On Veldkamp Lines

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Abstract

One says that Veldkamp lines exist for a point-line geometry Γ if, for any three distinct (geometric) hyperplanes A, B and C (i) A is not properly contained in B and (ii) $A \cap B \subseteq C$ implies $A \subset C$ or $A \cap B = A \cap C$. Under this condition, the set $\mathcal{V}$ of all hyperplanes of Γ acquires the structure of a linear space – the *Veldkamp space* – with intersections of distinct hyperplanes playing the role of lines. It is shown here that an interesting class of strong parapolar spaces (which includes both the half-spin geometries and the Grassmannians) possess Veldkamp lines. Combined with other results on hyperplanes and embeddings, this implies that for most of these parapolar spaces, the corresponding Veldkamp spaces are projective spaces.

The arguments incorporate a model of partial matroids based on intersections of sets.

1 Introduction

Let Γ be a point-line geometry, that is, a rank two incidence system $(\mathcal{P}, \mathcal{L})$ with each object incident with at least two others. The objects of $\mathcal{P}$ are called "points"; those of $\mathcal{L}$ are called "lines"; nothing is assumed by this nomenclature. A *subspace of* Γ is a subset S of $\mathcal{P}$ such that any line L with two of its incident points in S has all its incident points in S. We assume without any real loss that distinct lines possess distinct sets of incident points (distinct *point-shadows*) and so may themselves be regarded as subsets of $\mathcal{P}$.

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A *(geometric) hyperplane* is a proper subspace H of Γ such that each line L intersects H non-trivially. We may note that if A and B are hyperplanes with A properly contained in B, then $\mathcal{P} - A$ cannot have a connected collinearity graph, and conversely. Thus for any hyperplane A, the two conditions

(i) A is not properly contained in B for any hyperplane B,

(ii) $\mathcal{P} - A$ has a connected collinearity graph

are equivalent, and if this happens for *any* hyperplane A, then we say *Veldkamp points exist*. We say that *Veldkamp lines exist*, if

(i) *Veldkamp points exist*

(ii) *For any three distinct hyperplanes A, B and C, $A \cap B \subseteq C$ implies $A \cap B = A \cap C$.*

If this condition holds, then the incidence system $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$, where $\mathcal{V}_1$ is all hyperplanes of Γ, $\mathcal{V}_2$ is the set of intersections of pairs of distinct hyperplanes of $\mathcal{V}_1$, and incidence is inclusion, becomes a linear space since any two "points" of a "line" determine that line. The linear space $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ is then called the *Veldkamp space*. (The term is intended to be meaningless unless Veldkamp lines already exist.)

In some cases, the Veldkamp space $\mathcal{V}$ is a projective space $\mathbb{P}$ and in some of these cases there is an embedding $e : (\mathcal{P}, \mathcal{L}) \to (\mathcal{V}_1, \mathcal{V}_2)$ provided by the Veldkamp space. Historically this step was first taken by F.D. Veldkamp for "embeddable polar spaces" – that is, polar spaces of rank at least three whose planes are Desarguesian (see J. Tits [Ti], for details of this characterization of Veldkamp's polar spaces). If $(\mathcal{P}, \mathcal{L})$ is such a space, then the mapping $p \to p^{\perp}$, $p \in \mathcal{P}$, provides the embedding $\mathcal{P} \to \mathcal{V}$.

There are quite a few geometries, Γ, of diameter k in which the mapping $p \to \Delta^*_{k-1}(p)$, the set of all points at distance at most $k-1$ from p, defines a mapping $e : \mathcal{P} \to \mathcal{V}$. But in many of these cases, for example, in the half-spin geometries $D_{2n,2n}(F)$, and the Grassmannians $A_{2n-1,n}(F)$, where F is a field, it has never even been established that Veldkamp lines exist! Thus it is not clear that the mapping $e : \mathcal{P} \to \mathcal{V}$ *constitutes* an embedding into a Veldkamp space in these cases.

In this note, we rectify this by showing that Veldkamp lines exist for a class of strong parapolar spaces which include *all* half-spin geometries and *all* Grassmannians, $A_{n,k}(F)$ (Corollary 6.3).

The results here are generalized somewhat, to yield a modest contribution to an outstanding question, proposed to me by Professor Arjeh Cohen ([Co1]).

QUESTION: If $\Gamma = (\mathcal{P}, \mathcal{L})$ is a strong parapolar space of singular rank at least 3, for which Veldkamp lines exist, is the Veldkamp space a projective space?

The results of this paper show that the hypothesis that Veldkamp lines exist in the above question, is automatically true. In fact, in the case of half-spin geometries and the exceptional Lie geometry $E_{6,1}(F)$, the following stronger condition holds:

(Veldkamp Planes Exist):

(i) *Veldkamp lines exist.*

(ii) *For any four hyperplanes $A_1, \ldots, A_4$ with A_1, A_2 and A_3 independent (i.e. $A_1 \cap A_2 \cap A_3$ is not the intersection of two of the A_i, $i \in \{1, 2, 3\}$), then $A_1 \cap A_2 \cap A_3 \subseteq A_4$ implies either $A_1 \cap A_2 \subseteq A_4$ or $A_1 \cap A_2 \cap A_3 = A_1 \cap A_2 \cap A_4$.*

The relevance of this condition to Cohen's question is manifest in the following

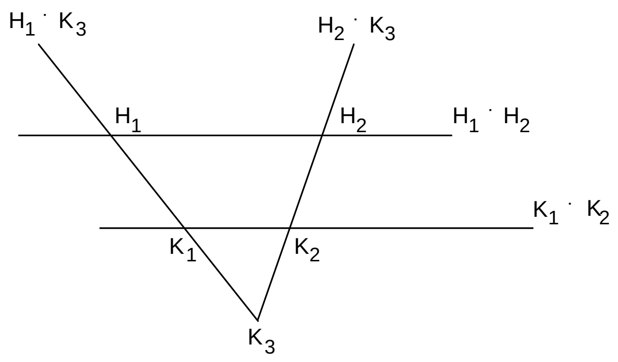
Lemma 1.1 *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a geometry for which the following two conditions hold:*

(i) Veldkamp planes exist.

(ii) (Teirlinck's Condition) *For any two distinct hyperplanes A and B and a point p in $\mathcal{P} - (A \cup B)$, there is a unique hyperplane C containing $\{p\} \cup (A \cap B)$.*

Then the Veldkamp space is a generalized projective space.

Although this proof seems to be well-known, at least for polar spaces (see Chapter 15 of the forthcoming book of Buekenhout and Cohen [BC]), we sketch it for the sake of completeness. It suffices to verify the so-called Pasch's Axiom for $\mathcal{V}$. Let H_1, H_2 and K_3 be hyperplanes with K_3 not containing $H_1 \cap H_2$. Suppose K_1 and K_2 are hyperplanes containing $H_1 \cap K_3$ and $H_2 \cap K_3$, respectively. In the Veldkamp space we have the configuration :



with the Veldkamp lines indicated. Clearly $K_1 \cap K_2$ is not contained in $H_1 \cap H_2$, so there is a point p in $K_1 \cap K_2 - (H_1 \cap H_2)$. By (ii) there is a hyperplane X containing p and $H_1 \cap H_2$. Then $X \cap K_1 \cap K_2$ properly contains $W = K_1 \cap K_2 \cap H_1$. Thus as Veldkamp planes exist, $K_1 \cap K_2 \cap H_1 \subseteq X$ implies $K_1 \cap K_2 \subseteq X$. So X is on the intersection of the Veldkamp lines $H_1 \cap H_2$ and $K_1 \cap K_2$. Thus $\mathcal{V}$ is a linear space with thick lines satisfying Pasch's axiom and so is projective.

There is a condition which implies Teirlinck's axiom, namely the condition that every hyperplane arises from an embedding.

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a geometry of points and lines with no repeated lines, and let $\mathbb{P}$ be a projective space (we also denote the points of $\mathbb{P}$ by the same symbol $\mathbb{P}$). A *projective embedding* is an injective map $e : \mathcal{P} \to \mathbb{P}$ such that

(i) *$e(\mathcal{P})$ spans $\mathbb{P}$*

(ii) *for each line L of $\mathcal{L}$ (regarded as a subset of $\mathcal{P}$), $e(L)$ is the point set of some projective line of $\mathbb{P}$.*

If $\mathbb{H}$ is a projective hyperplane of $\mathbb{P}$ it is easy to see that $H = e^{-1}(e(\mathcal{P}) \cap \mathbb{H})$ is a geometric hyperplane of Γ. In that case, we say that the hyperplane H of Γ, *arises from the embedding e*.

One can easily argue at this point that if Veldkamp planes exist, and all hyperplanes arise from some embedding $e : \mathcal{P} \to \mathbb{P}$, then Teirlinck's condition holds and so by Lemma 1.1, the Veldkamp space is a projective space.

But, as was pointed out to me by Peter Johnson [Jh], the hypothesis that all hyperplanes arise from some embedding is so overwhelmingly strong, that we in fact have

Proposition (P. Johnson) *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a geometry of points and lines such that*

(i) Veldkamp lines exist

(ii) For some projective embedding $e : \mathcal{P} \to \mathbb{P}$ of the geometry Γ, every geometric hyperplane of Γ arises from the embedding.

Then the Veldkamp space $\mathcal{V}$ of Γ is a projective space.

Proof. From Lemma 4.2, to be proved in Section 4 of this paper, the hypothesis that Veldkamp lines exist implies

(1.1) *Every subspace W of codimension at most 2 in $\mathbb{P}$ is spanned by the image points of $e(\mathcal{P})$ contained in it.*

It follows (see Lemma 4.2(ii)) that e provides an isomorphism between $(\mathcal{V}_1, \mathcal{V}_2)$ and the points and lines of the dual projective space $\mathbb{P}^*$. Hence $\mathcal{V}$ is a projective space. ∎

We then obtain several answers to the posed question:

Corollary *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be one of the following geometries*

(i) a Grassmann space, $A_{n,d}(D)$, D a division ring

(ii) a half-spin geometry $D_{n,n}(F)$, F a field

(iii) a Lie incidence geometry of type $E_{6,1}(F)$, F a field.

Then the Veldkamp space of Γ is a projective space.

Proof. By Corollary 6.3 of this paper, Veldkamp lines exist for the listed geometries. By the Proposition it suffices to show that there exists a projective embedding $e : \Gamma \to \mathbb{P}$ from which every geometric hyperplane of Γ arises. This result for the half-spin geometries appears in [Sh4]. The result for Grassmann spaces is in [Sh3]. The result for $E_{6,1}$ appears in [CooSh]. ∎

2 Partial Matroids

A *dependence theory* on a set X is normally regarded as a relation D between X and its power set $P(X)$ satisfying these three axioms:

(i) (reflexivity) *Any element x depends on any finite set which contains it.*

(ii) (transitivity) *If x depends on A and every member of A depends on B, where A and B are finite sets, then x depends on B.*

(iii) (the exchange axiom) *If y depends on $\{x_1, \dots, x_n\}$ but not $\{x_1, \dots, x_{n-1}\}$, then x_n depends on $\{x_1, \dots, x_{n-1}, y\}$.*

We wish to consider here a *partial dependence theory* where the relation D satisfies (i) and (ii) above and

(iii-r) (Partial exchange axiom) *For each positive integer $s \leq r$, if x depends on $\{x_1, \dots, x_s\}$ but not $\{x_1, \dots, x_{s-1}\}$, then x_s depends on $\{x_1, \dots, x_{s-1}, x\}$.*

One can lift most of the definitions used for dependence theories to the partial case. We say that a set $\{x_1, \dots, x_n\}$ is *independent* if no x_i is dependent on its complement in this set. We say U is *spanned by a set* X if and only if every element of U depends on a finite subset of X; we say U is a *flat* if and only if U is the set of all elements spanned by some set X.

If F is a flat consisting of all elements spanned by an s-element set X, where $s \leq r$, then the restriction of D to $F \times P(F)$ is an ordinary dependence theory. If X is an independent set X is called a *basis* for F. Thus using (i) and (ii) it is easy to show

Lemma 2.1 *Let D be a partial dependence theory on set X satisfying the partial exchange axiom (iii-r). Then the following hold:*

(i) Suppose A and B are finite sets with $A \subseteq B$. If x depends on A then x depends on B.

(ii) If $s \leq r$ and $U = \{x_1, \dots, x_{s-1}\}$ is an independent set and x_s does not depend on U, then $\{x_1, \dots, x_{s-1}, x_s\}$ is also an independent set.

(iii) If F is the flat of all elements dependent on an s-set X where $s \leq r$, then any independent set of fewer than r elements extends to a basis and any two bases of F have the same cardinality.

3 The Set-intersection Model of a Partial Matroid

Let $\mathcal{F} = \{A_\sigma \mid \sigma \in I\}$ be a family of subsets of a set P indexed by I. We shall declare that a set $A \in \mathcal{F}$ *depends* on subset $\{A_1, \dots, A_n\}$ of $\mathcal{F}$, if and only if $A_1 \cap \dots \cap A_n \subseteq A$. The following observation is immediate:

(3.1) *The relation of "dependence" defined on $\mathcal{F} \times P(\mathcal{F})$ satisfies the axioms (i) (reflexivity) and (ii) (transitivity) of a dependence theory.*

Fix a positive integer r. Consider next the

(r-fold intersection property.) *For each positive integer s with $s \leq r$, and members $A, A_1, \ldots, A_s$ of $\mathcal{F}$, if*

$$A_1 \cap \cdots \cap A_s \subseteq A,$$

then either

$$A_1 \cap \cdots \cap A_{s-1} \subseteq A \quad \text{or} \quad A_1 \cap \cdots \cap A_{s-1} \cap A = A_1 \cap \cdots \cap A_s.$$

Here, if $s = 1$, the intersection $A_1 \cap \cdots \cap A_{s-1}$ is over an empty collection of sets and by convention is understood to mean the entire set P. Thus for $r = 1$, the r-fold intersection property asserts that in $\mathcal{F}$ it is impossible to have

$$A_\sigma \subset A_\tau \subset P \quad \text{(proper inclusions)}$$

for any indices σ and τ of I.

Lemma 3.1 *Suppose the family $\mathcal{F} = \{A_\sigma\}$ of subsets of P satisfies the r-fold intersection property. Then with respect to the definition of "dependence" preceding (3.1), one obtains a partial dependence theory satisfying the partial exchange axiom (iii-r).*

Proof. Axioms (i) and (ii) hold as noted in (3.1). Suppose $s \leq r$ and that A depends on $\{A_1, \ldots, A_s\}$, but not on $\{A_1, \ldots, A_{s-1}\}$. This means

$$A_1 \cap \cdots \cap A_s \subseteq A$$

but

$$A_1 \cap \cdots \cap A_{s-1} \not\subseteq A.$$

By the r-fold intersection property $A_1 \cap \cdots \cap A_{s-1} \cap A$ is contained in A_s, so A_s depends on $\{A_1, \ldots, A_{s-1}, A\}$. Thus the partial exchange axiom holds. ∎

Note that in this model, to say that $\{A_1, \ldots, A_n\}$ is an independent set is equivalent to saying that the intersection $A_1 \cap \cdots \cap A_n$ *does not remain the same* if any A_i is omitted from the intersection.

4 Veldkamp $(r-1)$-Spaces and Embeddings

We wish to apply the ideas of the previous section to the case that P is the set $\mathcal{P}$ of points of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ and $\mathcal{F}$ is the set $\mathcal{V}$ of all geometric hyperplanes of Γ. In section 1 we introduced three properties affecting the structure of $\mathcal{V}$: (1) Veldkamp points exist, (2) Veldkamp lines exist and (3) Veldkamp planes exist. But these three concepts are simply the r-fold intersection property for hyperplanes in the respective cases $r = 1$, 2 and 3.

For $r \geq 4$, we say *Veldkamp* $(r-1)$-*spaces exist* if and only if $\mathcal{V}$ satisfies the r-fold intersection property.

A sufficient condition for this is given in

Lemma 4.1 *Let $\mathcal{V}$ be the set of hyperplanes of a geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ and fix a positive integer r. Suppose, for any subset $\{A_1, \ldots, A_s\}$ of $\mathcal{V}$, where $s \leq r$, that*

$$A_1 \cap \cdots \cap A_{s-1} - A_1 \cap \cdots \cap A_s$$

has a connected collinearity graph (this includes the case that it is empty).

Then Veldkamp $(r-1)$-spaces exist.

Proof. We need only verify the r-fold intersection property. So fix $s \leq r$, and suppose $A, A_1, \ldots, A_s$ are hyperplanes of Γ, with $A_1 \cap \cdots \cap A_s \subseteq A$. Then we have

$$A_1 \cap \cdots \cap A_s \subseteq A_1 \cap \cdots \cap A_{s-1} \cap A \subseteq A_1 \cap \cdots \cap A_{s-1}.$$

The last set is a subspace S of Γ and the first set, $H = A_1 \cap \cdots \cap A_s$, is either equal to S or is a hyperplane of it. If $H = S$ all three sets are equal. But if they are not equal, H is a hyperplane of S and $S - H$ has a connected collinearity graph. It follows that H is a maximal subspace of S. Thus in either case, $H = A_1 \cap \cdots \cap A_{s-1} \cap A$ or else $A_1 \cap \cdots \cap A_{s-1} \cap A = A_1 \cap \cdots \cap A_{s-1}$, which implies $A_1 \cap \cdots \cap A_{s-1} \subseteq A$. But these two alternatives form the conclusion of the r-fold intersection property. Thus Veldkamp $(r-1)$-spaces exist. ■

Lemma 4.2 *Suppose Veldkamp $(r-1)$ spaces exist for $\Gamma = (\mathcal{P}, \mathcal{L})$.*

(i) If $e : \Gamma \to \mathbb{P}$ is an embedding of Γ, then every subspace $\mathbb{K}$ of codimension at most r in $\mathbb{P}(V)$ is spanned by the image points contained in it.

(ii) If, moreover, every hyperplane of Γ arises from the embedding $e : \Gamma \to \mathbb{P}$, then there is an incidence-preserving bijection between the subspaces of $\mathbb{P}$ of codimension at most r and the flats of $\mathcal{V}$ of dimension at most r. (The map converts codimension to dimension.)

Proof. (i) For each subspace U of $\mathbb{P}$ set $S(U) = \{p \in \mathcal{P} | e(p) \in U\}$. Clearly $S(U)$ is a subspace of Γ and if $\mathrm{codim}(U) = 1$, $S(U)$ is a hyperplane of Γ. We proceed by induction on r.

Assume $r = 1$ so Veldkamp points exist. Then all geometric hyperplanes are maximal subspaces. Let $\mathbb{H}$ be a hyperplane of $\mathbb{P}$ and suppose $\mathbb{K} = \langle e(\mathcal{P}) \cap \mathbb{H} \rangle_{\mathbb{P}}$ has codimension at least 2. There exists a point $p \in \mathcal{P} - S(\mathbb{K})$ and so there is a hyperplane $\mathbb{H}'$ containing $\langle \mathbb{K}, e(p) \rangle_{\mathbb{P}}$. Then $S(\mathbb{H})$ is properly contained in $S(\mathbb{H}')$ while both are geometric hyperplanes. This contradicts $S(\mathbb{H})$ maximal.

Now assume the result holds for all values of r less than k, and suppose Veldkamp $(r-1)$-spaces exist for Γ. Suppose by way of contradiction that $\mathbb{U}$ is a subspace of codimension k in $\mathbb{P}$ such that $\mathbb{K} = \langle e(\mathcal{P}) \cap \mathbb{U} \rangle_{\mathbb{P}}$ has codimension $> k$. Again choose $p \in \mathcal{P} - S(\mathbb{U})$. Then there is a subspace $\mathbb{U}'$ of codimension k in $\mathbb{P}$ containing

$\langle \mathbb{K}, e(p)\rangle_{\mathbb{P}}$. Since $\mathbb{U}$ and $\mathbb{U}'$ each have codimension k in $\mathbb{P}$, each can be expressed as the intersection of k independent hyperplanes $\mathbb{H}_i$ (or $\mathbb{H}'_i$) of $\mathbb{P}$, so

$$\mathbb{U} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_k, \quad \mathbb{U}' = \mathbb{H}'_1 \cap \cdots \cap \mathbb{H}'_k.$$

Setting $A_i = S(\mathbb{P}_i)$ and $A'_i = S(\mathbb{P}'_i)$ we see that

$$S(\mathbb{U}) = A_1 \cap \cdots \cap A_k \subsetneqq S(\mathbb{U}') = A'_1 \cap \cdots \cap A'_h. \tag{5.1}$$

Since Veldkamp $(k-1)$ spaces exist, the collection $\mathcal{V}$ forms a partial matroid with the partial exchange axiom (iii-k). The set F of all hyperplanes of Γ containing $S(\mathbb{U})$ is a flat spanned by $\{A_1, \ldots, A_k\}$.

Now the geometric hyperplanes $\{A'_i\}$ are independent since, by induction, any intersection over a *proper* subset of the $\mathbb{P}'_i$ is spanned by the points of $e(\mathcal{P})$ which it contains, and so the intersection over the corresponding A'_is must properly contain the full intersection.

Now a similar argument shows that the A_i are also independent. Thus $\{A_i\}$ is a basis of k-elements of the flat F, and $\{A'_i\}$ is an independent set of k elements which does *not* span F. This contradicts Lemma 2.3(iii).

(ii) Since Veldkamp $(r-1)$ spaces exist, the "points" of $\mathcal{V}$ form a partial matroid satisfying axiom (iii-r). Let $\mathbb{U}$ be a subspace of codimension $k \leq r$ in $\mathbb{P}$. Then as $\mathbb{U}$ is the intersection of k independent hyperplanes of $\mathbb{P}$, we can conclude as in part (i), that $S(\mathbb{U})$ is a k-dimensional flat. Since each such space $\mathbb{U}$ is generated by its image points, S induces an injection

$$S^* : \{\mathbb{U} \leq \mathbb{P} | \mathrm{codim}\mathbb{U} \leq r\} \to \{k - \text{flats of } V | k \leq r\}.$$

But this map is onto. Suppose X is a k-flat of V. Then X is all hyperplanes containing an intersection $A_1 \cap \cdots \cap A_k$ of independent hyperplanes. But each hyperplane A_i is $S(\mathbb{H}_i)$ for some projective hyperplane $\mathbb{H}_i$ of $\mathbb{P}$, since all hyperplanes arise from the embedding $e : \Gamma \to \mathbb{P}$. Thus

$$A_1 \cap \cdots \cap A_k = S(\mathbb{H}_1 \cap \cdots \cap \mathbb{H}_k := \mathbb{U})$$

so $X = S^*(\mathbb{U})$.

Thus S^* is a bijection. ■

5 Veldkamp Points and Lines for Polar Spaces

By a *polar space* we mean a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ satisfying the familiar "one or all" axiom for points and lines, namely:

(5.1) *For any point-line pair (p, L), p is collinear with either* 1 *or all points of L.*

The *radical*, $\mathrm{Rad}(\mathcal{P})$, is the set of points collinear with all other points; the polar space is called *non-degenerate* if $\mathrm{Rad}(\mathcal{P})$ is empty. There is a canonical procedure described in [BSh] for obtaining a non-degenerate polar space $\rho(\Gamma)$ from any polar space Γ. Its points are the equivalence classes $\rho(\mathcal{P})$ on $\mathcal{P} - \mathrm{Rad}(\mathcal{P})$ for the equivalence relation defined by the equation $x^\perp = y^\perp$. The restriction of the mapping $*$:

$\mathcal{P} - \mathrm{Rad}(\mathcal{P}) \to \rho(\mathcal{P})$ to any line not meeting $\mathrm{Rad}(\mathcal{P})$ is injective and the images of such restrictions form the lines $\rho(\mathcal{L})$ of the polar space $\rho(\Gamma)$.

A subspace S of $\Gamma = (\mathcal{P}, \mathcal{L})$ is said to be *singular* if any two of its points are collinear. The *singular rank* of Γ is the minimal length of an unrefinable chain of singular subspaces beginning with the empty subspace. (To avoid confusion on this point, the *length* of any properly ascending chain is the number of "upward hops" in it – that is, one less than the number of members in the chain.) If Γ is a polar space, the *reduced rank* of Γ is the singular rank of $\rho(\Gamma)$; if Γ is non-degenerate, the singular rank is just called the *rank*. Thus a non-degenerate polar space of rank 2 is a (non-degenerate) generalized quadrangle; a polar space of reduced rank zero is a singular space.

A polar space *has thick lines* if and only if each of its lines is incident with at least three distinct points.

The following well-known Lemma is seminal for what follows.

Lemma 5.2 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a polar space of reduced rank at least two having thick lines. Then for any hyperplane H, the set $\mathcal{P} - H$ has a connected collinearity graph – that is, Veldkamp points exist.*

Proof. Choose by way of contradiction, two points x and y lying in distinct connected components of $\mathcal{P} - H$. Then $x^\perp \cap y^\perp \subseteq H$ and, as neither x nor y lies in $\mathrm{Rad}(\mathcal{P})$ and Γ has reduced rank at least 2, $x^\perp \cap y^\perp$ is not a linear space. There is thus a non-collinear pair of points (u, v) in $x^\perp \cap y^\perp$. Let A and B be lines on $\{x, v\}$ and $\{x, u\}$ and C a line containing $\{v, y\}$. Then $H \cap A = \{u\}$ and $H \cap C = \{v\}$. Since C is thick, there is a point y' in $C - \{y, v\}$, which clearly belongs to the same connected component of $\mathcal{P} - H$ as does y. But by the axiom (5.1), y' is collinear to one or all points of A. But if y' were collinear with u then $u^\perp \cap C$ would contain $\{y, y'\}$ and hence all of C by (5.1), contradicting the fact that u is not collinear with v on C. Thus y' is not collinear with u and so is collinear with a point x' of $A - \{u\} = A - H$. But then (x, x', y', y) is a path of length three connecting x and y. This contradicts the choice of x and y and proves the Lemma. ∎

Corollary 5.3 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a polar space of reduced rank at least $r + 1$. Then Veldkamp $(r - 1)$-spaces exist for Γ.*

Proof. This proof depends on the fact that any subspace of a polar space is a polar space. A hyperplane of a polar space of reduced rank at least k is in fact a polar space of reduced rank at least k minus one. Thus for any integer $s \leq r$, and hyperplanes $A_1, \ldots, A_s$ of Γ, the intersection $A_1 \cap \cdots \cap A_{s-1}$ is a polar space of reduced rank at least 2 which is either equal to $A_1 \cap \cdots \cap A_s$ or contains the latter as a hyperplane. In either case (in the latter Lemma 5.2 must be employed) $A_1 \cap \cdots \cap A_{s-1} - A_1 \cap \cdots \cap A_s$ has a connected collinearity graph. The result now follows from Lemma 4.1. ∎

6 A Class of Strong Parapolar Spaces

If p and q are distinct points of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, a *path from p to q of length n* is a sequence of points $(x_0, x_1, \ldots, x_n)$ with $x_0 = p$, $x_n = q$ and $i = 1, \ldots, n$, x_{i-1} distinct from, but collinear with x_i. A path of minimal length from p to q is called a *geodesic* and its length $d_\Gamma(p,q)$ is called *the distance from p to q in Γ*. A subspace S of Γ is called *convex* if it contains every geodesic connecting any two of its points. $\mathcal{P}$ itself is a convex subspace, and clearly the intersection of any family of convex subspaces is a convex subspace. Thus, given any subset X of $\mathcal{P}$, the intersection of all convex subspaces of Γ containing X is a convex subspace denoted $\langle X \rangle_\Gamma$ and called *the convex closure of the set X*.

We introduce a family $\{\mathcal{E}_n \mid n = 1, 2, \ldots\}$ of classes $\mathcal{E}_n$ of point-line geometries. If $\Gamma \in \mathcal{E}_n$, $n \geq 2$, then Γ satisfies the following four axioms:

(E1) Γ is connected (i.e., $\mathcal{P}$ has a connected collinearity graph) and has thick lines.

(E2) (i) For any positive integer $k \leq n$, every geodesic $(x_0, \ldots, x_k)$ of length k completes to a geodesic $(x_0, \ldots, x_k, \ldots, x_n)$ of length n.

(ii) $\operatorname{diam}\Gamma := \max\{d_\Gamma(p,q) \mid p, q \in \mathcal{P}\}$ is n exactly and for each point p, the set $\Delta^*_{n-1}(p) := \{q \in \mathcal{P} \mid d_\Gamma(p,q) \leq n-1\}$ is a geometric hyperplane of Γ.

(E3) If p and q are distinct points of Γ with $d_\Gamma(p,q) = k$, then the convex closure $\langle p, q \rangle_\Gamma$ is a member of $\mathcal{E}_k$.

Remarks 1) The members of the set $\mathcal{E}_1$, strictly speaking, are not point-line geometries: Each $\Gamma \in \mathcal{E}_1$ is just a single thick line, since by, (E2)(ii) each point p, the sole member of $\Delta^*_0(p)$, must be a geometric hyperplane.

2) The geometries of $\mathcal{E}_2$ have diameter 2 and for each point p of a geometry Γ of $\mathcal{E}_2$, $p^\perp$ is a hyperplane of Γ. Thus they are non-degenerate polar spaces of rank at least two.

3) Similarly, if $n > 2$, and $\Gamma \in \mathcal{E}_n$, it is true that the convex closure of every pair (p,q) of points at distance 2 is a convex non-degenerate polar space of rank at least 2. We call such convex polar subspaces *symplecta*.

4) If all symplecta have rank at least 3, it is easy to see that Γ satisfies the axioms of a parapolar space without having special pairs (p,q) such that $|p^\perp \cap q^\perp| = 1$. (Such spaces are called *strong parapolar spaces*: see [CooSh] for definitions.)

If $\Gamma \in \mathcal{E}_n$, we denote the subcollection of all convex subspaces of the form $\langle p, q \rangle_\Gamma$ where $d_\Gamma(p,q) = k \leq n$, by the symbol $\mathcal{E}_k(\Gamma)$.

There are four "classical" models of geometry classes $\mathcal{E}_n$, $n \geq 2$.

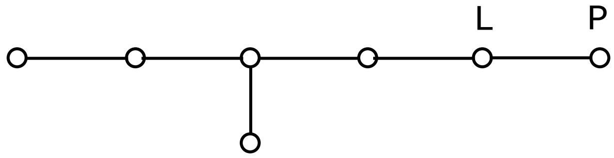
1. **The dual polar spaces.** Points are maximal singular subspaces of classical polar spaces of rank at least two (polar spaces of type D_n are excluded here since dual polar spaces of this type yield thin lines). These have been characterized by P. Cameron ([Ca]). Here, symplecta are generalized quadrangles.

2. **The Grassmann spaces of the form** $A_{2n-1,n}$. Points are n-dimensional subspaces of a $2n$-dimensional vector space; lines are rank 2 flags of dimension $(n-1, n+1)$. There are several characterizations of these geometries (see Proposition

6.1 of [Sh1] and Bichara and Tallini [BT]; but probably the best is that of Cohen ([Co2])). Here symplecta are rank 3 polar spaces of type $A_{3,2}$.

3. **The half-spin geometries** $D_{n,n}$ *where* n *is even.* Points are the members of one of the two classes of maximal singular subspaces of a vector space of dimension $2n$ admitting a quadratic form of maximal Witt index. Lines are totally singular $(n-2)$-dimensional subspaces. There are several characterizations of these geometries (Cooperstein [Coo], Cooperstein and Cohen [CooCo]; see also Shult [Sh2] for one based on singular subspaces rather than symplecta). Here symplecta are classical polar spaces of type D_4 and rank 4.

4. **The exceptional geometry** $E_{7,1}$ whose points and lines are the cosets of the maximal parabolic subgroups of $E_7(k)$ corresponding to the nodes "$\mathcal{P}$" and "$\mathcal{L}$" in the diagram



(This is called $E_{7,7}$ in [BCN].)

We shall require some elementary results on these geometries.

Lemma 6.1 *Let* Γ *be a member of* $\mathcal{E}_n$, $n \geq 2$, *and let* m *be any positive integer less than* n.

(i) For any point p, $\Delta^*_m(p)$ *is a subspace of* Γ.

(ii) For any hyperplane H *of* Γ, $\mathcal{P} - H$ *has a connected collinearity graph.*

Proof. (i) Let L be a line and suppose x and y are two distinct points of L in $\Delta^*_m(p)$. Then $L \subseteq \Delta^*_m(p)$ unless L contains a point v with $d_\Gamma(p, v) = m + 1$. Then $Y = \langle p, v\rangle_\Gamma \in \mathcal{E}_{m+1}(\Gamma)$ and contains x, y – and hence L – by convexity. Then as $\Delta^*_m(p)$ is a hyperplane of Y, we have $L \subseteq \Delta^*_m(p)$.

(ii) If $n = 2$, this is Lemma 4.2. We assume by way of contradiction that $n > 2$ is minimal such that (ii) is false. So there exists a geometry $\Gamma \in \mathcal{E}_n$, a hyperplane H of Γ with two points x and y in distinct connected components of $\mathcal{P} - H$, with n minimal with respect to these conditions. Then $d_\Gamma(x, y) = n$. Let $(x_0 = x, x_1, \ldots, x_n = y)$ be a geodesic connecting x and y and let $Y = \langle x_1, y\rangle_\Gamma \in \mathcal{E}_{n-1}(\Gamma)$. Let L be a line of Y on x_1 and x_2 and let $R = \langle x, x_2\rangle_\Gamma \in \mathcal{E}_2(\Gamma)$. Now, let A be a line on $\{x, x_1\}$ and let B be a line of $\mathrm{Res}(x)$ not in $A^\perp$. Then as $x^\perp \cap Y$ is, by the convexity of Y, a singular subspace, $B \cap Y = \emptyset$. Since B is thick, there is a point v in $B - \{\{x\} \cup H\}$, and $B \not\subseteq A^\perp$ implies v is not collinear with x_1. Thus $v^\perp \cap L = \{u\} \neq \{x_1\}$.

Now no point z in $\mathcal{P} - H$ exists, with $\max(d_\Gamma(x, z), d_\Gamma(y, z)) < n$, since, by minimality of n, $\langle x, z\rangle_\Gamma - H$ and $\langle y, z\rangle_\Gamma - H$ would both be connected. Thus $\Delta^*_{n-2}(x_1) \cap Y \subseteq H$. Since $\Delta^*_{n-2}(x_1) \cap Y$ is a hyperplane of Y, minimality of n forces $Y - \Delta^*_{n-2}(x_1)$ connected, whence $\Delta^*_{n-2}(x_1) \cap Y$ is a maximal subspace of Y. Thus $\Delta^*_{n-2}(x_1) \cap Y = H \cap Y$. Since v lies in the same connected component of $\mathcal{P} - H$ as x, then similarly $\Delta^*_{n-2}(u) \cap H = H \cap Y$. Thus $\Delta^*_{n-2}(x_1) \cap Y = \Delta^*_{n-2}(u) \cap Y$. But this is impossible since by axiom $E2(i)$, (u, x_1) completes to a geodesic $(u, x_1, a_2, \ldots, a_{n-1})$ of Y whence $a_{n-1} \in \Delta^*_{n-2}(x_1) \cap Y$ but is not in $\Delta^*_{n-2}(u) \cap Y$. This contradiction completes the proof. ∎

In the following, we wish to show that if $\mathcal{E}_n$ is a class of geometries satisfying the axioms (E1)-(E3), and the members of $\mathcal{E}_2$ all have reduced rank r, then there are r points $x_1, \ldots, x_r$ such that the hyperplanes $H_i = \Delta^*_{n-1}(x_i)$, $i = 1, \ldots, r$, are independent in the sense of section 3. In order to do this we must introduce the concept of gated sets.

Let $G = (V, E)$ be any connected graph and let $X \subseteq V$ define an induced subgraph and suppose y is any vertex. We say that X *is strongly gated with respect to* y if X contains a unique vertex g nearest y (called the *gate*) (with respect to the distance metric $d_G : V \times V \to \mathbb{Z}$) and for each vertex x in X we have

$$d_G(y, x) = d_G(y, g) + d_X(g, x).$$

Of course, if X is embedded isometrically as a subgraph of G, – as, for example, when X is convex in G – then the second term $d_X(g, x) = d_G(g, x)$ in which case we say that X *is gated with respect to* y. The latter concept seems to have been introduced by Dress and Scharlau ([DrSch]). Near polygons, for example, are point-line geometries in which each line is gated with respect to every point. Buildings are chamber systems in which residues are gated with respect to every chamber (Scharlau, [Sch]).

Lemma 6.2 *Suppose $\Gamma \in \mathcal{E}_n$, satisfying (E1) through (E3).*

(i) Suppose $Y \in \mathcal{E}_{n-1}(\Gamma)$ and that x is at distance 1 and at distance n from points of Y. Then Y is gated with respect to point x.

(ii) Suppose $T \in \mathcal{E}_m(\Gamma)$, $1 \le m < n$, and x is a point such that there are points a and b in T with $d_\Gamma(x, a) = n$, $d_\Gamma(x, b) = n - m$. Then T is gated with respect to x with gate b.

(iii) If $S \in \mathcal{E}_2(\Gamma)$ and $g \in S$, then there exists a point u with $d_\Gamma(g, u) = n - 2$ and S is gated with respect to u.

Proof. (i) Let $W = \{y \in Y \mid d_\Gamma(x, y) = n\}$. By hypothesis W is a non-empty subset of $Y - \Delta^*_{n-2}(g)$ where $g \in x^\perp \cap Y$, also given to be non-empty. Since Y is convex, $x^\perp \cap Y$ is a singular subspace. If $x^\perp \cap Y$ contained a line L, then for any $w \in W$, $\Delta^*_{n-2}(w) \cap Y$, being a hyperplane of Y, would meet L at a point t, so $n = d_\Gamma(x, w) \le d_\Gamma(x, t) + d_\Gamma(t, w) = 1 + n - 2 = n - 1$, an absurdity. Thus the singular space $x^\perp \cap Y$ consists of a single point g. Choose $w \in W$ and let N be any line of Y on w. Then $N \cap \Delta^*_{n-2}(g)$ is a point, and any point of $N - \Delta^*_{n-2}(g)$ also belongs to W. Thus W is a connected component of the collinearity graph on $Y - (\Delta^*_{n-2}(g) \cap Y)$. By Lemma 6.1(ii), the latter is connected. Thus $W = Y - (\Delta^*_{n-2}(g) \cap Y)$ – i.e., every point of Y at distance $n - 1$ from g is distance n from x.

Now suppose v is any point of Y at distance d from g. Then a geodesic path p from g to v can be extended to a geodesic path $p * q$ of length $n-1$ terminating at a point t. Since $t \in W$, $d(x,t) = n$ by the previous paragraph so $(x,g) * p * q$ is a geodesic from x to t. Thus its first part $(x,g) * p$ is a geodesic, whence $d_\Gamma(x,v) = 1 + d = d_\Gamma(x,g) + d_\Gamma(g,v)$. Thus Y is gated with respect to point x.

(ii) The result is true by (i) if $n - m = 1$; that is, $x^\perp \cap T \neq \emptyset$. We proceed by induction on $n-m$. Let $(b, x_1, \ldots, x_{n-m} = x)$ be a geodesic from b to x. Then as $d(a,x) = n$, $d(a, x_{n-m-1}) = n-1$ so $Y = \langle a, x_{n-m-1} \rangle \in \mathcal{E}_{n-1}(\Gamma)$ and x is at distance 1 and n from points of Y. Thus by (i) Y is gated with respect to x with gate x_{n-m-1}. It follows by induction on $n-m$ that T is gated (in Y) with respect to x_{n-m-1}. Noting that T and Y are convex in Γ, this means that for each $t \in T$,

$$\begin{aligned} d_\Gamma(t,b) + d_\Gamma(b,x) &= d_\Gamma(t,b) + d_\Gamma(b, x_{n-m-1}) + d_\Gamma(x_{n-m-1}, x) \\ &= d_Y(b, x_{n-m-1}) + d_\Gamma(x_{n-m-1}, x) = d_\Gamma(t,x). \end{aligned}$$

Thus T is gated with respect to x with gate b.

(iii) There is a geodesic (a,b,g) in S, which can be extended to a geodesic $(a,b,g,x_3,\ldots,x_n = u)$ of Γ. Then $d(a,u) = n$, $d(g,u) = n-2$ and (ii) applies to yield the result.

■

Corollary 6.3 *Suppose $\Gamma \in \mathcal{E}_n$, and that S is a symplecta of $\mathcal{E}_2(\Gamma)$. Suppose points $x_1, \ldots, x_k$ of S are such that the sets $\{x_i^\perp \cap S\}$ are independent sets of S. Then the k hyperplanes $H_i = \Delta^*_{n-1}(x_i)$, are independent sets of Γ, in the sense of section 3.*

Proof. By hypothesis, for each $(k-1)$-subset $U_j = \{x_1, \ldots, x_k\} - \{x_j\}$, there exists a point g_j of S such that $g_j^\perp \cap \{x_1, \ldots, x_k\} = U_j$. By Lemma 6.2(ii), there exists a point u_j in Γ, such that S is gated with respect to u_j with gate g_j. Then u_j lies in $\Delta^*_{n-1}(x_i)$ if and only if $i \neq j$. It is thus impossible for H_j to contain the intersection of the remaining H_i's for $i \neq j$, whence the sets $\{H_i \mid i = 1, \ldots, k\}$ are independent.
■

Corollary 6.4 *Suppose either (i), $X = \{x_1, \ldots, x_r\}$ are points of a symplecton S which span an $(r-1)$-dimensional projective space or (ii), $X = \{x_1, x_2, x_3, x_4\}$ are 4 points; any two of which are collinear except for the pair $\{x_1, x_4\}$. Then the sets $\{H_x = \Delta^*_{n-1}(x) \mid x \in X\}$ are independent.*

Proof. In case (i) $\langle X \rangle \simeq PG(r-1)$ and S has rank $\geq r$, and for any hyperplanes H of $\langle X \rangle$ there is a point s_H of S with $s_H^\perp \cap \langle X \rangle = H$. Thus $\{x_i^\perp \cap S \mid i = 1, \ldots, r\}$ are independent in S and Corollary 6.3 applies.

In case (ii) let $S = \langle x_1, x_4 \rangle$. Since S has rank ≥ 3, $x_1^\perp \cap x_4^\perp$ is a non-degenerate polar space in which there is a point u collinear with x_2 but not x_3 or vice versa. Then $u^\perp \cap X = \{x_1 x_4 x_2\}$ or $\{x_1 x_4 x_3\}$ as desired. Also $x_1^\perp \cap X = \{x_1 x_2 x_3\}$ and $x_4^\perp \cap X = \{x_2 x_3 x_4\}$. Thus the sets $\{x_i^\perp \cap S \mid i \simeq 1, \ldots, 4\}$ are independent in S, and the result again follows from Corollary 6.3. ■

The goal of the next few lemmas is to show that if A and B are subspaces of Γ with $A - B$ non-empty, then there exists a point x in $A - B$ which lies on a line not contained in A. This is not true for all geometries Γ, but it is true for strong parapolar spaces with thick lines, and that is the context for these lemmas. The proof uses the arguments on deep points outlined in [Sh3].

Throughout the rest of this section let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a strong parapolar space – that is, Γ is connected, the convex closure of every distance 2 pair of points is a symplecton, and every line lies in a symplecton. Suppose further that Γ has thick lines.

Let S be a proper subspace of Γ. A point x of S is said to be a *deep point* if $x^\perp \subseteq S$. We let $D_0(S)$ be the set of points of S which are not deep. Inductively, we define for $i \geq 1$,

$$D_i(S) := \{x \in S - (D_0(S) + \cdots + D_{i-1}(S)) \bigm| x^\perp \cap D_{i-1}(S) \neq \emptyset\},$$

the points not in any previous $D_j(S)$ which are collinear with a point of $D_{i-1}(S)$. Then there is a complete partition of the points of S:

$$S = D_0(S) + D_1(S) + \cdots$$

into sets of points of "increasing depth". For each positive integer k we set

$$D_k^*(S) = D_k(S) + D_{k+1}(S) + \cdots .$$

Lemma 6.5 *The sets $D_k^*(S)$ are subspaces of S.*

Proof. This is Lemma 2.2.2 of [Sh3]. If false, there is a line L carrying two points x and y of $D_k^*(S)$ and a point z in $D_{k-1}(S)$. By definition, z lies on a line N carrying a point u of $D_{k-2}(S)$ at distance 2 from x or y. (Note that by convention $D_{-1}(S) = \mathcal{P} - S$.) Form the symplecton $R = \langle x, u\rangle_\Gamma$. Then by induction on k, $D_{k-1}^*(S) \cap R$ is a subspace of R with two distinct deep points x and y. But that is impossible for a nondegenerate polar space R since the hyperplanes $x^\perp \cap R$ and $y^\perp \cap R$ are both maximal subspaces of R (see Lemma 6.1(ii) applied to R). ∎

Lemma 6.6 *Let A and B be proper subspaces of the strong parapolar space Γ having thick lines. If $A - B$ is non-empty, it contains a point lying on a line not in A.*

Proof. It suffices to show that $D_0(A) - B$ is non-empty. Since A is partitioned into the sets $D_i(A)$, there exists an integer k such that $D_k(A) - B$ is non-empty. But then we are done if we can show

(6.1) *For $j \geq 1$, $D_j(A) - B$ non-empty implies $D_{j-1}(A) - B$ is also non-empty.*

Let y be a point of $D_j(A) - B$, $j \geq 1$. Like any other point of $D_j(A)$, y must lie on a line – say L – carrying a point u of $D_{j-1}(A)$. Since L is thick and $D_j^*(A)$ is a subspace of A, $L \cap D_j^*(A) = \{y\}$ and $L - \{y\}$ contains at least two points, at most one of which lies in B. Thus L contains a point of $D_{j-1}(A) - B$. Thus the implication (6.1) holds and the proof is complete. ∎

Corollary 6.7 *A strong parapolar space with thick lines is never the union of two proper subspaces.*

Proof. Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ as above, and $\mathcal{P} = A \cup B$ where A and B are proper subspaces. Then $A - B$ is non-empty, and so by Lemma 6.5 there is a point x in $A - B$ on a line L not in A. Then as L is thick and meets A and B in at most one point each, L contains a point of $\mathcal{P} - (A \cup B)$, contrary to assumption. ■

7 Veldkamp Lines for Strong Parapolar Spaces

We can at least prove

Theorem 7.1 *Let $\{\mathcal{E}_n\}$, $n \geq 2$, be a family of point-line geometries satisfying axioms (E1)-(E3). Assume that for each $\Gamma \in \mathcal{E}_n$, all symplecta in $\mathcal{E}_2(\Gamma)$ have rank at least three. Then for any two hyperplanes A_1, A_2 of Γ, the set*

$$Z = A_1 - (A_1 \cap A_2)$$

is connected.

For any geometry $\Gamma \in \mathcal{E}_n$, $n \geq 2$, Veldkamp lines exist.

Proof. The last assertion follows from the connectivity result by Lemma 4.1.

It remains, then to show the connectivity of the set Z. By way of contradiction, we assume it to be false. Then there exists a minimal integer n such that $\mathcal{E}_n$ contains a geometry Γ with two hyperplanes A_1, A_2 such that $Z = A_1 - (A_1 \cap A_2)$ is non-empty and has a non-connected collinearity graph. By hypothesis (and Corollary 5.3), $n \geq 3$.

By Lemma 6.6 there is a point x in $A_1 - A_2$ lying on a line L not in A_1. Let y be a point of Z lying in a connected component of Z distinct from that containing x. Now if $d_\Gamma(x, y) = d < n$, then $R = \langle x, y \rangle_\Gamma \in \mathcal{E}_d(\Gamma)$ and so by minimality of n, $R \cap Z$ is connected, contrary to the choice of x and y. Thus we see

$$d_\Gamma(x, y) = n \tag{7.1}$$

and

(7.2) *there exists no point $z \in Z$ such that* $\max(d_\Gamma(x, z), d_\Gamma(y, z)) < n$.

Now the line L carries a point x_1 at distance $n-1$ from y. Then $Y = \langle x_1, y \rangle_\Gamma \in \mathcal{E}_{n-1}(\Gamma)$ and by (7.2)

$$Y \cap \Delta^*_{n-2}(x_1) \cap Z = \emptyset.$$

That is,

$$(Y \cap \Delta^*_{n-2}(x_1)) \cap A_1 \subseteq A_2. \tag{7.3}$$

Now choose a line N_i on x_1 in Y, and note that as Y is gated with respect to x, $N_i \not\subseteq x^\perp$, so the symplecton $R_i = \langle L, N_i \rangle_\Gamma$ can be formed.

Suppose, first, that $Z \cap R_i \subseteq x_1^\perp$. This means

$$R_i \cap A_1 \subseteq A_2 \cup x_1^\perp.$$

$R_i \cap A_1$ is itself a polar space. If it is degenerate, then $R_i \cap A_1 = p^\perp \cap R_i$ for some point p in R_i. As x_1 is not in A_1 each line on p in R_i contains just one point of $x_1^\perp$ and hence at least two points of A_2 whence $p^\perp \subseteq A_2$. But this contradicts $x \in A_1 - A_2$. Thus $R_i \cap A_1$ is a non-degenerate polar space of rank at least two. By Corollary 6.7, $R_i \cap A_1$ is contained in A_2 or in $x_1^\perp$. The first case is out because $x \in A_1 - A_2$. So assume $R_i \cap A_1 \subseteq x_1^\perp$. Now as Veldkamp points exist for R_i, and $R_i \cap A_1$ is a hyperplane of R_i, it is in fact a maximal subspace of R_i. But x_1 was chosen not in A_1 so $R_i = x_1^\perp$ against R_i non-degenerate.

Thus we can find a point z_i in $Z \cap R_i - x_1^\perp$. Then $z_i^\perp \cap N_i = \{v_i\} \neq x_1$. Since z_i belongs to the same connected component of Z as does x, we have

$$(Y \cap \Delta^*_{n-2}(v_i)) \cap A_1 \subseteq A_2.$$

Now $Y \cap A_1 \neq Y$ because of x_1, and $Y \cap A_2 \neq Y$ because of y. Thus $Y \cap A_1$ and $Y \cap A_2$ are distinct hyperplanes of Y, and as Y has Veldkamp lines

$$Y \cap \Delta^*_{n-2}(v_i) \supseteq A_1 \cap A_2 \cap Y. \tag{7.4}$$

Similarly

$$Y \cap \Delta^*_{n-2}(x) \supseteq A_1 \cap A_2 \cap Y.$$

Now there is a symplecton of rank ≥ 3 on x_1 within Y, and so there is a plane π of Y on x_1, and we can choose the lines N_i, $i = 1, 2$ distinct and in π. Then the three points $\{x_1, v_1, v_2\}$ span the plane π. Then by Lemma 6.3, the three hyperplanes

$$Y \cap \Delta^*_{n-2}(x_1), \quad Y \cap \Delta^*_{n-2}(v_i), \quad i = 1, 2$$

are independent in the sense of section 3. But that is a contradiction, since each of these contain $A_1 \cap A_2 \cap Y$ and Veldkamp lines exist for Y. ∎

Any result in this direction can be extended to geometries whose convex closures $\langle x, y\rangle_\Gamma$ belong to $\mathcal{E}_d$.

Specifically we have

Theorem 7.2 *Let Γ be a geometry with the property that for every point-pair (x, y) at mutual distance d, the convex closure $\langle x, y\rangle_\Gamma$ belongs to the class $\mathcal{E}_d$ satisfying axioms (E1)-(E3).*

If all the convex subgeometries $\langle x, y\rangle_\Gamma$ of diameter two or more, satisfy the condition

(7.5) For any $s \leq r$ hyperplanes $A_1, \ldots, A_s$,

$$A_1 \cap \cdots \cap A_{s-1} - A_1 \cap \cdots \cap A_s$$

is connected

then Γ itself satisfies condition (7.5).

Proof. Let $A_1, \ldots, A_s$ be hyperplanes of Γ, set $Z = A_1 \cap \cdots \cap A_{s-1} - A_1 \cap \cdots \cap A_s$ and suppose x and y belong to distinct connected components of the collinearity graph on Z. This is an impossible supposition since $\langle x, y\rangle_\Gamma \cap Z$ is connected by hypothesis. ■

Corollary 7.3 *All Grassmann spaces $A_{n,d}(D)$, all half-spin geometries $D_{n,n}(F)$ (n odd or even), and all Lie incidence geometries of type $E_{6,1}(F)$ or $E_{7,1}(F)$, (D a division ring, F a field) possess Veldkamp lines.*

Proof. For the geometries listed, all convex subgeometries $\langle x, y\rangle_\Gamma$ of diameter at least 2 belong to $\mathcal{E}_d$, $d = d_\Gamma(x, y)$ and have symplecta of rank 3. The conclusion thus follows from Theorem 7.1 and 7.2. ■

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