Classification of Surfaces in $\mathbb{R}^3$ which are centroaffine-minimal and equiaffine-minimal

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Abstract

We classify all surfaces which are both, centroaffine-minimal and equiaffine-minimal in $\mathbb{R}^3$.

1 Introduction.

In equiaffine differential geometry, the variational problem for the equiaffine area integral leads to the equiaffine minimal surfaces, such surfaces have zero equiaffine mean curvature $H(e) = 0$. These surfaces were called affine minimal by Blaschke and his school ([1]). Calabi [2] pointed out that, for locally strongly convex surfaces with $H(e) = 0$, the second variation of the area integral is negative, so he suggested that the surfaces with $H(e) = 0$ should be called affine maximal surfaces. Wang [13] studied the variation of the centroaffine area integral and introduced the centroaffine minimal hypersurfaces, such hypersurfaces have the property that $\mathrm{trace}_G \widehat{\nabla}\widehat{T} \equiv 0$, where G is the centroaffine metric, $\widehat{\nabla}$ the centroaffine metric connection and $\widehat{T}$ the centroaffine Tchebychev form (see the definitions in §2). The study of Wang [13] leads to the more general definitions (and the generalizations) of the Tchebychev operator and Tchebychev hypersurfaces, see [5], [8], [9] and [10].

In this paper, we consider the centroaffine surfaces which are centroaffine-minimal and equiaffine-minimal in $\mathbb{R}^3$. We give the following classification theorem.

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Theorem. *Let $x : \mathbf{M} \to \mathbb{R}^{\mathbf{3}}$ be a surface both centroaffine-minimal and equiaffine-minimal. Then x is centroaffinely equivalent to one of the following surfaces in $\mathbb{R}^3$:*

(i)
$$x_3 = x_1^{\alpha} x_2^{\beta},$$

where α and β are constants satisfying:

$$\alpha = 1, \beta \neq 0 \quad \text{or} \quad \beta = 1, \alpha \neq 0 \quad \text{or} \quad \alpha \neq 0, \alpha + \beta = 0;$$

(ii)
$$x_3 = [\exp(-\alpha \arctan \frac{x_1}{x_2})](x_1^2 + x_2^2)^{\beta},$$

where α and β are constants satisfying:

$$\beta = 0, \alpha \neq 0 \quad \text{or} \quad \alpha = 0, \beta = 1;$$

(iii)
$$x_3 = -x_1(\alpha \log x_1 + \beta \log x_2),$$

where α and β are constants satisfying:

$$\alpha = 0, \beta \neq 0;$$

(iv)
$$x = (v, a(v)e^u, b(v)e^u), \quad v > 0,$$

where $\{a(v), b(v)\}$ are the fundamental solutions of the differential equation $y''(v)$ - $\vartheta(v)y(v) = 0$ and $\vartheta(v)$ is an arbitrary differential function of v.

2 The centroaffine surfaces in $\mathbb{R}^3$.

Let $x : \mathbf{M} \to \mathbb{R}^{\mathbf{3}}$ be a surface and $[\ ,\ ,\ ]$ the standard determinant in $\mathbb{R}^3$. x is said to be a centroaffine surface if the position vector of x, denoted again by x, is always transversal to the tangent plane $x_*(\mathbf{TM})$ at each point of $\mathbf{M}$ in $\mathbb{R}^3$. We define a symmetric bilinear form G on $\mathbf{TM}$ by

$$G = -\sum_{i,j=1}^{2} \frac{[e_1(x), e_2(x), e_i e_j(x)]}{[e_1(x), e_2(x), x]}\, \theta^i \otimes \theta^j, \tag{2.1}$$

where $\{e_1, e_2\}$ is a local basis for $\mathbf{TM}$ with the dual basis $\{\theta^1, \theta^2\}$. Note that G is globally defined. A centroaffine surface x is said to be nondegenerate if G is nondegenerate. We call G the centroaffine metric of x. We say that a surface is definite (indefinite) if G is definite (indefinite).

For the centroaffine surface x, let $\nabla = \{\Gamma_{ij}^k\}$ and $\widehat{\nabla} = \{\widehat{\Gamma}_{ij}^k\}$ be the induced connection and the Levi-Civita connection of the centroaffine metric G. We define the cubic form C by (in the following, we use the Einstein summation convention)

$$\Gamma_{ij}^k - \widehat{\Gamma}_{ij}^k =: C_{ij}^k,\ \ C_{ijk} := G_{km} C_{ij}^m,\ \ i, j, k, m = 1, 2. \tag{2.2}$$

We know that $C = C_{ijk}\theta^i\theta^j\theta^k$ is the centroaffine Fubini-Pick form for x which is totally symmetric. The Tchebychev vector field and the Tchebychev form are defined by

$$T := T^j e_j = \frac{1}{2}G^{ik}C^j_{ik}e_j, \tag{2.3}$$

$$\widehat{T} := T_j\theta^j = G_{ij}T^i\theta^j. \tag{2.4}$$

It is well-known that T and $\widehat{T}$ are centroaffine invariants.

Definition. *Let $x : \mathbf{M} \to \mathbb{R}^{\mathbf{3}}$ be a centroaffine surface. If* $\mathrm{trace}_G\widehat{\nabla}\widehat{T} = G^{ij}\widehat{\nabla}_i T_j \equiv 0$, *$x$ is called a centroaffine minimal surface.* ([13])

Remark 2.1. $\widehat{\nabla}\widehat{T}$ is symmetric ([13]).

Remark 2.2. From the definition of the centroaffine minimal surface we know that

1. (1) the proper affine spheres are centroaffine minimal surfaces;
2. (2) the centroaffine surfaces with parallel Tchebychev form are centroaffine minimal surfaces ([9]).

Example 2.1. The surface defined by

$$x_3 = x_1^\alpha x_2^\beta, \tag{2.5}$$

for any $\alpha, \beta \in \mathbb{R}$, $\alpha\beta(\alpha+\beta-1) \neq 0$, is a centroaffine minimal surface in $\mathbb{R}^3$. It can be written as

$$x = (e^u, e^v, e^{\alpha u+\beta v}).$$

The centroaffine metric is flat; it is given by

$$G = \frac{\alpha^2-\alpha}{\alpha+\beta-1}\mathrm{d}u^2 + 2\frac{\alpha\beta}{\alpha+\beta-1}\mathrm{d}u\mathrm{d}v + \frac{\beta^2-\beta}{\alpha+\beta-1}\mathrm{d}v^2.$$

When $0 < \alpha < 1$, $0 < \beta < 1-\alpha$ or $\alpha < 0$, $\beta > 1-\alpha$ or $\alpha > 1$, $0 > \beta > 1-\alpha$, the surface is positive definite; when $\alpha < 0$, $\beta < 0$, the surface is negative definite; otherwise, the surface is indefinite. For the Tchebychev form $\widehat{T}$ we have

$$T_1 = \frac{1}{2}(1+\alpha),\ \ T_2 = \frac{1}{2}(1+\beta),$$

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \frac{1}{4}(6 - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha} + \alpha + \beta).$$

Obviously, $\widehat{\nabla}\widehat{T} \equiv 0$.

Example 2.2. The surface defined by

$$x_3 = [\exp(-\alpha\arctan\frac{x_1}{x_2})](x_1^2+x_2^2)^\beta, \tag{2.6}$$

for any α, $\beta \in \mathbb{R}$, $(2\beta - 1)(\alpha^2 + \beta^2) \neq 0$, is a centroaffine minimal surface in $\mathbb{R}^3$. It can be written as

$$x = (e^u \sin v, e^u \cos v, e^{2\beta u - \alpha v}).$$

The centroaffine metric is flat; it is given by

$$G = 2\beta \mathrm{d}u^2 - 2\alpha \mathrm{d}u\mathrm{d}v + \frac{2\beta + \alpha^2}{2\beta - 1}\mathrm{d}v^2.$$

When $2\beta > 1$, the surface is positive definite; otherwise, the surface is indefinite. For the Tchebychev form $\widehat{T}$ we have

$$T_1 = 1 + \beta, \ T_2 = -\frac{1}{2}\alpha,$$

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \frac{4\alpha^2 + 8\beta^2 + 4\beta^3 + \alpha^2\beta + 4\beta}{2(4\beta^2 + \alpha^2)}.$$

Obviously, $\widehat{\nabla}\widehat{T} \equiv 0$.

Example 2.3. The surface defined by

$$x_3 = -x_1(\alpha \log x_1 + \beta \log x_2), \tag{2.7}$$

for any α, $\beta \in \mathbb{R}$, $\beta(\alpha + \beta) \neq 0$, is a centroaffine minimal surface in $\mathbb{R}^3$. It can be written as

$$x = (e^u, e^v, -e^u(\alpha u + \beta v)).$$

The centroaffine metric is flat; it is given by

$$G = \frac{\alpha}{\alpha + \beta}\mathrm{d}u^2 + 2\frac{\beta}{\alpha + \beta}\mathrm{d}u\mathrm{d}v - \frac{\beta}{\alpha + \beta}\mathrm{d}v^2.$$

When $\alpha > -\beta > 0$ or $\alpha < -\beta < 0$, the surface is positive definite; otherwise, the surface is indefinite. For the Tchebychev form $\widehat{T}$ we have

$$T_1 = 1, \ T_2 = \frac{1}{2},$$

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \frac{8\beta - \alpha}{4\beta}.$$

Obviously, $\widehat{\nabla}\widehat{T} \equiv 0$.

Example 2.4. The surface defined by

$$x = (e^v, a(v)e^u, b(v)e^u), \tag{2.8}$$

where $\{a(v), b(v)\}$ are the fundamental solutions of the differential equation $y''(v) - y'(v) - \vartheta(v)y(v) = 0$ and $\vartheta(v)$ is an arbitrary differential function of v, is a centroaffine minimal surface in $\mathbb{R}^3$. The centroaffine metric is flat; it is given by

$$G = 2\mathrm{d}u\mathrm{d}v.$$

The surface is indefinite. For the Tchebychev form $\widehat{T}$ we have

$$T_1 = 1, \ T_2 = 1,$$

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = 2.$$

Obviously, $\widehat{\nabla}\widehat{T} \equiv 0$. Let $w = e^v$, then (2.8) can be written as the surface (iv) given by Theorem.

3 The Proof of the Theorem.

We need the following lemmata.

Lemma 3.1. *Let $x : \mathbf{M} \to \mathbb{R}^3$ be an immersed surface. Then we have the following relations between the equiaffine quantities and the centroaffine quantities (We mark the equiaffine quantities by (e)):*

$$G(e) = \rho(e)G, \tag{3.1}$$

$$\rho(e)S(e)(X) = \varepsilon \mathrm{id}(X) - \widehat{\nabla}_X T - C(T, X) + \widehat{T}(X)T, \tag{3.2}$$

where $\rho(e)$ is the equiaffine support function, $\varepsilon = \pm 1$, $X \in \mathbf{TM}$.

Proof: (3.1) comes from (3.1) of [8]. By (5.1.3.iv) of [12] and (1.3.iv), (4.3.2) of [8], we can get (3.2). ∎

Lemma 3.2. *Let $x : \mathbf{M} \to \mathbb{R}^3$ be a centroaffine surface. If x is both centroaffine-minimal and equiaffine-minimal, its Tchebychev form satisfies*

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \pm 2. \tag{3.3}$$

Proof: From (3.2) we have

$$2\rho(e)H(e) = 2\varepsilon - \mathrm{trace}_G \widehat{\nabla}\widehat{T} - 2\|\widehat{T}\|^2 + \|\widehat{T}\|^2, \tag{3.4}$$

where $H(e)$ is the equiaffine mean curvature of x. By the definition, x is centroaffine-minimal and equiaffine-minimal if and only if $H(e) \equiv \mathrm{trace}_G \widehat{\nabla}\widehat{T} \equiv 0$. Therefore from (3.4) we get $2\varepsilon = \|\widehat{T}\|^2$. ∎

Lemma 3.3. *Let $x : \mathbf{M} \to \mathbb{R}^3$ be a centroaffine minimal surface with $\|\widehat{T}\|^2 =$ constant $\neq 0$. Then $\widehat{T}$ is parallel with respect to the centroaffine metric connection $\widehat{\nabla}$.*

Proof: If the metric of x is definite, we can choose a complex coordinate $z = u + iv$, $\bar{z} = u - iv$ such that

$$G = \varepsilon e^w(\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z). \tag{3.5}$$

If the metric of x is indefinite, we can choose the asymptotic parameter (u, v) such that

$$G = e^w(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u). \tag{3.6}$$

In both cases, from the condition $\|\widehat{T}\|^2 = G^{ij}T_iT_j =$ constant $\neq 0$ we know that $T_1 \neq 0$ and $T_2 \neq 0$. The condition $\mathrm{trace}_G \widehat{\nabla}\widehat{T} = 0$ means

$$\widehat{\nabla}_1 T_2 \equiv \widehat{\nabla}_2 T_1 \equiv 0. \tag{3.7}$$

$\|\widehat{T}\|^2 =$ constant yields

$$\begin{cases}(\widehat{\nabla}_1 T_1)T_2 + (\widehat{\nabla}_1 T_2)T_1 \equiv 0\\ (\widehat{\nabla}_2 T_1)T_2 + (\widehat{\nabla}_2 T_2)T_1 \equiv 0.\end{cases} \tag{3.8}$$

(3.7), (3.8) and $T_1 \neq 0$, $T_2 \neq 0$ give

$$\widehat{\nabla}_1 T_1 \equiv \widehat{\nabla}_1 T_2 \equiv \widehat{\nabla}_2 T_1 \equiv \widehat{\nabla}_2 T_2 \equiv 0.$$

So $\widehat{\nabla}\widehat{T} \equiv 0$. ∎

The Proof of the Theorem

By Lemma 3.2 and Lemma 3.3, we know that, in $\mathbb{R}^3$, the surfaces which are centroaffine -minimal and equiaffine-minimal are the centroaffine surfaces with parallel Tchebychev form $\widehat{T}$ and $\|\widehat{T}\|^2 = \pm 2$. Therefore, from [9], we obtain the surfaces given in Theorem. If we define the centroaffine metric by (2.1), we have $\varepsilon = 1$ in Lemma 3.1. Therefore, from Example 2.1-2.3,

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \frac{1}{4}(6 - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha} + \alpha + \beta) = 2$$

yields

$$(\alpha - 1)(\beta - 1)(\alpha + \beta) = 0;$$

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \frac{4\alpha^2 + 8\beta^2 + 4\beta^3 + \alpha^2\beta + 4\beta}{2(4\beta^2 + \alpha^2)} = 2$$

yields

$$\beta[\alpha^2 + 4(\beta - 1)^2] = 0;$$

$$\|\widehat{T}\|^2 = G^{ij}T_iT_j = \frac{8\beta - \alpha}{4\beta} = 2$$

yields

$$\alpha = 0.$$

This completes the proof of the Theorem. ∎

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