Canonical embedding of function spaces into the topological bidual of $\mathrm{C}(K;E)$

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Abstract

Let K be a Hausdorff compact space and E be a real Banach lattice with order continuous norm. In this paper, we essentially prove the existence of canonical embeddings of (vector) sublattices of $\mathrm{FB}(K;E)$, the Banach lattice of $E-$valued bounded functions on K, into the topological bidual of $\mathrm{C}(K;E)$, the usual Banach lattice of $E-$valued continuous functions on K. This is related and extends some results in the real case of H. H. Schaefer ([14], [15]).

1 Introduction

We refer to [4] for general topological spaces, to [1], [9], [10], [13] and [19] for ordered spaces theory and to [16] and [17] for spaces of continuous functions.

Let us fix a few notations and properties.

Following the classical lattice notation, if there exists, the supremum of a majorized subset D of a vector lattice (or a Riesz space) is denoted by $\vee D$ or $\sup D$.

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If $D = \{e, f\}$, we will denote it $e \vee f$ or $\sup\{e, f\}$. We use similar notations for a minorized subset of a vector lattice.

The zero element of a vector space will be denoted by θ. For an element e of a vector lattice, the *positive part* of e is defined by $e_+ = e \vee \theta$, its *negative part* by $e_- = (-e) \vee \theta$, and its *absolute value* by $|e| = e \vee (-e)$. The *positive cone* of a vector lattice $(E, \leq)$ is the set $E_+ = \{\, e \in E : \theta \leq e\}$.

Unless specifically stated, throughout this paper, K denotes a Hausdorff compact space and E a real Banach lattice with *order norm continuous* (or equivalently, with *Lebesgue property*). This Lebesgue property in E means that *every monotone increasing net to θ norm converges to θ.* Note that there are many examples and characterizations of such Banach lattices E. (cf. [3], [8] and [13] for these results.) This Lebesgue property is an essential key in our work.

The space of $E-$ valued bounded functions on K is denoted by $\mathrm{FB}(K; E)$. It is clear that, endowed with the canonical order and supremum norm denoted by $\|\cdot\|_K$, this space is a Dedekind complete (or order complete) Banach lattice.

We denote by $\mathrm{C}(K; E)$, $\mathrm{C}(K; E)'$ and $\mathrm{C}(K; E)''$ respectively, the usual Banach space of $E-$ valued continuous functions on K, its dual Banach space and its topological bidual. These three spaces are Banach lattices under their canonical orders. Of course, the Banach lattices $\mathrm{C}(K; E)'$ and $\mathrm{C}(K; E)''$ are Dedekind complete.

A *lower semi-continuous* (in short l.s.c.) function is a function F defined on K with values in E such that the following two properties are satisfied:

$$\text{(L)} \quad \exists h \in \mathrm{C}(K; E) \ : \ \theta \leq F \leq h.$$

$$\text{(SC)} \quad F(x) = \sup\{\, f(x) : f \in \mathrm{C}(K; E)_+,\ f \leq F\}, \quad \forall x \in K.$$

We denote by $\mathrm{LSC}(K; E)$ the set of all l.s.c. functions. Note that, for every l.s.c. function F, there is a net in $\mathrm{C}(K; E)$ which increases to F.

It is easy to see that the set $\mathrm{LSC}(K; E)$ is a Dedekind complete convex cone and a sublattice of $\mathrm{FB}(K; E)$.

In fact, the set $\mathrm{LSC}(K; E)$ is the second key of our work.

Let us set $\mathrm{LS}(K; E) = \mathrm{LSC}(K; E) - \mathrm{LSC}(K; E)$. Of course, the set $\mathrm{LS}(K; E)$ is a normed vector sublattice of $\mathrm{FB}(K; E)$ containing the Banach lattice $\mathrm{C}(K; E)$.

As it is well known, the evaluation map

$$\Psi : \mathrm{C}(K; E) \to \mathrm{C}(K; E)'' \quad f \mapsto \int f \, d\cdot$$

is an isometric vector lattice isomorphism (for the norm topologies).

In the next sections, we will introduce extensions of this mapping Ψ. In fact, the obtaining of these extensions constitutes an answer to a question asked by H.H. Schaefer in [14] and [15].

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2 Integral functional on $\mathrm{LS}(K;E)\times\mathrm{C}(K;E)'$

The following theorem is a direct consequence of the Lebesgue property of the space E.

Theorem 2.1 (Dini) *Every monotone increasing net to θ in $\mathrm{C}(K;E)$ uniformly converges to θ in $\mathrm{C}(K;E)$.*

For every l.s.c. function F, let us set $s_F=\{\,f\in\mathrm{C}(K;E)_+ : f\le F\}$. Moreover, if we consider $m\in\mathrm{C}(K;E)'_+$, it is clear that the set defined by $\{\,\int_K f\,dm : f\in s_F\}$ is a majorized subset in $\mathbb{R}$. We denote by $\int_K F\,dm$ or $\int F\,dm$ its supremum. Hence, we obtain the following proposition:

Proposition 2.2 *For every $m\in\mathrm{C}(K;E)'_+$, the mapping*

$$\int\cdot\,dm\ :\ \mathrm{LSC}(K;E)\to\mathbb{R}_+\qquad F\mapsto\int F\,dm$$

is positive homogeneous and additive; moreover, it is increasing and one has the inequalities

$$0\le\int F\,dm\le\|F\|_K\,\|m\|\,,\quad\forall F\in\mathrm{LSC}(K;E).$$

Proof. We only prove the additivity, the rest is clear. Let F, G be l.s.c. functions. On the one hand, the inequality

$$\int F\,dm+\int G\,dm\le\int(F+G)\,dm$$

is clear. On the other hand, there are nets (f_α) and (g_β) in $\mathrm{C}(K;E)_+$ such that $f_\alpha\uparrow F$ and $g_\beta\uparrow G$. If h is fixed in s_{F+G}, it is clear that $(f_\alpha+g_\beta)\wedge h\uparrow h$ in $\mathrm{C}(K;E)_+$. Hence, by Dini's theorem 2.1, this latter convergence is uniform on K. Finally, we have

$$\begin{aligned}\int h\,dm &\le \sup\left\{\int(f+g)\,dm : f\in s_F, g\in s_G\right\}\\ &\le \sup\left\{\int f\,dm+\int g\,dm : f\in s_F, g\in s_G\right\}\\ &\le \int F\,dm+\int G\,dm;\end{aligned}$$

so, we obtain

$$\int(F+G)\,dm\le\int F\,dm+\int G\,dm.$$

∎

The proof of the following proposition is straightforward.

Proposition 2.3 *For every $F \in \mathrm{LSC}(K;E)$, the mapping*

$$\int F\,d\cdot \;:\; \mathrm{C}(K;E)'_+ \to \mathbb{R} \qquad m \mapsto \int F\,dm$$

is positive homogeneous, additive and increasing.

Remarks. a) For every l.s.c. function F, one has

$$\|F\|_K = \sup\Big\{ \int F\,dm : m \in \mathrm{C}(K;E)'_+, \|m\| \le 1 \Big\}.$$

b) For every $m \in \mathrm{C}(K;E)'_+$, one has

$$\|m\| = \sup\Big\{ \int F\,dm : F \in \mathrm{LSC}(K;E), \|F\|_K \le 1 \Big\}.$$

Definition. Let $F \in \mathrm{LS}(K;E)$ and $m \in \mathrm{C}(K;E)'$. If F_1 and F_2 are l.s.c. functions such that $F = F_1 - F_2$, it is clear that the real number

$$\int F_1\,dm_+ - \int F_2\,dm_+ - \int F_1\,dm_- + \int F_2\,dm_-$$

is independent of the choice of the functions F_1 and F_2. We again denote by $\int F\,dm$ this real number.

The following lemma is a direct consequence of the Proposition 2.3.

Lemma 2.4 *For every $F \in \mathrm{LS}(K;E)$, the mapping*

$$\int F\,d\cdot \;:\; \mathrm{C}(K;E)'_+ \to \mathbb{R} \qquad m \mapsto \int F\,dm$$

is positive homogeneous and additive.

Remark. If we endow the space $\mathrm{LS}(K;E)$ with the supremum norm, it does not seem to exist a canonical linear continuous injection from this space into the bidual $\mathrm{C}(K;E)''$. That is the reason why we introduce an auxiliary norm $\|\cdot\|_o$ on $\mathrm{LS}(K;E)$.

Definition. Let $F \in \mathrm{LS}(K;E)$. We know that there are l.s.c. functions G and H such that $F = G - H$. Hence, the set

$$\{\, \|F_1\|_K + \|F_2\|_K : F = F_1 - F_2;\; F_1,\; F_2 \in \mathrm{LSC}(K;E)\}$$

is minorized in $\mathbb{R}$ and we denote by $\|F\|_o$ its infimum.

The following proposition may be easily established.

Proposition 2.5 *The mapping*

$$\|\cdot\|_o \;:\; \mathrm{LS}(K;E) \to \mathbb{R} \qquad F \mapsto \|F\|_o$$

is a norm on $\mathrm{LS}(K;E)$ *such that*

$$\|\cdot\|_K \le \|\cdot\|_o \;\; \textit{on} \;\; \mathrm{LS}(K;E) \qquad \textit{and} \qquad \|\cdot\|_K = \|\cdot\|_o \;\; \textit{on} \;\; \mathrm{LSC}(K;E).$$

Furthermore, the norms $\|\cdot\|_K$ *and* $\|\cdot\|_o$ *are equivalent on* $\mathrm{C}(K;E)$*; more precisely, one has*

$$\|\cdot\|_K \le \|\cdot\|_o \le 2\,\|\cdot\|_K \qquad \textit{on} \qquad \mathrm{C}(K;E).$$

In the following result, we suppose that the space $\mathrm{LS}(K;E)$ is endowed with the norm $\|\cdot\|_o$.

Theorem 2.6 *The mapping*

$$\int \cdot\, d\cdot \;:\; \mathrm{LS}(K;E) \times \mathrm{C}(K;E)' \to \mathbb{R} \qquad (F,m) \mapsto \int F\,dm$$

is a bilinear functional and one has

$$\left|\int F\,dm\right| \le \|F\|_o \,\|m\|\,, \quad \forall F \in \mathrm{LS}(K;E),\ \forall m \in \mathrm{C}(K;E)'.$$

Proof. The linearity with respect to the first variable (resp. second variable) is a direct consequence of the Proposition 2.2 (resp. Lemma 2.4 and Proposition 3.6.1 of [19]).

Now we finally prove the inequality. Let $F \in \mathrm{LS}(K;E)$ be such that $F = F_1 - F_2$, where F_1 and F_2 are l.s.c. functions. Then, by Propositions 2.2 and 2.3, we obtain

$$\begin{aligned}
\left|\int F\,dm\right| &\le \int F_1\,dm_+ + \int F_2\,dm_+ + \int F_1\,dm_- + \int F_2\,dm_- \\
&\le \int (F_1+F_2)\,dm_+ + \int (F_1+F_2)\,dm_- \\
&\le \int (F_1+F_2)\,d(m_+ + m_-) \\
&\le \|F_1+F_2\|_K\,\||m|\| \\
&\le (\|F_1\|_K + \|F_2\|_K)\,\|m\|\,.
\end{aligned}$$

So, the real number $|\int F\,dm|$ is a minorant of the set

$$\{\,(\|F_1\|_K + \|F_2\|_K)\,\|m\| \,:\, F = F_1 - F_2;\ F_1, F_2 \in \mathrm{LSC}(K;E)\}\,;$$

hence the conclusion. ∎

3 Fundamental result

Let $F \in \mathrm{LSC}(K;E)$. Of course, the set $\{\int f\, d\cdot : f \in s_F\}$ is a majorized subset of the bidual $\mathrm{C}(K;E)''$. Moreover, we have

$$\int F\, d\cdot = \sup\left\{\int f\, d\cdot : f \in s_F\right\} \in \mathrm{C}(K;E)''_+$$

by virtue of the Theorem 2.6.

Proposition 3.1 *The mapping*

$$I \;:\; \mathrm{LSC}(K;E) \to \mathrm{C}(K;E)''_+ \qquad F \mapsto \int F\, d\cdot$$

is positive homogeneous, additive, increasing and injective; moreover, it preserves finite suprema and infima, and keeps the norm.

Proof. The fact that the map I is positive homogeneous, additive and increasing is a direct consequence of the Proposition 2.2.

We now show that I is injective. Let $F,\ G \in \mathrm{LSC}(K;E)$ be such that $F \neq G$. Then, there is $x \in K$ such that $F(x) \neq G(x)$. Of course, there are $e_1,\ e_2 \in E_+$ satisfying $e_1 \wedge e_2 = 0$ and $F(x) - G(x) = e_1 - e_2$; so, we have $e_1 \neq e_2$. Hence, we distinguish two cases.

First case: $e_1 = \theta$ or $e_2 = \theta$. Of course, there is $e' \in E'_+$ such that $\langle e_2, e'\rangle > 0$ or $\langle e_1, e'\rangle > 0$ according to the case and so we have $\langle F(x), e'\rangle \neq \langle G(x), e'\rangle$. That is, we get $\delta_x \otimes e' \in \mathrm{C}(K;E)'_+$ and then we have

$$\langle \delta_x \otimes e', I(F)\rangle = \langle F(x), e'\rangle \neq \langle G(x), e'\rangle = \langle \delta_x \otimes e', I(G)\rangle\,;$$

what suffices.

Second case: $e_1 \neq \theta$ and $e_2 \neq \theta$. Of course, there is $e' \in E'_+$ such that $\langle e_1, e'\rangle > 0$ and $\langle e_2, e'\rangle = 0$. Hence, we have

$$\langle F(x) - G(x), e'\rangle = \langle e_1, e'\rangle \neq 0. \qquad (*)$$

That is, we get $\delta_x \otimes e' \in \mathrm{C}(K;E)'_+$ and by virtue of $(*)$, it is clear that $I(F) \neq I(G)$.

Let $F,\ G \in \mathrm{LSC}(K;E)$. Let us prove that I preserves finite suprema. Since I is increasing, it is clear that $I(F) \vee I(G) \leq I(F \vee G)$. Furthermore, there are nets (f_α) and (g_β) in $\mathrm{C}(K;E)_+$ such that $f_\alpha \uparrow F$ and $g_\beta \uparrow G$. Then for fixed h in $s_{F\vee G}$, we have $(f_\alpha \vee g_\beta) \wedge h \uparrow h$ in $\mathrm{C}(K;E)_+$. So again by Dini's theorem 2.1, this latter convergence is uniform on K. Hence, we have

$$\begin{aligned}
\int h\, d\cdot &\leq \sup\left\{\int (f \vee g)\, d\cdot : f \in s_F,\ g \in s_G\right\} \\
&\leq \sup\left\{\left(\int f\, d\cdot\right) \vee \left(\int g\, d\cdot\right) : f \in s_F,\ g \in s_G\right\} \\
&\leq I(F) \vee I(G)
\end{aligned}$$

and so, we obtain

$$I(F \vee G) \leq I(F) \vee I(G).$$

Let us prove that I preserves finite infima. Of course, we have $I(F \wedge G) \leq I(F) \wedge I(G)$. Since Ψ is a lattice homomorphism and so preserves finite infima, we successively have

$$\begin{aligned} I(F \wedge G) &\geq \sup\Big\{ \int (f \wedge g)\, d\cdot : f \in s_F,\ g \in s_G \Big\} \\ &\geq \sup\Big\{ \Big(\int f\, d\cdot\Big) \wedge \Big(\int g\, d\cdot\Big) : f \in s_F,\ g \in s_G \Big\} \\ &\geq I(F) \wedge I(G). \end{aligned}$$

Finally, we have on the one hand

$$\begin{aligned} \|I(F)\| &= \sup\Big\{ \Big|\int F\, dm\Big| : m \in \mathrm{C}(K;E)',\ \|m\| \leq 1 \Big\} \\ &\leq \sup\{ \|F\|_K \|m\| : m \in \mathrm{C}(K;E)',\ \|m\| \leq 1 \} = \|F\|_K\,; \end{aligned}$$

and on the other hand

$$\|F\|_K = \sup\Big\{ \int F\, dm : m \in \mathrm{C}(K;E)'_+,\ \|m\| \leq 1 \Big\} \leq \|I(F)\|\,.$$

Hence the conclusion. ∎

Theorem 3.2 *The mapping*

$$\tilde{I} \,:\, (\mathrm{LS}(K;E), \|\cdot\|_o) \to \mathrm{C}(K;E)'' \qquad F \mapsto \int F\, d\cdot$$

is a linear continuous injection and a vector lattice homomorphism which extends Ψ*.*

Proof. By virtue of the Theorem 2.6, it is clear that $\tilde{I}$ is a linear continuous operator which extends I, and so Ψ.

Since I is injective, it is immediate that $\tilde{I}$ is too.

To conclude we prove that $\tilde{I}$ is a vector lattice homomorphism. Let $F \in \mathrm{LS}(K;E)$. Of course, there are $F_1,\ F_2 \in \mathrm{LSC}(K;E)$ such that $F = F_1 - F_2$ so that $|F| = (F_1 \vee F_2) - (F_1 \wedge F_2)$, where $F_1 \vee F_2,\ F_1 \wedge F_2 \in \mathrm{LSC}(K;E)$. Because of this latter identity and the lattice preserving properties of I, we obtain

$$\begin{aligned} \tilde{I}(|F|) &= \tilde{I}(F_1 \vee F_2) - \tilde{I}(F_1 \wedge F_2) = I(F_1 \vee F_2) - I(F_1 \wedge F_2) \\ &= I(F_1) \vee I(F_2) - I(F_1) \wedge I(F_2) = |I(F_1) - I(F_2)| = \Big|\tilde{I}(F)\Big|. \end{aligned}$$

∎

Proposition 3.3 *For every monotone increasing net* (F_α) *to* F *in the space* $\mathrm{LSC}(K;E)$*, one has* $I(F) = \sup_\alpha I(F_\alpha)$*.*

Proof. It is clear that $\sup_\alpha I(F_\alpha) \leq I(F)$. Let us set $\Phi = \cup_\alpha s_{F_\alpha}$. Then the set Φ is a monotone increasing and majorized net denoted (g_β) in $\mathrm{C}(K;E)_+$. Of course, we have $F = \sup_\beta g_\beta$ and if $f \in s_F$, we also have $f = \sup_\beta (g_\beta \wedge f)$. By virtue of Dini's theorem 2.1, the net $(g_\beta \wedge f)$ uniformly converges to f in $\mathrm{C}(K;E)_+$ and so, by the Proposition 3.1, the net $I(g_\beta \wedge f)$ converges to $I(f)$ in $\mathrm{C}(K;E)''$. Accordingly, we obtain

$$I(f) = \sup_\beta I(g_\beta \wedge f) = \sup_\beta (I(g_\beta) \wedge I(f)).$$

Moreover for each β, there is $\alpha = \alpha(\beta)$ such that $g_\beta \in s_{F_\alpha}$ and so, we have

$$I(g_\beta) \wedge I(f) \leq I(g_\beta) \leq I(F_\alpha) \leq \sup_\alpha I(F_\alpha);$$

so we get $I(f) \leq \sup_\alpha I(F_\alpha)$ for every $f \in s_F$.

Hence the conclusion. ∎

Remark. On the space $\mathrm{C}(K;E)''$, we also consider the topology τ of the uniform convergence on the all order bounded subsets of $\mathrm{C}(K;E)'$. It is well known that this topology is generated by the system of semi-norms $\{ p_m : m \in \mathrm{C}(K;E)'_+ \}$ defined by

$$p_m \ : \ \mathrm{C}(K;E)'' \to \mathbb{R} \qquad \varphi \mapsto \langle m, |\varphi| \rangle = \sup \{ |\langle \varphi, n \rangle| : |n| \leq m \}.$$

Of course, every p_m is a continuous lattice semi-norm (for the canonical norm).

The following proposition gives some desirable properties of this topology τ.

Proposition 3.4 a) *On the space* $\mathrm{C}(K;E)''$, *the weak*-topology, the* τ*-topology and the norm topology are finer and finer.*

b) *In the space* $\mathrm{C}(K;E)''$, *the* τ*-bounded subsets and the norm bounded subsets are identical.*

c) *The space* $\mathrm{C}(K;E)''$ *is* τ*-quasi complete.*

d) *For every monotone increasing* (resp. *decreasing) and majorized* (resp. *minorized) net* (φ_α) *of* $\mathrm{C}(K;E)''$, *one has*

$$\sup_\alpha \varphi_\alpha = \lim_\tau \varphi_\alpha \qquad (\text{resp. } \inf_\alpha \varphi_\alpha = \lim_\tau \varphi_\alpha\ ) \ .$$

In particular, one has

d1) *Every monotone net converging to* θ *in* $\mathrm{C}(K;E)''$ *is* τ*-converging to* θ.

d2) *Every filter on* $\mathrm{C}(K;E)''$ *which order converges to* $\varphi \in \mathrm{C}(K;E)''$ *is* τ*-converging to* φ.

d3) *Every sequence which order converges to* θ *in* $\mathrm{C}(K;E)''$ *is* τ*- converging to* θ.

Proof. The proofs of a) and b) are clear.

The proof of c) is due to the following three properties:

c1) the topology τ is finer than the weak*-topology.

c2) the space $\mathrm{C}(K;E)''$ is quasi-complete for the weak*-topology.
c3) every closed semi-ball in $\mathrm{C}(K;E)''$ is closed for the weak*-topology.

The proof of d) is a direct consequence of the Proposition IV.1.15 of [10]. Furthermore, it is a direct matter to establish the particular cases. ∎

Corollary 3.5 *For every monotone increasing net* (F_α) *to* F *in the space* $\mathrm{LSC}(K;E)$*, one has*

$$I(F) = \sup_\alpha I(F_\alpha) = \lim_\tau I(F_\alpha).$$

4 Embedding theorem

The Theorem 3.2 gives a first example of a canonical embedding of function spaces into the bidual of $\mathrm{C}(K;E)$. Moreover, it constitutes the source of our motivation to search for other sublattices of $\mathrm{FB}(K;E)$ which may be embedded into this bidual. In this section, we construct a theoretical example of such canonical embeddings (cf. Theorm 4.9)

Notation. We consider the set

$$\mathrm{B}(K;E) := \{\, F \in \mathrm{FB}(K;E)_+ : \exists H \in \mathrm{LSC}(K;E),\ F \le H \}\,.$$

Of course, the set $\mathrm{B}(K;E)$ contains the positive cone of $\mathrm{LS}(K;E)$. Moreover, it is a Dedekind complete convex cone and a sublattice of $\mathrm{FB}(K;E)$. (By definition of $\mathrm{LSC}(K;E)$, this set $\mathrm{B}(K;E)$ is equal to the set

$$\{\, F \in \mathrm{FB}(K;E)_+ : \exists f \in \mathrm{C}(K;E)_+,\ F \le f \}\,;$$

but for technical reasons, we will often prefer the first definition.)

For every $F \in \mathrm{B}(K;E)$, let us set $j_F := \{\, H \in \mathrm{LSC}(K;E) : F \le H\}$; clearly, this latter set is non void. Hence for every $F \in \mathrm{B}(K;E)$, the set$\{\, I(H) : H \in j_F\}$ is non void and minorized in $\mathrm{C}(K;E)''_+$; we denote by $J(F)$ its infimum.

Remark. Of course, for every $F \in \mathrm{B}(K;E)$, there exists a monotone decreasing net (F_α) in $\mathrm{LSC}(K;E)$ satisfying $\theta \le F \le F_\alpha$ for every α and $J(F) = \inf_\alpha I(F_\alpha)$. Hence $I(F_\alpha) \downarrow J(F)$ in $\mathrm{C}(K;E)''$ and so, the net $I(F_\alpha)$ converges (for τ) to $J(F)$ by Proposition 3.4 d).

Proposition 4.1 *The mapping*

$$J \ :\ \mathrm{B}(K;E) \to \mathrm{C}(K;E)''{}_+ \qquad F \mapsto J(F)$$

is positive homogeneous, subadditive, increasing, preserves finite suprema and extends I*. Furthermore, one has*

a) *if* $F \in \mathrm{B}(K;E)$ *with* $J(F) = \theta$*, then* $F = \theta$*.*

b) $J(F+G) = J(F \vee G) + J(F \wedge G) = J(F) + J(G)$ *for all* $F \in \mathrm{LSC}(K;E)$ *and all* $G \in \mathrm{B}(K;E)$*.*

c) $J(G-F) = J(G) - J(F)$ *for all* $F,\ G \in \mathrm{LSC}(K;E)$ *such that* $F \le G$*.*

Proof. It is clear that the map J is positive homogeneous, increasing and extends I.

Let $F,\ G \in \mathrm{B}(K;E)$. Then for all $H \in j_F,\ L \in j_G$ we have

$$J(F+G) \leq I(H+L) = I(H) + I(L).$$

Thus we get

$$J(F+G) \leq \inf\{\, I(H) + I(L) : H \in j_F,\ L \in j_G\} \leq J(F) + J(G).$$

Let us prove that the map J preserves finite suprema. Of course, we have $J(F) \vee J(G) \leq J(F \vee G)$. Furthermore, for all $H \in j_F,\ L \in j_G$ we have

$$J(F \vee G) \leq I(H \vee L) = I(H) \vee I(L)$$

and so we get

$$J(F \vee G) \leq \inf\{\, I(H) \vee I(L) : H \in j_F,\ L \in j_G\} \leq J(F) \vee J(G).$$

We prove the property a). Let (F_α) be a monotone decreasing net in $\mathrm{LSC}(K;E)$ satisfying $\theta \leq F \leq F_\alpha$ for all α and $\inf_\alpha I(F_\alpha) = \theta$. Let $x \in K$. There is $e'_x \in E'_+$ such that $\|e'_x\| = 1$ and $\langle F(x), e'_x\rangle = \|F(x)\|$. Hence $\delta_x \otimes e'_x \in \mathrm{C}(K;E)'_+$ and so, the net $\int F_\alpha \, d(\delta_x \otimes e'_x)$ norm converges to θ. Since one has $0 \leq \langle F(x), e'_x\rangle \leq \langle F_\alpha(x), e'_x\rangle$ for all α, it is immediate that $\|F(x)\| = 0$; hence the conclusion.

To prove the property b), observe that $F + G = (F \vee G) + (F \wedge G)$. Then we clearly get

$$\begin{aligned} J(F+G) &\leq J(F \vee G) + J(F \wedge G) \\ &\leq J(F) \vee J(G) + J(F) \wedge J(G) = J(F) + J(G). \end{aligned}$$

That is, it suffices to establish that

$$J(F) + J(G) \leq J(F+G)$$

for all $F \in \mathrm{LSC}(K;E)$ and $G \in \mathrm{B}(K;E)$. In fact for all $H \in j_{F+G}$, we have $H = F + (H - F)$ and then we successively get

$$J(F) + J(G) \leq J(F) + J(H-F) = I(H);$$

what suffices.

Finally, we prove the property c). Of course, there is a net (f_α) in $\mathrm{C}(K.E)_+$ such that $f_\alpha \uparrow F$. Then we have $G - F = \inf_\alpha (G - f_\alpha)$. Now for all α we also have $G - f_\alpha \in \mathrm{LSC}(K;E)$ and so, we successively get $I(G) = I(f_\alpha) + I(G - f_\alpha)$ so that

$$\inf_\alpha I(G - f_\alpha) = I(G) - \sup_\alpha I(f_\alpha) = I(G) - I(F).$$

By increase of J, we get

$$J(G-F) \leq \inf_\alpha I(G - f_\alpha) = J(G) - J(F).$$

The other inequality is immediate by subadditivity of J. ∎

Proposition 4.2 *For every sequence* (F_r) *increasing to* F *in* $\mathrm{B}(K;E)$*, one has*

$$J(F) = \sup_r J(F_r) = \lim_\tau J(F_r).$$

Proof. Let V denote any closed, absolutely convex, solid τ-neighborhood of θ in $\mathrm{C}(K;E)''$. For each $r \in \mathbb{N}$ there exists $H_r \in \mathrm{LSC}(K;E)$ satisfying $F_r \leq H_r$ and $I(H_r) \in J(F_r) + 2^{-(r+1)}V$. Moreover, there is $H \in \mathrm{LSC}(K;E)$ such that $F \leq H$. Let us set $L_r = (H_1 \vee \ldots \vee H_r) \wedge H$ for all $r \in \mathbb{N}$. Of course, the sequence (L_r) is increasing and majorized in $\mathrm{LSC}(K;E)$ and so admits a supremum denoted by L. Hence, by virtue of Corollary 3.5, the sequence $I(L_r)$ τ-converges to $I(L)$. Then there is $s \in \mathbb{N}$ such that $I(L_r) \leq I(L) \in \frac{1}{2}V$ for all $r \in \mathbb{N}$ with $r \geq s$.

Now for all $r \in \mathbb{N}$, we successively have

$$\begin{aligned} \theta &\leq I(L_r) - J(F_r) \leq \bigvee_{k=1}^{r} I(H_k \wedge H) - \bigvee_{k=1}^{r} J(F_k) \\ &\leq \sum_{k=1}^{r} [I(H_k) \wedge I(H) - J(F_k)] \leq \sum_{k=1}^{r} [I(H_k) - J(F_k)] \in \frac{1}{2}V. \end{aligned}$$

It thus follows that

$$\theta \leq I(L) - J(F_r) \quad (r \geq s)$$

and since V is solid and $I(L) \geq J(F) \geq J(F_r)$ for all r, we get $I(L) - J(F) \in V$. But since V is τ-closed, we also have

$$\theta \leq I(L) - \sup_r J(F_r) \in V$$

and this shows that $J(F) = \sup_r J(F_r)$.

Finally, the relation $J(F) = \lim_\tau J(F_r)$ holds by Proposition 3.4 d). ∎

Remark. It is well known that, in general, it is not true that

$$J(F+G) = J(F) + J(G) \tag{$*$}$$

for all $F,\ G \in \mathrm{B}(K;E)$. (cf. [[2]; Exercise 8d, p.239] or [[7]; note 2, p.122].)

However, there exists subsets of $\mathrm{B}(K;E)$ on which the equality $(*)$ holds. Note that the positive cone of $\mathrm{LS}(K;E)$ is a such subset. Our next purpose is to search for other subsets of $\mathrm{B}(K;E)$ on which the map J is additive.

Definition. We denote by $\widehat{\mathrm{B}}(K;E)$ the set $\mathrm{B}(K;E) - \mathrm{B}(K;E)$. It is clear that this set is a normed vector sublattice of $\mathrm{FB}(K;E)$. Furthermore, the set $\mathrm{B}(K;E)$ is the positive cone of $\widehat{\mathrm{B}}(K;E)$.

Convention. Throughout the sequel of this paper, unless specifically stated, M will always denote *a vector sublattice of* $\widehat{\mathrm{B}}(K;E)$ *satisfying the following two properties:*

(1) $\mathrm{C}(K;E)_+ \subset M_+ \subset \mathrm{B}(K;E)$.

(2) $J(F+G) = J(F) + J(G), \quad \forall F,\ G \in M_+$.

Definition. *We define* $\overline{\mathrm{M}}_+$ *to be the set of all functions* $F \in \mathrm{B}(K;E)$ *for which there exists a sequence* (F_r) *in* M_+ *increasing to* F. *Furthermore, we define* $\overline{\overline{\mathrm{M}}}_+$ *to be the set of all functions* $F \in \mathrm{B}(K;E)$ *for which there exists a sequence* (F_r) *in* $\overline{\mathrm{M}}_+$ *decreasing to* F.

It is obvious that the sets $\overline{\mathrm{M}}_+$ and $\overline{\overline{\mathrm{M}}}_+$ are convex cone and sublattices of $\widehat{\mathrm{B}}(K;E)$. Moreover, the set $\widetilde{\mathrm{M}} := \overline{\overline{\mathrm{M}}}_+ - \overline{\overline{\mathrm{M}}}_+$ is a vector sublattice of $\widehat{\mathrm{B}}(K;E)$. But in general, the sets M_+, $\overline{\mathrm{M}}_+$, $\overline{\overline{\mathrm{M}}}_+$ and $\widetilde{\mathrm{M}}_+$ become strictly bigger and bigger.

The proof of the following lemma is easily established.

Lemma 4.3 *The map* J *satisfies the following properties:*

a) $J(G-F) = J(G) - J(F)$ *for all* $F,\ G \in \mathrm{M}_+$ *such that* $F \le G$.

b) $J(F \wedge G) = J(F) \wedge J(G)$ *for all* $F,\ G \in \mathrm{M}_+$.

c) $J(F) = \sup_r J(F_r) = \lim_\tau J(F_r)$ *for all* $F \in \overline{\mathrm{M}}_+$ *and all sequence* (F_r) *in* M_+ *such that* $F_r \uparrow F$.

d) $J(F+G) = J(F) + J(G)$ *and* $J(F \wedge G) = J(F) \wedge J(G)$ *for all* $F,\ G \in \overline{\mathrm{M}}_+$.

Lemma 4.4 *The map* J *satisfies the following properties:*

a) *For all* $F \in \overline{\overline{\mathrm{M}}}_+$ *and all sequence* (F_r) *in* $\overline{\mathrm{M}}_+$ *such that* $F_r \downarrow F$, *one has*

$$J(F) = \inf_r J(F_r) = \lim_\tau J(F_r).$$

b) *For all* $F,\ G \in \overline{\overline{\mathrm{M}}}_+$, *one has*

$$J(F+G) = J(F) + J(G) \quad \text{and} \quad J(F \wedge G) = J(F) \wedge J(G).$$

Proof. a) Of course, there is $g \in \mathrm{C}(K;E)_+$ such that $F_r \le g$ for all r. Let us prove that for each r

$$J(g) = J(F_r) + J(g - F_r). \qquad (i)$$

Indeed, there exists a sequence $(G_{r,k})_{k\in\mathbb{N}}$ in M_+ such that $G_{r,k} \uparrow F_r$. To simplify the notations, we set $G_{r,k} = G_k$ for each k. By hypothesis, $J(g) = J(G_k) + J(g - G_k)$ for each k. Now we have $J(F_r) = \lim_\tau J(G_k)$ by Proposition 4.2. Then the sequence $J(g - G_k)$ τ-converges to its limit φ so that $J(g) = J(F_r) + \varphi$. By subadditivity of J, we also have $J(g) \le J(F_r) + J(g - F_r)$ hence $\varphi \le J(g - F_r)$. Furthermore, we have $g - F_r \le g - G_k$ and since the map J is isotone, we get $J(g - F_r) \le J(g - G_k)$ for each k. Hence $J(g - F_r) \le \varphi = J(g) - J(F_r)$. This prove (i).

Now since $g - F_r \uparrow g - F$ in $\mathrm{B}(K;E)$, it follows that $J(g-F) = \sup_r J(g - F_r)$ by Proposition 4.2.

By subadditivity of J, we have $J(g) \le J(F) + J(g-F)$ and so, $\inf_r J(F_r) \le J(F)$. But here the equality must hold, since $F \le F_r$ for all r and since J is isotone.

Again, the relation $J(F) = \lim_\tau J(F_r)$ is true by Proposition 3.4 d).

Finally, the proofs of b) and c) are easy to establish. ∎

Lemma 4.5 *The map J satisfies the following properties:*

a) $J(G-F)=J(G)-J(F)$ *for all* $F,\ G\in\overline{\mathrm{M}}_+$ *such that* $F\leq G$.

b) $J(G-F)=J(G)-J(F)$ *for all* $F,\ G\in\overline{\overline{\mathrm{M}}}_+$ *such that* $F\leq G$.

c) $J(F+G)=J(F)+J(G)$ *for all* $F,\ G\in\widetilde{\mathrm{M}}_+$.
In particular, one has

$$J(G-F)=J(G)-J(F)$$

for all $F,\ G\in\widetilde{\mathrm{M}}_+$ *such that* $F\leq G$ *and for all* $F,\ G\in\widetilde{\mathrm{M}}_+$,

$$J(F\wedge G)=J(F)\wedge J(G).$$

d) *Forall* $F,\ G\in\widetilde{\mathrm{M}}_+$ *such that* $J(F)=J(G)$, *one has* $F=G$.

e) *For all* $F\in\widetilde{\mathrm{M}}_+$ *and all sequence* (F_r) *in* $\widetilde{\mathrm{M}}_+$ *such that* $F_r\downarrow F$, *one has*

$$J(F)=\inf_r J(F_r)=\lim_\tau J(F_r).$$

Proof. a) Note that there are sequences (F_r) and (G_r) in M_+ such that $F_r\uparrow F$ and $G_r\uparrow G$. By considering the sequences $F_r\vee G_r,\ F_r\wedge G_r$ if necessary, we can suppose that $F_r\leq G_r$ for all r. Of course, we have $J(G_r-F_r)=J(G_r)-J(F_r)$ and also $J(G-F_r)=J(G)-J(F_r)$ for all r by Lemma 4.3 d). Then, by Lemma 4.4 a) and Lemma 4.3 c), we successively get

$$J(G-F)=\lim_\tau J(G-F_r)=J(G)-\lim_\tau J(F_r)=J(G)-J(F).$$

b) Again there are sequences (F_r), (G_r) in $\overline{\mathrm{M}}_+$ such that $F_r\downarrow F$, $G_r\downarrow G$ and $F_r\leq G_r$ for all r. So for all $r,\ s\in\mathbb{N}$ with $r\geq s$, we have $J(G_s-F_r)=J(G_s)-J(F_r)$ by virtue of a). Then Proposition 4.2 and Lemma 4.3 d) imply that $J(G_s)=J(F)+J(G_s-F)$ for all s. Now $\lim_\tau J(G_r)=J(G)$ by Lemma 4.4 a) and so we get $J(G)=\lim_\tau J(G_s-F)+J(F)$. But the subadditivity of J implies that $J(G)\leq J(F)+J(G-F)$ and so we have $\lim_\tau J(G_s-F)\leq J(G-F)$. Since $G-F\leq G_s-F$ for all s, we get $J(G-F)=\lim_\tau J(G_s-F)$. Thus, finally, $J(G-F)=J(G)-J(F)$.

The additivity of J on $\widetilde{\mathrm{M}}_+$ is immediate

The property d) is a direct consequence of the Proposition 4.1.

e) is immediate by use of a similar argument to the one of the proof of the Lemma 4.4 a).

Hence the conclusion. ∎

Note. The remark after the Lemma 2.4 also applies to the vector lattice $\widetilde{\mathrm{M}}$. That is, we introduce a suitable norm on this space $\widetilde{\mathrm{M}}$.

Definition. Let $F\in\widetilde{\mathrm{M}}$. We know that there are $F_1,\ F_2\in\widetilde{\mathrm{M}}_+$ such that $F=F_1-F_2$ and so $L_1,\ L_2\in\mathrm{LSC}(K;E)$ with $F_1\leq L_1$ and $F_2\leq L_2$. That is, the set

$$\left\{\|H\|_K+\|L\|_K : F=F_1-F_2;\ F_1,\ F_2\in\widetilde{\mathrm{M}}_+,\ H\in j_{F_1},\ L\in j_{F_2}\right\}$$

is minorized in $\mathbb{R}$ and we denote by $\|F\|_\sim$ its infimum.

It is easy to establish the following

Proposition 4.6 *The mapping*

$$\|\cdot\|_\sim : \widetilde{\mathrm{M}} \longrightarrow \mathbb{R} \qquad F \longmapsto \|F\|_\sim$$

is a norm on $\widetilde{\mathrm{M}}$ *such that*

$$\|\cdot\|_K \leq \|\cdot\|_\sim \text{ on } \widetilde{\mathrm{M}} \quad \text{and} \quad \|\cdot\|_K = \|\cdot\|_\sim \text{ on } \mathrm{C}(K;E)_+.$$

Furthermore, the norms $\|\cdot\|_K$ *and* $\|\cdot\|_\sim$ *are equivalent on* $\mathrm{C}(K;E)$*; more precisely, one has*

$$\|\cdot\|_K \leq \|\cdot\|_\sim \leq 2\,\|\cdot\|_K \text{ on } \mathrm{C}(K;E).$$

Definition. Let $F \in \widetilde{\mathrm{M}}$ be with the decompositions $F = F_1 - F_2 = G_1 - G_2$, where $F_1,\ F_2,\ G_1,\ G_2 \in \widetilde{\mathrm{M}}_+$. Of course, we have $J(F_1 + G_2) = J(G_1 + F_2)$ and so $J(F_1) + J(G_2) = J(G_1) + J(F_2)$. Hence the element

$$J(F_1) - J(F_2) = J(G_1) - J(G_2) \in \mathrm{C}(K;E)''$$

is independent of decomposition choice of F; we denote it by $\widetilde{J}(F)$.

Lemma 4.7 *The mapping*

$$\widetilde{J} : (\widetilde{\mathrm{M}}, \|\cdot\|_\sim) \longrightarrow \mathrm{C}(K;E)'' \qquad F \longmapsto \widetilde{J}(F)$$

is an injective vector lattice homomorphism such that

$$\left\|\widetilde{J}(F)\right\| \leq \|F\|_\sim, \quad \forall F \in \widetilde{\mathrm{M}},$$

$$\widetilde{J}(F) = J(F), \quad \forall F \in \widetilde{\mathrm{M}}_+$$

and

$$\widetilde{J}(F) = \Psi(F), \forall F \in \mathrm{C}(K;E).$$

Furthermore,

a) *For every increasing* (resp. *decreasing) sequence* (F_r) *in* $\widetilde{\mathrm{M}}$ *with pointwise limit* $F \in \widetilde{\mathrm{M}}$*, one has* $\widetilde{J}(F) = \lim_\tau \widetilde{J}(F_r)$.

b) *For every order bounded subset* D *of* $\widetilde{\mathrm{M}}$*,* $\widetilde{J}(D)$ *is a* τ*-bounded* (resp. $\|\cdot\|$*-bounded) subset of* $\mathrm{C}(K;E)''$.

Proof. The linearity of $\widetilde{J}$ follows from Proposition 4.1 and Lemma 4.5 c); its injectivity follows from Lemma 4.5 d).

It is immediate that $\widetilde{J}$ is a vector lattice homomorphism which extends both the map Ψ and the restriction of J on $\widetilde{\mathrm{M}}_+$.

The assertion a) is a direct consequence of Proposition 4.2 and Lemma 4.5 e).

The assertion b) holds because of following three properties:

b_1) the map $\widetilde{J}$ is order bounded.

b_2) every order bounded subset of $\mathrm{C}(K;E)''$ is τ-bounded.

b_3) in the space $\mathrm{C}(K;E)''$, the norm bounded and τ-bounded subsets coincide.

Hence the conclusion. ∎

Remark. The norms $\|\cdot\|_o$ on $\mathrm{LS}(K;E)$ and $\|\cdot\|_\sim$ on $\widetilde{\mathrm{M}}$ allowed us to obtain a continuous linear canonical injection of each of these two spaces into the bidual $\mathrm{C}(K;E)''$. (cf. Theorem 3.2 and Lemma 4.7) However, in general, these two norms are not lattice norms.

Definition. Let w_1 denote the smallest uncountable ordinal and α denote a countable ordinal (i.e. $\alpha < w_1$). We denote by $\mathbb{M}_o$ the space M of our Convention (see above) and we define, by transfinite induction, $\mathbb{M}_\alpha$ to be the set $(\cup_{\beta<\alpha}\mathbb{M}_\beta)^\sim$. That is, we set $\mathbb{M} := \cup_{\alpha<w_1}\mathbb{M}_\alpha$.

It is clear that the set $\mathbb{M}$ contains the space $\widetilde{\mathrm{M}}$.

The proof of the following proposition is immediate.

Proposition 4.8 *For all ordinal* $\alpha < w_1$*, the set* $\mathbb{M}_\alpha$ *is a normed vector sublattice of* $\widehat{\mathrm{B}}(K;E)$ *which contains the space* $\mathrm{C}(K;E)$*.*

Furthermore, the set $\mathbb{M}$ *is a normed vector sublattice of* $\widehat{\mathrm{B}}(K;E)$ *which contains the space* $\mathrm{C}(K;E)$*.*

The following theorem gives a theoretical solution to the embedding problem which we investigate.

Theorem 4.9 *The mapping*

$$J : \mathbb{M}_+ \longrightarrow \mathrm{C}(K;E)''_+ \qquad F \longmapsto J(F)$$

is positive homogeneous, additive and injective.

That is, the mapping

$$\widetilde{J} : \mathbb{M} \longrightarrow \mathrm{C}(K;E)'' \qquad F \longmapsto \widetilde{J}(F)$$

is an injective vector lattice homomorphism which extends Ψ*.*

Proof. It is a direct consequence of the Lemma 4.7. ∎

5 Applications

In the previous section, we got an "abstract" result of our embedding problem. (cf. Theorem 4.9) Now, we are going to give some practical examples of the abstract space $\mathbb{M}$.

Definition. We define the *Baire classes* $\mathrm{Ba}(K;E)_\alpha$ ($\alpha < w_1$) as follows: let $\mathrm{Ba}(K;E)_0 = \mathrm{C}(K;E)$ and, for each ordinal $\alpha < w_1$, let $\mathrm{Ba}(K;E)_\alpha$ denote the set of all functions $F \in \widehat{\mathrm{B}}(K;E)$ that are pointwise limits of uniformly bounded sequences in $\cup_{\beta<\alpha}\mathrm{Ba}(K;E)_\beta$ and finally, we set $\mathrm{Ba}(K;E) = \cup_{\alpha<w_1}\mathrm{Ba}(K;E)_\alpha$.

For all $C \in \mathbb{R}_+$, we introduce the continuous mapping

$$\theta_C : \mathbb{R}_+ \longrightarrow [0,1] \qquad c \longmapsto \begin{cases} 1, & \text{if } c \in [0,C], \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is easy to establish.

Lemma 5.1 *For every* $C \in \mathbb{R}_+$*, ordinal* $\alpha \in]0, w_1[$ *and* $F \in \mathrm{Ba}(K;E)_\alpha$*,* $(\theta_C \circ \|\cdot\| \circ F) \cdot F$ *belongs to* $\mathrm{Ba}(K;E)_\alpha$*.*

In particular, $(\theta_C \circ \|\cdot\| \circ F) \cdot F$ *is an element of* $\mathrm{Ba}(K;E)$ *for all* $C \in \mathbb{R}_+$ *and* $F \in \mathrm{Ba}(K;E)$*.*

Lemma 5.2 *For every ordinal* $\alpha \in]0, w_1[$ *and* $F \in \mathrm{Ba}(K;E)_\alpha$*, there is an uniformly bounded sequence* (F_r) *in* $\cup_{\beta<\alpha}\mathrm{Ba}(K;E)_\beta$ *with pointwise limit* F *and such that*

$$\|F_r\|_K \leq \|F\|_K\,, \quad \forall r \in \mathbb{N}.$$

Proof. Of course, there exists an uniformly bounded sequence (H_r) of the set $\mathrm{B} := \cup_{\beta<\alpha}\mathrm{Ba}(K;E)_\beta$ which pointwise converges to F. Let us set $F_r = (\theta_{\|F\|} \circ \|\cdot\| \circ H_r) \cdot H_r$ for all $r \in \mathbb{N}$. Hence the sequence (H_r) is uniformly bounded in B, by Lemma 5.1. That is, for all $x \in K$, one successively has

$$\lim F_r(x) = \lim \theta_{\|F\|_K}(\|H_r(x)\|) \cdot H_r(x) = \theta_{\|F\|_K}(\|F(x)\|) \cdot F(x) = F(x).$$

Furthermore, for all $x \in K$ and $r \in \mathbb{N}$, one also has

$$\|F_r(x)\| = \left\|\theta_{\|F\|_K}(\|H_r(x)\|) \cdot H_r(x)\right\|.$$

Thus on the one hand, if $\|H_r(x)\| \in [0, \|F\|_K]$,

$$\|F_r(x)\| = \|H_r(x)\| \leq \|F\|_K$$

and the other hand, if $\|H_r(x)\| \in]\|F\|_K\,, +\infty[$,

$$\|F_r(x)\| = (\|F\|_K \,/\, \|H_r(x)\|) \cdot \|H_r(x)\| = \|F\|_K\,.$$

Finally, we get $\|F_r\|_K \leq \|F\|_K$ for all $r \in \mathbb{N}$. ∎

Proposition 5.3 *For every ordinal* $\alpha \in]0, w_1[$*, the set* $\mathrm{Ba}(K;E)_\alpha$ *is a vector sublattice of* $\widehat{\mathrm{B}}(K;E)$ *containing the space* $\mathrm{C}(K;E)$ *and the space* $\mathrm{Ba}(K;E)_\alpha$ *is a Banach lattice under the supremum norm.*

Furthermore, the set $\mathrm{Ba}(K;E)$ *is a vector sublattice of* $\widehat{\mathrm{B}}(K;E)$ *containing the space* $\mathrm{C}(K;E)$ *and the space* $\mathrm{Ba}(K;E)$ *is a Banach lattice under the supremum norm.*

Proof. It is clear from the above definition and the Theorem 5.2 of [1] that each set $\mathrm{Ba}(K;E)_\alpha$, as well as the set $\mathrm{Ba}(K;E)$, is a vector sublattice of $\widehat{\mathrm{B}}(K;E)$ containing the space $\mathrm{C}(K;E)$.

Next, we show that each space $\mathrm{Ba}(K;E)_\alpha$ is complete under the supremum norm. It suffices to prove that every absolutely convergent series norm converges. More precisely, we show that if a sequence (F_r) in $\mathrm{Ba}(K;E)_{\alpha,+}$ satisfies $\|F_r\|_K \leq 2^{-r}$ for all $r \in \mathbb{N}$, then the series $\sum_{r=1}^{\infty} F_r$ belongs to $\mathrm{Ba}(K;E)_\alpha$. By virtue of the

Lemma 5.2, for every $r \in \mathbb{N}$, there exists an uniformly bounded sequence $(F_{r,k})_{k\in\mathbb{N}}$ in $\mathrm{B} := \cup_{\beta<\alpha}\mathrm{Ba}(K;E)_\beta$ with pointwise limit F_r and such that

$$\|F_{r,k}\|_K \leq \|F_r\|_K\,, \quad \forall k \in \mathbb{N}.$$

Let us set $k(x,0) = 1$ for all $x \in K$. That is, for all $s \in \mathbb{N}$, there is a natural number $k(x,s) > k(x,s-1)$ such that

$$\left\|\sum_{r=1}^{s} F_{r,k}(x) - \sum_{r=1}^{s} F_r(x)\right\| \leq 2^{-s}, \quad \forall k \geq k(x,s).$$

Now, we consider the sequence $\left(\sum_{r=1}^{s} F_{r,s}\right)_{s\in\mathbb{N}}$: it is clear that this sequence belongs to the set B and, of course, it is uniformly bounded since, for every $s \in \mathbb{N}$, one has

$$\sum_{r=1}^{s} \|F_{r,s}\|_K \leq \sum_{r=1}^{\infty} \|F_r\|_K \leq 1.$$

Let $\varepsilon \in\,]0,+\infty[$. Then there is $s_0 \in \mathbb{N}$ such that $3 \cdot 2^{-s_0} \leq \varepsilon$. For every natural number $s \geq k(x,s_0)$, we successively have

$$\begin{aligned}\left\|\sum_{r=1}^{s} F_{r,s}(x) - \sum_{r=1}^{\infty} F_r(x)\right\| &\leq \left\|\sum_{r=1}^{s_0} F_{r,s}(x) - \sum_{r=1}^{s_0} F_r(x)\right\| \\ &\qquad + \left\|\sum_{r=s_0+1}^{s} F_{r,s}(x)\right\| + \left\|\sum_{r=s_0+1}^{\infty} F_r(x)\right\| \\ &\leq 2^{-s_0} + 2^{-s_0} + 2^{-s_0} \leq \varepsilon.\end{aligned}$$

Hence the sequence $\left(\sum_{r=1}^{s} F_{r,s}\right)_{s\in\mathbb{N}}$ pointwise converges to $\sum_{r=1}^{\infty} F_r$.

Finally, let us show that the space $\mathrm{Ba}(K;E)$ is complete under the supremum norm. We prove that if the sequence (F_r) in $\mathrm{Ba}(K;E)_+$ verifies $\|F_r\|_K \leq 2^{-r}$ for all $r \in \mathbb{N}$, then one has $\sum_{r=1}^{\infty} F_r \in \mathrm{Ba}(K;E)$. Observe that, for every $r \in \mathbb{N}$, there is anordinal $\alpha_r < w_1$ such that $F_r \in \mathrm{Ba}(K;E)_{\alpha_r,+}$. Moreover, it is well known that there exists an ordinal $\alpha < w_1$ such that $\alpha_r < \alpha$ for all $r \in \mathbb{N}$. Of course the cone $\mathrm{Ba}(K;E)_{\alpha,+}$ contains the set $\{\, F_r : r \in \mathbb{N}\}$ and since we already know that the space $\mathrm{Ba}(K;E)_\alpha$ is a Banach lattice, we have that $\sum_{r=1}^{\infty} F_r \in \mathrm{Ba}(K;E)_\alpha$.

Hence the conclusion. ∎

We now show that the mapping J (cf. this notation at the beginning of Section 4) can be additive on the set $\mathrm{Ba}(K;E)_+$, the positive cone of the Banach lattice $\mathrm{Ba}(K;E)$. For this purpose, we need the following definition.

Definition. The space E has *the condition* $(*)$ if the norm convergence and the order convergence for the sequences of E are equivalent. (cf. [20], [21] and [22] for the examples of such spaces E.)

Lemma 5.4 *If the space E has the condition $(*)$, then the space* $\mathrm{Ba}(K;E)$ *is a vector sublattice of the vector lattice* $\mathbb{M}$.

In particular, the mapping J is additive on the cone $\mathrm{Ba}(K;E)_+$ *and one has* $\mathrm{Ba}(K;E)^{\sim} = \mathrm{Ba}(K;E)$.

Proof. We first show that the inclusion $\mathrm{Ba}(K;E) \subset \mathbb{M}$ holds. It suffices to prove that one has $\mathrm{Ba}(K;E)_{\alpha,+} \subset \mathbb{M}_+$ for every ordinal $\alpha < w_1$.

The case $\alpha = 0$ is trivial. If α differs from 0, we proceed by recurrence. That is, suppose that $\mathrm{Ba}(K;E)_{\beta,+} \subset \mathbb{M}_{\beta,+}$ for all ordinal $\beta < \alpha$. Let us prove that one has $\mathrm{Ba}(K;E)_{\alpha,+} \subset \mathbb{M}_{\alpha,+}$. Let $F \in \mathrm{Ba}(K;E)_{\alpha,+}$. Then there exists an uniformly bounded sequence (F_r) in $\left(\cup_{\beta<\alpha}\mathrm{Ba}(K;E)_\beta\right)_+$ which pointwise converges to F. That is, by hypothesis, the sequence (F_r) belongs to the set $Z_\alpha := (\cup_{\beta<\alpha}\mathbb{M}_\beta)_+$ with pointwise limit F. Since the space E has the condition $(*)$, this sequence pointwise order converges to F. In particular, one has $\bigvee_{k\geq r} F_k(x) \downarrow F(x)$ for all $x \in K$. Hence one gets $F \in \overline{\overline{Z_\alpha}}$; what suffices.

It is clear that the space $\mathrm{Ba}(K;E)$ is a vector sublattice of $\mathbb{M}$.

The additivity of the mapping J on the cone $\mathrm{Ba}(K;E)_+$ is a direct consequence of the Theorem 4.9.

The equality of the lemma is straightforward. ∎

Theorem 5.5 *If the space E has the condition $(*)$, then the mapping*

$$\tilde{J} : (\mathrm{Ba}(K;E), \|\cdot\|_K) \longrightarrow \mathrm{C}(K;E)'' \qquad F \mapsto \tilde{J}(F)$$

is a linear continuous injection and a Banach lattice homomorphism such that

$$\tilde{J}(F) = J(F), \quad \forall F \in \mathrm{Ba}(K;E)_+$$

and

$$\tilde{J}(f) = \Psi(f), \quad \forall f \in \mathrm{C}(K;E).$$

In particular, the mapping

$$\tilde{J} : (\mathrm{Ba}(K;E), \|\cdot\|_\sim) \longrightarrow \mathrm{C}(K;E)'' \qquad F \mapsto \tilde{J}(F)$$

is a linear continuous injection and a vector lattice homomorphism.

Furthermore,

a) *For all increasing* (resp. *decreasing) sequence* (F_r) *in* $\mathrm{Ba}(K;E)$ *with pointwise limit* $F \in \mathrm{Ba}(K;E)$, *one has* $\tilde{J}(F) = \lim_\tau \tilde{J}(F_r)$.

b) *For all order bounded subset D of* $\mathrm{Ba}(K;E)$, $\tilde{J}(D)$ *is a* $\tau-$*bounded* (resp. $\|\cdot\|-$*bounded) of* $\mathrm{C}(K;E)''$.

Proof. By virtue of Lemma 4.7, Proposition 5.3 and Lemma 5.4, it is clear that the mapping $\tilde{J}$ is a linear injection and a Banach lattice homomorphism. That is, this mapping $\tilde{J}$ is continuous by the Theorem II.5.3. of [13].

The particular case is a direct consequence of Proposition 4.6 and the assertions a) and b) are immediate by Lemma 4.7. ∎

Corollary 5.6 *If the space E has the condition $(*)$, then the Banach lattice $\mathrm{Ba}(K;E)$ is algebraically isomorphic to a vector sublattice of $\mathrm{C}(K;E)''$.*

In the sequel, we give a second example of the space $\mathbb{M}$. (cf. the next definition and the Theorem 5.12)

Notation. We denote by $\mathrm{S}_0(K;E)$ the convex conical hull

$$\left\{ \sum_{k=1}^{r} \chi_{A_k} e_k : A_k = \text{ open in } K,\ e_k \in E_+,\ r \in \mathbb{N} \right\}.$$

Of course, this set is contained in the space $\mathrm{LSC}(K;E)$.

We also denote by $\mathrm{B}_0(K;E)$ the uniform closure of the vector sublattice $\mathrm{S}_0(K;E) - \mathrm{S}_0(K;E)$ in the Banach lattice $\mathrm{FB}(K;E)$.

Proposition 5.7 *The set $\mathrm{B}_0(K;E)$ is a Banach lattice contained in the space $\mathrm{BS}(K;E)^{\sim}$ and containing the space $\mathrm{C}(K;E)$.*

In particular, the mapping J is additive on the cone $\mathrm{B}_0(K;E)_+$.

Proof. Of course, the set $\mathrm{B}_0(K;E)$ is a Banach lattice. (cf. [1], Theorem 5.4(iii).)

Furthermore, it is well known that one has the inclusion $\mathrm{C}(K;E) \subset \mathrm{B}_0(K;E)$. ([2], Proposition IV.4.19)

Let us show that the inclusion $\mathrm{B}_0(K;E) \subset \mathrm{BS}(K;E)^{\sim}$ holds. So, we prove that one has $\mathrm{B}_0(K;E)_+ \subset \overline{\overline{\mathrm{BS}(K;E)_+}}$. Let $F \in \mathrm{B}_0(K;E)_+$. Then there exists a sequence (F_r) in $\mathrm{S}_0(K;E)$ that uniformly converges to F. Hence some subsequence $(F_{r_k})_{k\in\mathbb{N}}$ of the sequence (F_r) order converges to F. In particular, one has $H_k := \bigvee_{s\geq k} F_{r_s} \downarrow F$. Consequently, one has $H_k \in \mathrm{LSC}(K;E)$ and $F \leq H_k$ for all $k \in \mathbb{N}$. Finally, (H_k) is a sequence in $\mathrm{LS}(K;E)_+$ such that $H_k \downarrow F$ and so, one gets $F \in \overline{\overline{\mathrm{BS}(K;E)_+}}$.

That is, the additivity of the mapping J on the cone $\mathrm{B}_0(K;E)_+$ becomes clear.∎

Definition. Denoting by $\mathrm{Bo}(K;E)_0$ the Banach lattice $\mathrm{B}_0(K;E)$ (cf. the above notation) we again define by transfinite induction, the *Borel class* $\mathrm{Bo}(K;E)_\alpha$ ($\alpha < w_1$) to be the set of all functions $F \in \widehat{\mathrm{B}}(K;E)$ that are pointwise limits of uniformly bounded sequences in $\cup_{\beta<\alpha}\mathrm{Bo}(K;E)_\beta$. Finally, we set $\mathrm{Bo}(K;E) = \cup_{\alpha<w_1}\mathrm{Bo}(K;E)_\alpha$.

Lemma 5.8 *For every $C \in \mathbb{R}_+$, ordinal $\alpha \in]0,w_1[$ and $F \in \mathrm{Bo}(K;E)_\alpha$, $(\theta_C \circ \|\cdot\| \circ F)\cdot F$ belongs to $\mathrm{Bo}(K;E)_\alpha$.*

In particular, $(\theta_C \circ \|\cdot\| \circ F)\cdot F$ is an element of $\mathrm{Bo}(K;E)$ for all $C \in \mathbb{R}_+$ and $F \in \mathrm{Bo}(K;E)$.

Proof. a) Suppose that $\alpha = 0$. For every $F \in \mathrm{B}_0(K;E)$, there exists a sequence (F_r) in the vector lattice $\mathrm{M}_0(K;E) \equiv \mathrm{S}_0(K;E) - \mathrm{S}_0(K;E)$ that uniformly converges to F. Then there is $M \in]0,+\infty[$ such that $\|F_r\|_K \leq M$ and hence, one has $(\theta_C \circ \|\cdot\| \circ F_r) \cdot F_r \in \mathrm{M}_0(K;E)$ for all $r \in \mathbb{N}$. That is, one successively gets

$$\begin{aligned}
&\|(\theta_C \circ \|\cdot\| \circ F_r) \cdot F_r - (\theta_C \circ \|\cdot\| \circ F) \cdot F\|_K \\
&\quad \leq \sup_{x\in K} |\theta_C(\|F_r(x)\|) - \theta(\|F(x)\|)| \cdot \|F_r\|_K \\
&\qquad + \sup_{x\in K} \theta_C(\|F(x)\|) \cdot \|F_r - F\|_K \\
&\quad \leq M \sup_{x\in K} |\theta_C(\|F_r(x)\|) - \theta_C(\|F(x)\|)| + \|F_r - F\|_K
\end{aligned}$$

and the last right side of these inequalities converges to 0. Finally, the sequence $(\theta_C \circ \|\cdot\| \circ F_r) \cdot F_r$ uniformly converges to $(\theta_C \circ \|\cdot\| \circ F) \cdot F$ and so, one has $(\theta_C \circ \|\cdot\| \circ F) \cdot F \in \mathrm{B}_0(K;E)$ by virtue of the Proposition 5.7

To show the case $\alpha \neq 0$, one proceeds by transfinite recurrence. ∎

Lemma 5.9 *For every ordinal $\alpha \in]0, w_1[$ and $F \in \mathrm{Bo}(K;E)_\alpha$, there is an uniformly bounded sequence (F_r) in $\cup_{\beta<\alpha}\mathrm{Bo}(K;E)_\beta$ with pointwise limit F and such that*

$$\|F_r\|_K \leq \|F\|_K, \quad \forall r \in \mathbb{N}.$$

Proof. The proof is similar to that of the Lemma 5.2. ∎

Proposition 5.10 *For every ordinal $\alpha \in [0, w_1[$, the set $\mathrm{Bo}(K;E)_\alpha$ is a vector sublattice of $\widehat{\mathrm{B}}(K;E)$ containing the space $\mathrm{C}(K;E)$ and the space $\mathrm{Bo}(K;E)_\alpha$ is a Banach lattice under the supremum norm.*

Furthermore, the set $\mathrm{Bo}(K;E)$ is a vector sublattice of $\widehat{\mathrm{B}}(K;E)$ containing the space $\mathrm{C}(K;E)$ and the space $\mathrm{Bo}(K;E)$ is a Banach lattice under the supremum norm.

Proof. The proof is similar to the one of Proposition 5.3. ∎

Lemma 5.11 *If the space E has the condition $(*)$, then the space $\mathrm{Bo}(K;E)$ is a vector sublattice of the vector lattice $\mathbb{M}$ with $M = \mathrm{B}_0(K;E)$.*

In particular, the mapping J is additive on the cone $\mathrm{Bo}(K;E)_+$ and one has $\mathrm{Bo}(K;E)^\sim = \mathrm{Bo}(K;E)$.

Proof. The proof is similar to the one of Lemma 5.4. ∎

Theorem 5.12 *If the space E has the condition $(*)$, then the mapping*

$$\tilde{J} : (\mathrm{Bo}(K;E), \|\cdot\|_K) \longrightarrow \mathrm{C}(K;E)'' \qquad F \mapsto \tilde{J}(F)$$

is a linear continuous injection and a Banach lattice homomorphism such that

$$\tilde{J}(F) = J(F), \quad \forall F \in \mathrm{Bo}(K;E)_+$$

and

$$\tilde{J}(f) = \Psi(f), \quad \forall f \in \mathrm{C}(K;E).$$

In particular, the mapping

$$\tilde{J} : (\mathrm{Bo}(K;E), \|\cdot\|_{\sim}) \longrightarrow \mathrm{C}(K;E)'' \qquad F \mapsto \tilde{J}(F)$$

is a linear continuous injection and a vector lattice homomorphism.

Furthermore,

a) *For every increasing* (resp. *decreasing) sequence* (F_r) *in the space* $\mathrm{Bo}(K;E)$ *with pointwise limit* $F \in \mathrm{Bo}(K;E)$*, one has* $\tilde{J}(F) = \lim_{\tau} \tilde{J}(F_r)$*.*

b) *For every order bounded subset* D *of* $\mathrm{Bo}(K;E)$*,* $\tilde{J}(D)$ *is a* $\tau-$*bounded* (resp. $\|\cdot\|-$*bounded) subset of* $\mathrm{C}(K;E)''$*.*

Proof. The proof is similar to the one of Theorem 5.5. ∎

Corollary 5.13 *If the space* E *has the condition* (∗)*, then the Banach lattice* $\mathrm{Bo}(K;E)$ *is algebraically isomorphic to a vector sublattice of* $\mathrm{C}(K;E)''$*.*

Remarks. a) There are other examples of spaces $\mathbb{M}$ without the condition (∗) of the Banach lattice E. In fact, with the same initial space as in the construction of the space $\mathrm{Ba}(K;E)$ (resp. $\mathrm{Bo}(K;E)$),one introduces for every ordinal $\alpha \in]0, w_1[$ the class $\mathbb{A}_\alpha$ (resp. $\mathbb{B}_\alpha$) as the set of all functions $F \in \widehat{\mathrm{B}}(K;E)$ that are pointwise order limits of sequences in $\cup_{\beta<\alpha}\mathbb{A}_\beta$ (resp. $\cup_{\beta<\alpha}\mathbb{B}_\beta$); and afterwards one sets $\mathbb{M} = \cup_{\alpha<w_1}\mathbb{A}_\alpha$ (resp. $\cup_{\alpha<w_1}\mathbb{B}_\alpha$).

b) By virtue of a), the vector lattices $\mathrm{C}(K;E)$, $\mathbb{A}$, $\mathbb{B}$ and $\mathrm{C}(K;E)''$ are bigger and bigger.

Moreover, by virtue of Condition (∗) on the Banach lattice E, Corollaries 5.6 and 5.13, it is clear that the Banach lattices $\mathrm{C}(K;E)$, $\mathrm{Ba}(K;E)$, $\mathrm{Bo}(K;E)$ and $\mathrm{C}(K;E)''$ are bigger and bigger.

c) Every function in the space $\mathrm{Ba}(K;E)$ (resp. $\mathrm{Bo}(K;E)$) is Baire (resp. Borel) -mesurable. However, we do not know if the converse is true.

d) It would be interesting to get a generalization of our results for K a (locally) compact space and E a complete locally convex lattice with the Lebesgue property.

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