=H_\alpha$ and 2 to obtain $\overline{H_\alpha}=-\lambda_\alpha\lambda_{-\alpha}H_{\widetilde{\alpha}}$, $[H,E_\alpha]=\alpha(H)E_\alpha$ and 2 to obtain $\overline{\alpha(H)}=-\widetilde{\alpha}(\overline{H})$ and $\beta(H_\alpha,H)=\alpha(H)$ to obtain $\lambda_\alpha\lambda_{-\alpha}=1$.

$*$) for 5: from theorem 1 we have

$\widetilde{Q}E_\alpha=a(E_\alpha+2\sum_{\nu>\alpha}E_\nu)\ \forall\alpha\in\nu_+$ where $\sum_{\nu>\alpha}E_\nu=0$ if $\alpha\in\Delta_+\backslash\hat{\Gamma}_+$

To determine $\widetilde{Q}E_{-\alpha}$ we use: $\beta(\widetilde{Q}X,Y)=-\beta(X,\widetilde{Q}Y)$, $\widetilde{Q}(\oplus_{\alpha\in\Delta^+}\mathfrak{g}^{-\alpha})\subset\oplus_{\alpha\in\Delta^+}\mathfrak{g}^{-\alpha}$ and $\beta(E_\alpha,E_{-\gamma})=\delta_{\alpha\gamma}$. □

Remark 1. The equality 5 of Lemma 1. does not depend of the sign of λ.

Theorem 2. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and let $Q\in\Lambda^2\mathfrak{g}_0$ satisfying*

$$\begin{cases}[\widetilde{Q}X,\widetilde{Q}Y]-\widetilde{Q}[\widetilde{Q}X,Y]-\widetilde{Q}[X,\widetilde{Q}Y]=\lambda[X,Y]\\ \beta(\widetilde{Q}X,Y)=-\beta(X,\widetilde{Q}Y)\ \textit{with}\ \lambda>0\end{cases}$$

Then, there exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ which is as in a.proposition 1, a system of positive roots Δ^+ of $({\mathfrak{g}_0}^{\mathbb{C}},{\mathfrak{h}_0}^{\mathbb{C}})$, one subset Γ_+ of the set Φ of simple roots corresponding to Δ^+ and a map $\tau:\Gamma_+\to\Gamma_-=\{\widetilde{\alpha}$ when $\alpha\in\Gamma_+\}$ satisfying

(1) $<\tau(\alpha),\tau(\nu)>=<\alpha,\nu>,\ \forall\alpha,\nu\in\Gamma_+$;

(2) $\forall\alpha\in\Gamma_+$, *there exists a positive integer k such that* $\tau^l(\alpha)\in\Gamma_+,\ \forall l<k$ *and* $\tau^k(\alpha)\notin\Gamma_+$

(3) $\tau(\alpha)=\widetilde{\nu}\ \Rightarrow\ \tau(\nu)=\widetilde{\alpha}\ \forall\alpha\in\Gamma_+$

such that, for a choice of Weyl bases E_α in $\mathfrak{g}^\alpha$ where $\overline{E_\alpha}=\lambda_\alpha E_{-\widetilde{\alpha}}$ with $\beta(E_\alpha,E_{-\alpha})=1$ and

(4) $\lambda_{\tau(\alpha)}=\lambda_\alpha\ \forall\alpha\in\Gamma_+$ *we have*

$$Q=Q_0+a(\sum_{\alpha\in\Delta^+}E_{-\alpha}\wedge E_\alpha+2\sum_{\alpha\in\hat{\Gamma}_+,\alpha<\nu}E_{-\nu}\wedge E_\alpha)$$

where $a^2=-\lambda$ and $Q_0\in\Lambda^2\mathfrak{h}_0$ is determined by $Q(\alpha,\nu),\ \forall\alpha,\nu\in\Phi$ and those must verify:

(5) $Q(\tau(\alpha),\nu)=Q(\alpha,\nu)-a(<\alpha,\nu>+<\tau(\alpha),\nu>),\ \forall\alpha\in\Gamma_+,\ \forall\nu\in\Phi$

(6) $\widetilde{Q}(H_{\widetilde{\alpha}})=-\overline{\widetilde{Q}(H_\alpha)}\ \forall\alpha$.

proof: we want that a Q given by theorem 1 satisfies $\widetilde{Q}(\overline{X})=\overline{\widetilde{Q}(X)}\ \forall X\in\mathfrak{g}$

In particular for $X=E_\alpha$ we obtain:

<u>1^{st} *case* :</u>

If $\alpha\in\hat{\Gamma}_+$ and $\tau(\alpha)=\widetilde{\nu}\in\hat{\Gamma}_-$, $\widetilde{\nu}\notin\hat{\Gamma}_+$ then $\overline{\widetilde{Q}(E_\alpha)}=-a(\lambda_\alpha E_{-\widetilde{\alpha}}+2\lambda_{\tau(\alpha)}E_{-\widetilde{\nu}})$ and $\widetilde{Q}(\overline{E_\alpha})=-a\lambda_\alpha(E_{-\widetilde{\alpha}}+2\sum_{\gamma<\widetilde{\alpha}}E_{-\gamma})$

There is equality if and only if $\lambda_{\tau(\alpha)}=\lambda_\alpha$ and $\tau(\nu)=\widetilde{\alpha}$

<u>2^{nd} *case* :</u>

We apply a recursive process in the case where $\alpha\in\hat{\Gamma}_+$ is such that $\tau^l(\alpha)\in\hat{\Gamma}_+,\ l=1,\cdots,k$ and $\tau^{k+1}(\alpha)=\widetilde{\nu}\notin\hat{\Gamma}_+$.

We apply recursive hypothese to $\tau(\alpha)$.

Then $\overline{\widetilde{Q}(E_\alpha)} = -a(\lambda_\alpha E_{-\widetilde{\alpha}} + 2\lambda_{\tau(\alpha)}E_{-\tau^k(\nu)} + \cdots + 2\lambda_{\tau^{k+1}(\alpha)}E_{-\nu})$ and

$\widetilde{Q}(\overline{E_\alpha}) = -a\lambda_\alpha(E_{-\widetilde{\alpha}} + 2\sum_{\gamma<\widetilde{\alpha}} E_{-\gamma})$

There is equality if and only if : $\lambda_\alpha = \lambda_{\tau(\alpha)} = \cdots = \lambda_{\tau^{k+1}(\alpha)}$ and $\tau(\tau^k(\nu)) = \widetilde{\alpha}$ that is

$\tau^{k+1}(\nu) = \widetilde{\alpha}$.

We also want that $\widetilde{Q}(\overline{X}) = \overline{\widetilde{Q}(X)}$ $\forall X \in \mathfrak{h}$, that gives immediately 6. □

Remarks

1) If $\tau(\alpha) = \widetilde{\alpha}$ then $\lambda_\alpha \in \mathbb{C}$.
2) If $\tau(\alpha) = \widetilde{\nu}$ and $\tau(\nu) = \widetilde{\alpha}$ then $\lambda_{\widetilde{\alpha}} = \overline{\lambda_\alpha}$ and $\lambda_{\widetilde{\nu}} = \overline{\lambda_\nu}$.
3) If $\tau(\alpha) = \widetilde{\alpha}$ then $\widetilde{Q}(H_\alpha - aH_\alpha) \in i\Lambda^2\mathfrak{h}_0$.

B. The case $\lambda < 0$.

The existence of a solution in this case has been studied in [1]:

Theorem 1. *There exists a solution of (3) for $\lambda < 0$ if and only if $\mathfrak{g}_0$ is the sum of simple ideals which are either split, complex or one of the following cases (using the notation of Helgason [5]):*

(i) $SU(p,p), SU(p,p+1)$;
(ii) $SO(p,p+2)$;
(iii) EII.

We extend $\widetilde{Q}$ $\mathbb{C}-$ linearly to $\mathfrak{g}$ as in A. and we use the same A.b.theorem 1.

Lemma 1. *As we work on $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ with $\lambda < 0$ we have:*

1) $\mathfrak{h} = {\mathfrak{h}_0}^{\mathbb{C}}$;
2) $\overline{\mathfrak{g}^\alpha} = \mathfrak{g}^{\widetilde{\alpha}}$ *where* $\widetilde{\alpha}_{|\mathfrak{h}_0} = \overline{\alpha}_{|\mathfrak{h}_0}$ *thus* $\overline{E_\alpha} = \lambda_\alpha E_{\widetilde{\alpha}}$;
3) $\overline{H_\alpha} = H_{\widetilde{\alpha}}$ *for* $\alpha \in \Delta$;
4) $\widetilde{Q}(E_{-\alpha}) = -aE_{-\alpha} - 2a\sum_{\nu<\alpha} E_{-\nu}$ *where* $\sum_{\nu<\alpha} E_{-\nu} = 0$ *if* $\nu \notin \hat{\Gamma}_+$

proof:

$*$) for 1: as $a^2 = -\lambda$, a is real,
thus $\mathfrak{g}_a = ({\mathfrak{g}_a}^{\mathbb{R}})^{\mathbb{C}}$, $\mathfrak{g}_-a = ({\mathfrak{g}_{-a}}^{\mathbb{R}})^{\mathbb{C}}$, $\mathfrak{g}' = ({\mathfrak{g}'}^{\mathbb{R}})^{\mathbb{C}}$, $\mathfrak{b}_+ = ({\mathfrak{b}_+}^{\mathbb{R}})^{\mathbb{C}}$, $\mathfrak{b}_- = ({\mathfrak{b}_-}^{\mathbb{R}})^{\mathbb{C}}$
so $\mathfrak{h} = {\mathfrak{h}_0}^{\mathbb{C}}$
$*$) for 2: we use $\mathfrak{g}^\alpha \subset \mathfrak{b}_+$ and $\overline{\mathfrak{b}_+} = \mathfrak{b}_+$.
$*$) for 3: we use $[E_\alpha, E_{-\alpha}] = H_\alpha$ and 2 to obtain $\overline{H_\alpha} = \lambda_\alpha\lambda_{-\alpha}H_{\widetilde{\alpha}}$; $[H, E_\alpha] = \alpha(H)E_\alpha$ and 2 to obtain $\overline{\alpha(H)} = \widetilde{\alpha}(\overline{H})$ and $\beta(H_\alpha, H) = \alpha(H)$ to obtain $\lambda_\alpha\lambda_{-\alpha} = 1$. □

Theorem 2. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra as in theorem 1 and let $Q \in \Lambda^2\mathfrak{g}_0$ satisfying*

$$\begin{cases} [\widetilde{Q}X, \widetilde{Q}Y] - \widetilde{Q}[\widetilde{Q}X, Y] - \widetilde{Q}[X, \widetilde{Q}Y] = \lambda[X, Y] \\ \beta(\widetilde{Q}X, Y) = -\beta(X, \widetilde{Q}Y) \text{ with } \lambda < 0 \end{cases}$$

Then, there exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, a system of positive roots Δ^+ of $({\mathfrak{g}_0}^{\mathbb{C}}, {\mathfrak{h}_0}^{\mathbb{C}})$, two subsets Γ_+ and Γ_- of the set Φ of simple roots corresponding to Δ^+ and a map $\tau : \Gamma_+ \to \Gamma_-$ satisfying

(1) $<\tau(\alpha),\tau(\nu)>=<\alpha,\nu>,\ \forall\alpha,\nu\in\Gamma_+$;

(2) $\forall\alpha\in\Gamma_+$, *there exists a positive integer* k *such that* $\tau^l(\alpha)\in\Gamma_+,\ \forall l<k$ *and* $\tau^k(\alpha)\notin\Gamma_+$

(3) $\tau^k(\alpha)\in\Gamma_+ \iff \tau^k(\widetilde{\alpha})\in\Gamma_+$;

(4) $\tau(\widetilde{\alpha})=\widetilde{\tau(\alpha)}\ \forall\alpha\in\Gamma_+$

such that, for a choice of Weyl bases E_α *in* $\mathfrak{g}^\alpha$ *where* $\overline{E_\alpha}=\lambda_\alpha E_{\widetilde{\alpha}}$ *with* $\beta(E_\alpha,E_{-\alpha})=1$ *and*

(5) $\lambda_{\tau(\alpha)}=\lambda_\alpha\ \forall\alpha\in\Gamma_+$ *we have*

$$Q=Q_0+a(\sum_{\alpha\in\Delta^+}E_{-\alpha}\wedge E_\alpha+2\sum_{\alpha\in\hat{\Gamma}_+,\alpha<\nu}E_{-\nu}\wedge E_\alpha)$$

where $a^2=-\lambda$ *and* $Q_0\in\Lambda^2\mathfrak{h}_0$ *is determined by* $Q(\alpha,\nu),\ \forall\alpha,\nu\in\Phi$ *and those must verify:*

(6) $Q(\tau(\alpha),\nu)=Q(\alpha,\nu)-a(<\alpha,\nu>+<\tau(\alpha),\nu>),\ \forall\alpha\in\Gamma_+,\ \forall\nu\in\Phi$

(7) $\widetilde{Q}(H_{\widetilde{\alpha}})=\overline{\widetilde{Q}(H_\alpha)}\ \forall\alpha$.

proof: we want that a Q given by b.theorem 1 satisfies $\widetilde{Q}(\overline{X})=\overline{\widetilde{Q}(X)}\ \forall X\in\mathfrak{g}$
In particular for $X=E_\alpha$ we obtain:

$\widetilde{Q}(\overline{E_\alpha})=a\lambda_\alpha(E_{\widetilde{\alpha}}+2\sum_{\gamma>\widetilde{\alpha}}E_\gamma)$ and $\overline{\widetilde{Q}(E_\alpha)}=a\lambda_\alpha E_{\widetilde{\alpha}}+2a\sum_{\nu>\alpha}\lambda_\nu E_{\widetilde{\nu}}$
Assume that $\tau^k(\widetilde{\alpha})\in\Gamma_+$ for $k=1,\cdots,l-1$ and $\tau^l(\widetilde{\alpha})\notin\Gamma_+$; and that $\tau^i(\alpha)\in\Gamma_+$ for $i=1,\cdots,j-1$ and $\tau^j(\alpha)\notin\Gamma_+$.
Then the previous equality reads:
$(*)\quad a\lambda_\alpha E_{\widetilde{\alpha}}+2a\lambda_\alpha E_{\tau(\widetilde{\alpha})}+\cdots+2a\lambda_\alpha E_{\tau^l(\widetilde{\alpha})}=a\lambda_\alpha E_{\widetilde{\alpha}}+2a\lambda_{\tau(\alpha)}E_{\widetilde{\tau(\alpha)}}+\cdots+2a\lambda_{\tau^j(\alpha)}E_{\widetilde{\tau^l(\alpha)}}$

We must have j=l this is 3.
We apply a recursive process to get 4 and 5.
<u>1^{st} *case* :</u> for l=1
If $\alpha\in\Gamma_+$ and $\tau(\alpha)\notin\Gamma_+$, then by 3 we have $\widetilde{\alpha}\in\Gamma_+$ and $\tau(\widetilde{\alpha})\notin\Gamma_+$
So $(*)$ gives: $\lambda_\alpha E_{\tau(\widetilde{\alpha})}=\lambda_{\tau(\alpha)}E_{\widetilde{\tau(\alpha)}}$.

<u>2^{nd} *case* :</u> Assume that ($\forall\alpha\in\Gamma_+$ such that $\tau^k(\alpha)\in\Gamma_+$ for $k=1,\cdots,l'-1$ and $\tau^{l'}(\alpha)\notin\Gamma_+$ we have: $\lambda_\alpha=\lambda_{\tau(\alpha)}$ and $\tau(\widetilde{\alpha})=\widetilde{\tau(\alpha)}$) for all $l'\le l$
We write $(*)$ for l+1: $\lambda_\alpha E_{\tau(\widetilde{\alpha})}+\cdots+\lambda_\alpha E_{\tau^{l+1}(\widetilde{\alpha})}=\lambda_{\tau(\alpha)}E_{\widetilde{\tau(\alpha)}}+\cdots+\lambda_{\tau^{l+1}(\alpha)}E_{\widetilde{\tau^{l+1}(\alpha)}}$
We successively apply the recursive hypothesis to $\tau^l(\alpha)$ for $l'=1$; to $\tau^{l-1}(\alpha)$ for $l'=2$; to $\tau^{l-2}(\alpha)$ for $l'=3;\cdots$; to $\tau(\alpha)$ for $l'=l$, this gives 4 and 5. □

4 Case of a complex structure.

Aim: We show that there exist solutions of the modified Yang-Baxter equation when $\mathfrak{g}_0$ is a simple real Lie algebra such that ${\mathfrak{g}_0}^{\mathbb{C}}$ is not simple, which are not preserving the ideals in ${\mathfrak{g}_0}^{\mathbb{C}}$.

We see ${\mathfrak{g}_0}^{\mathbb{C}}$ as $\mathfrak{g}_0+i\mathfrak{g}_0$, the conjugation is given by $\overline{(X,Y)}=(X,-Y)$
Here ${\mathfrak{g}_0}^{\mathbb{C}}=I_1\oplus I_2$ where I_1 and I_2 are two simple ideals of ${\mathfrak{g}_0}^{\mathbb{C}}$.

Let J be a complex structure on $\mathfrak{g}_0$, extended $\mathbb{C}-$lineareły to ${\mathfrak{g}_0}^{\mathbb{C}}$ it is given by $J = iId_{|I_1} - iId_{|I_2}$. Hence $I_1 = \{(X, -JX) \mid X \in \mathfrak{g}_0\}$ and $I_2 = \{(X, JX) \mid X \in \mathfrak{g}_0\}$.

Let $M : \mathfrak{g}_0 \to \mathfrak{g}_0$ satisfying $M \circ adX = adX \circ M\ \forall X \in \mathfrak{g}_0$; if we still denote by M its $\mathbb{C}-$linear extension to ${\mathfrak{g}_0}^{\mathbb{C}}$, we have $M = uId + vJ$.

Hence when we restricted M to $\mathfrak{g}_0$, the most general modified Yang Baxter's equation on $\mathfrak{g}_0$ is:

$$(6)\qquad \begin{cases} \beta(\widetilde{Q}X, Y) & = -\beta(X, \widetilde{Q}Y) \\ [\widetilde{Q}X, \widetilde{Q}Y] & -\widetilde{Q}[\widetilde{Q}X, Y] - \widetilde{Q}[X, \widetilde{Q}Y] = (uId + vJ)[X, Y] \\ & \text{with } u^2 + v^2 \neq 0 \end{cases}$$

We still denote by $\widetilde{Q}$ the $\mathbb{C}-$linear extension of $\widetilde{Q}$ to ${\mathfrak{g}_0}^{\mathbb{C}}$.
We extend (6) $\mathbb{C}-$linearely to ${\mathfrak{g}_0}^{\mathbb{C}}$, we obtain the same equation but for $X, Y \in {\mathfrak{g}_0}^{\mathbb{C}}$.

On ${\mathfrak{g}_0}^{\mathbb{C}}$ we can see $\widetilde{Q}$ as

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q_1} & \widetilde{Q_{12}} \\ \widetilde{Q_{21}} & \widetilde{Q_2} \end{pmatrix}$$

where $\widetilde{Q_1} : I_1 \to I_1, \widetilde{Q_2} : I_2 \to I_2, \widetilde{Q_{12}} : I_2 \to I_1, \widetilde{Q_{21}} : I_1 \to I_2$ are linear maps.

Remark 1. $\widetilde{Q}(\overline{X}) = \overline{\widetilde{Q}(X)}$, so $\widetilde{Q_2}(X) = \overline{\widetilde{Q_1}(\overline{X})}$ and $\widetilde{Q_{12}}(X) = \overline{\widetilde{Q_{21}}(\overline{X})}$

We obtain the following equations:
(7)

$$\begin{cases} [\widetilde{Q_1}X, \widetilde{Q_1}Y] - \widetilde{Q_1}[\widetilde{Q_1}X, Y] - \widetilde{Q_1}[X, \widetilde{Q_1}Y] = (u + iv)[X, Y]\ \forall X, Y \in I_1 & (a) \\ [\widetilde{Q_{21}}X, \widetilde{Q_{21}}Y] - \widetilde{Q_{21}}[\widetilde{Q_1}X, Y] - \widetilde{Q_{21}}[X, \widetilde{Q_1}Y] = 0\ \forall X, Y \in I_1 & (b) \\ [\widetilde{Q_1}X, \widetilde{Q_{12}}Y] - \widetilde{Q_{12}}[\widetilde{Q_{21}}X, Y] - \widetilde{Q_1}[X, \widetilde{Q_{12}}Y] = 0\ \forall X \in I_1,\ \forall Y \in I_2 & (c) \end{cases}$$

Our study of those equations is not an exhaustive one.

• <u>1^{st} *case :*</u> Suppose $\widetilde{Q_{12}} = \widetilde{Q_{21}} = 0$
We then have to solve the following problem:

$$(8)\qquad \begin{cases} \beta(\widetilde{Q}X, Y) & = -\beta(X, \widetilde{Q}Y) \\ [\widetilde{Q}X, \widetilde{Q}Y] & -\widetilde{Q}[\widetilde{Q}X, Y] - \widetilde{Q}[X, \widetilde{Q}Y] = \lambda[X, Y]\ \forall X, Y \in \mathfrak{g} \end{cases}$$

Where $\mathfrak{g}$ is a complex semisimple Lie algebra and $\lambda \in \mathbb{C}$.

Proposition 1. *In the case where $\widetilde{Q_{12}} = \widetilde{Q_{21}} = 0$ there exist solutions $\widetilde{Q}$ of (6) which are given by 3.A.b.theorem 1 with $\lambda \in \mathbb{C}$*

- <u>2^{nd} *case :*</u> Suppose $\widetilde{Q_1} = \widetilde{Q_2} = 0$.

Proposition 2. *There is no solution for (6) when $\widetilde{Q_1} = \widetilde{Q_2} = 0$.*

Indeed for $X, Y \in I_1$ we obtain from (a): $(u + iv)[X, Y] = 0$. This is impossible as $u + iv \neq 0$

- <u>3^{rd} *case :*</u>

Remark 2. For any $\widetilde{Q_1}$ solution of (a), $\widetilde{Q_{21}} \equiv 0$ is a trivial solution of (b) and (c).

From 3.A.b.theorem 1, we know that for a Cartan subalgebra $\mathfrak{h}_1$ of I_1, a system of positive roots Δ^+ of $(I_1, \mathfrak{h}_1)$ and if we note $\mathfrak{n}_{1\pm} = \sum_{\alpha\in\Delta^+}(I_1)^{\pm\alpha}$, a solution of (a) is given by:

$$\widetilde{Q_1}(X) = \begin{cases} aX \text{ if } X \in \mathfrak{n}_{1+} \\ 0 \text{ if } X \in \mathfrak{h}_1 \\ -aX \text{ if } X \in \mathfrak{n}_{1-} \text{ where } a \text{ satisfies } a^2 = u + iv \end{cases} \tag{9}$$

Proposition 3. *For $\widetilde{Q_1}$ given by (9), a solution $\widetilde{Q_{21}}$ of (b) and (c) is given by:*

$$\widetilde{Q_{21}}(X) = \begin{cases} a\overline{X} \textit{ if } X \in \mathfrak{h}_1 \\ 0 \textit{ else} \end{cases} \tag{10}$$

proof: we only have to check (b) and (c) for this $\widetilde{Q_{21}}$; writing for $X \in I_1$, $X = X_+ + X_0 + X_-$ where $X_+ \in \mathfrak{n}_{1+}$, $X_0 \in \mathfrak{h}_1$, $X_- \in \mathfrak{n}_{1-}$, this is an immediate result. □

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