A multidimensional analogue of the Denjoy-Perron-Henstock-Kurzweil integral

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Introduction

Several new multidimensional integration theories that generalize or are analogous to the Denjoy-Perron-Henstock-Kurzweil integral were developed in recent years by several authors (e.g. [H], [L] (Generalized Riemann integral), [CD] (Generalized Denjoy integral), [M1] (GP-integral), [KMP] (BV-integral)). The purpose in the development of these theories has been to examine more general versions of the classical theorems in Lebesgue integration theory, such as the Divergence theorem, Fubini's theorem, or convergence theorems.

In this paper, we propose a simple elementary multidimensional integration that has a number of advantages also. First, as remarked in [L], unlike the one-dimensional case, a drawback of the known multidimensional integrals is that one cannot develop in the same system both Divergence and Fubini type theorems. This can be done with the integral presented here. Second, one main goal of the above theories has been to weaken the smoothness condition on the vector fields in the Divergence theorem. The continuous differentiability of the vector fields was replaced by their continuity and their pointwise, or asymptotic, or a.e. differentiability (with some other supplementary conditions). Here, we can remove all hypotheses about differentiability and prove a Divergence theorem for the class of all continuous vector fields (in fact, for a larger class of distributions). The third point concerns convergence theorems. In some of the previous integration theories, the convergence theorems are rather complicated (see e.g. Ch. 5, [L]), and in some others, they seem

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to be incomplete (for example, in [M1] or [CD], where there were forms of monotone convergence theorems, but dominated convergence type theorems were missing). In the present paper, we prove some simple general convergence theorems that admit both monotone and dominated convergence theorems as direct consequences. Motivated by the earlier study of a one-dimensional integration ([ALV]), we construct our space of integrable distributions as the completion of the space of continuous functions with respect to a certain norm. This permits us to use the methods of Functional Analysis to investigate the structure of this space and the integral on it.

As an illustration we present the solution of a simple initial value problem for a hyperbolic equation, where very little smoothness is required of the initial data.

To simplify the presentation, we consider here integration in the plane. The general case can be carried out in much the same way.

The paper consists of four parts. In Part 1, we define the class of G-integrable distributions and the integral on it. Some basic properties such as ordering and relationships with the Lebesgue integral are also considered in this part. In Part 2, we prove some Fubini type theorems for G-integrable distributions. A Green's theorem is derived in Part 3 and Part 4 is devoted to convergence theorems.

1 Integration in the class $G(Q)$

1.1 Construction of G-integral

Let $a, b, c, d \in \mathbb{R},\ a < b$ and $c < d$. In the sequel, we denote by Q the (open) rectangle $(a, b) \times (c, d)$ in $\mathbb{R}^2$, and put, for simplicity, $\partial = \partial_{12} = \partial_{21}$ in $D'(Q)$ (the space of distributions on Q with $D(Q)$ the space of test functions). The class of mappings we shall work with is given by $G(Q)$, where

$$G(Q) = \{\partial F \in D'(Q) : F \in C(\overline{Q})\}.$$

To define an integral on $G(Q)$, we need the following:

Lemma 1 *Let $F \in C(\overline{Q})$. Then $\partial F = 0$ (in $D'(Q)$) if and only if there exist $H \in C([a, b])$, $K \in C([c, d])$ such that*

$$F(x, y) = H(x) + K(y),\ (x, y) \in \overline{Q}. \tag{1}$$

Proof. It is clear that if F is of the form (1) then $\partial F = 0$. Conversely, suppose $0 = \partial F$. From $\partial_1(\partial_2 F) = 0$, we have $\partial_2 F = 1_{(a,b)} \otimes g$ for some $g \in D'(c, d)$ (here $\otimes$ denotes the tensor product between $D'(a, b)$ and $D'(c, d)$ and $1_{(a,b)}$ is the constant function 1 on (a, b)). Letting K be a primitive of g, we have $\partial_2(F - 1_{(a,b)} \otimes K) = 0$. Hence, there exists $H \in D'(a, b)$ such that $F - 1_{(a,b)} \otimes K = H \otimes 1_{(c,d)}$, i.e., $F = 1_{(a,b)} \otimes K + H \otimes 1_{(c,d)}$. We prove that H and K can be identified with continuous functions. Indeed, choosing $g_0 \in D(c, d)$ such that $\int_c^d g_0 = 1$, we have $\langle F, f \otimes g_0\rangle = \langle 1_{(a,b)}, f\rangle\langle K, g_0\rangle + \langle H \otimes 1_{(c,d)},\ f \otimes g_0\rangle = \langle H, f\rangle + \langle\langle K, g_0\rangle 1_{(a,b)}, f\rangle$, $f \in D(a, b)$. Putting

$$F_1 = \int_c^d F(\cdot, y) g_0(y) dy \in C([a, b]),$$

we have
$$\langle H, f\rangle = -\langle\langle K, g_0\rangle 1_{(a,b)}, f\rangle + \langle F_1, f\rangle, \forall f \in D(a,b),$$
i.e.,
$$H = \langle K, g_0\rangle 1_{(a,b)} + F_1 \in C([a,b]).$$
Similarly, $K \in C([c,d])$. For $f \otimes g \in D(a,b) \otimes D(c,d)$, we have

$$\begin{aligned}
&\langle H(x) + K(y), f \otimes g\rangle \\
&= \int_a^b H(x)f(x)dx \int_c^d g(y)dy + \int_a^b f(x)dx \int_c^d K(y)dy \\
&= \langle H \otimes 1_{(c,d)}, f \otimes g\rangle + \langle 1_{(a,b)} \otimes K, f \otimes g\rangle \\
&= \langle F, f \otimes g\rangle.
\end{aligned}$$

Since $D(a,b) \otimes D(c,d)$ is dense in $D(Q)$ (cf. [Ho], [K]), we have $H(x)+K(y) = F(x,y)$ in $D'(Q)$. Since the functions in both sides are continuous on $\overline{Q}$, we must have $H(x) + K(y) = F(x,y), \forall (x,y) \in \overline{Q}$. ∎

For $f \in G(Q)$, we put
$$I(f) = \{F \in C(\overline{Q}) : \partial F = f \text{ in } D'(Q)\}.$$
From Lemma 1, we immediately deduce:

Lemma 2 *If $f \in G(Q)$, then*
$$\begin{aligned}
&F_1(x,y) + F_1(a,c) - F_1(a,y) - F_1(x,c) \\
&= F_2(x,y) + F_2(a,c) - F_2(a,y) - F_2(x,c)
\end{aligned}$$
for all $F_1, F_2 \in I(f), (x,y) \in \overline{Q}$. Moreover, there exists a unique $F(f) \in I(f)$ such that
$$F(f)(a,y) = F(f)(x,c) = 0,$$
$\forall x \in [a,b], y \in [c,d]$.

This leads to the following definition.

Definition 1 *Let $f \in G(Q)$ (f is said to be G-integrable on Q) and let $Q' = (a',b') \times (c',d') \subset Q$. We put*
$$\int_{Q'} f = F(f)(b',d') + F(f)(a',c') - F(f)(a',d') - F(f)(b',c'),$$
where $F(f)$ is given by Lemma 2.

From Lemma 2, it is seen that Definition 1 is still valid if we replace $F(f)$ by any $F \in I(f)$. In particular, $\int_Q f = F(f)(b,d)$. For $f \in G(Q)$, we define
$$\|f\| = \sup\left\{\left|\int_{(a,x)\times(c,y)} f\right| : (x,y) \in \overline{Q}\right\}.$$
Now, let
$$\hat{C}(Q) = \{f \in C(\overline{Q}) : f(a,y) = f(x,c) = 0,\ \forall x \in [a,b], y \in [c,d]\}.$$
$\hat{C}(Q)$ is closed with respect to the usual sup-norm $\|\cdot\|_\infty$. Since the mapping $G(Q) \to \hat{C}(Q)$, $f \mapsto F(f)$ can be checked to be linear, bijective, and norm-preserving, we have:

Theorem 1 *$(G(Q), \|\cdot\|)$ is a separable Banach space, isomorphic to $(\hat{C}(Q), \|\cdot\|_\infty)$.*

We next consider some basic properties of the integral just defined.
Since the mapping $f \mapsto F(f)$ is linear, we immediately deduce

$$\int_Q (sf + tg) = s\int_Q f + t\int_Q g, \forall f, g \in G(Q), s, t \in \mathbb{R}.$$

More properties are given below.

1.2 On ordering G(Q)

The ordering in this case is much more complicated than that in the one-dimensional case. As usual, for $f, g \in G(Q)$, we say that $f \geq g$ if and only if $f - g$ is a positive measure on Q. For $f \in D'(Q)$, we know, by the Riesz Representation Theorem, that if f is a positive or bounded (Radon) measure on Q then it can be identified with a Borel measure (in the set function sense) $\mu = \mu_f$, which is also positive or bounded. We can define in these cases

$$F(x, y) = F_f(x, y) = \mu_f((a, x) \times (c, y)),\ (x, y) \in \overline{Q}.$$

We need the following lemma for our study of ordering on $G(Q)$.

Lemma 3 *(i) If f is a bounded measure on Q then $F_f \in L^\infty(Q)$ is the distributional primitive of f, i.e., $\partial F_f = f$ in $D'(Q)$.*

(ii) If in (i), we assume furthermore that $f \in G(Q)$, then

$$F_f(x, y) = F(f)(x, y),\ \forall (x, y) \in Q. \tag{2}$$

(iii) If $f \in G(Q)$ is a positive measure then (2) holds.

Proof. We sketch here only the main features, and leave the details to the reader.

(i) We first define

$$Z((x, y), (s, t)) = \chi_{(a,x)\times(c,y)}(s, t),\ (x, y), (s, t) \in \overline{Q}.$$

We have

$$F(x, y) = \int_Q Z((x, y), (s, t)) d\mu(s, t)$$

and $F \in L^\infty(Q)$. For $g \in D(Q)$,

$$\begin{aligned} \langle F, \partial g\rangle &= \int_Q \Big(\int_Q Z((x, y), (s, t))\partial g(x, y) d\mu(s, t)\Big) dxdy \\ &= \int_Q \bigg(\int_Q Z((x, y), (s, t))\partial g(x, y) dxdy\bigg) d\mu(s, t). \end{aligned}$$

But

$$\begin{aligned} \int_Q Z((x, y), (s, t))\partial g(x, y) dxdy &= \int_s^b \int_t^d \partial_{12} g(x, y) dxdy \\ &= -\int_s^b \partial_1 g(x, t) dx = g(s, t), \end{aligned}$$

hence

$$\begin{aligned}\langle F, \partial g\rangle &= \int_Q g(s,t)d\mu(s,t)\\ &= \langle f, g\rangle,\ \forall g \in D(Q).\end{aligned}$$

(ii) Let $f \in G(Q)$, and let $J = F - F(f)$. Then $J \in L^\infty(Q)$ and $\partial J = 0$ in $D'(Q)$. We first prove that

$$\lim_{x\to a+} F(x, y_0) = \lim_{y\to c+} F(x_0, y) = 0,\ \forall x_0 \in [a,b], \forall y_0 \in [c,d].$$

In fact, we have the Hahn decomposition $\mu = \mu^+ - \mu^-$, where $\mu^\pm$ are positive bounded measures on Q. Let $\{x_n\}$ be any sequence decreasing to a, since $\cap_{n\geq 1}[(a, x_n) \times (c, y_n)] = \emptyset$,

$$\lim \mu^\pm((a, x_n) \times (c, y_0)) = 0.$$

Hence

$$F(x_n, y_0) = \mu^+((a, x_n) \times (c, y_0)) - \mu^-((a, x_n) \times (c, y_0)) \to 0.$$

On the other hand, we have, as in Lemma 1, $H \in D'(a,b), K \in D'(c,d)$ such that $J = 1_{(a,b)} \otimes K + H \otimes 1_{(c,d)}$. Since $F, F(f)$ are bounded, we have $H \in L^\infty(a,b), K \in L^\infty(c,d)$. As in the last part of the proof of Lemma 1, this implies that

$$J(x,y) = H(x) + K(y), \quad \text{a.e. } (x,y) \in Q. \tag{3}$$

By Fubini's theorem, there exists $B \subset [c,d]$, $m_1([c,d]\backslash B) = 0$ (m_1 is the one-dimensional Lebesgue measure) such that (3) holds for all $y \in B$, a.e. $x \in [a,b]$. Let $(a', b') \subset (a,b)$. Since $\lim_{y\to c} J(x,y) = 0, \forall x \in [a,b]$ and $|J(x,y)| \leq |\mu|(G) + \|F(f)\|_\infty$, $(x,y) \in Q$, the dominated convergence theorem gives

$$\lim_{y\to c}\int_{a'}^{b'} J(x,y)dx = 0.$$

This implies that

$$(b'-a')^{-1}\int_{a'}^{b'} H(x)dx = \lim_{y\to c,\, y\in B} K(y)$$

for all subintervals $(a', b') \subset (a,b)$. Applying Lebesgue's theorem, one has

$$H(x) = \lim_{y\to c, y\in B} K(y)$$

fora.e. $x \in (a,b)$, i.e., H=const a.e. on (a,b). Similarly, K=const a.e. on (c,d). Thus, J=const a.e. on Q. This constant must be 0 since $\lim_{y\to c} J(x,y) = 0$. Hence $F(x,y) = F(f)(x,y)$ a.e. on Q. Using the continuity of $F(f)$ and the Hahn decomposition of μ, we see that this equality actually holds for all $(x,y) \in \overline{Q}$.

(iii) Let $f \in G(Q)$ be a positive measure on Q. We choose sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ such that $a_n < b_n,\ c_n < d_n, [a_n, b_n] \times [c_n, d_n] \subset Q,\ \forall n$ and $a_n \searrow a$, $b_n \nearrow b$, $c_n \searrow c$, $d_n \nearrow d$ as $n \to \infty$. Put $Q_n = (a_n, b_n) \times (c_n, d_n)$ and $f_n = f|_{Q_n}$, $\mu_n = \mu|_{Q_n}$. μ_n is a positive measure on Q_n and μ_n represents f_n. Since $\overline{Q_n}$ is compact, μ_n is bounded. Applying (ii), we have

$$\begin{aligned}F(f_n)(x,y) &= \mu_n((a_n, x) \times (c_n, y))\\ &= \mu((a_n, x) \times (c_n, y)),\end{aligned}$$

for all $(x,y) \in \overline{Q_n}$. Let $(x,y) \in Q$. Choosing m large enough such that for $(x,y) \in Q_m$, we have

$$\mu((a_n,x)\times(c_n,y)) = F(f)(x,y) - F(f)(a_n,y) - F(f)(x,c_n) + F(f)(a_n,c_n), \forall n \geq m.$$

On the other hand, we have

$$\begin{aligned} \lim \mu((a_n,x)\times(c_n,y)) &= \mu\left(\bigcup_{n\geq m}(a_n,x)\times(c_n,y)\right) \\ &= \mu((a,x)\times(c,y)). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} F(x,y) &= \mu((a,x)\times(c,y)) \\ &= F(f)(x,y) - F(f)(a,y) - F(f)(x,c) + F(f)(a,c) \\ &= F(f)(x,y). \end{aligned}$$

This show that $F = F(f)$ in Q. The extension of this equality to the boundary of Q is straightforward. ■

We immediately obtain from Lemma 3 the order preserving property of the G-integral, i.e.,

$$\int_Q f \geq \int_Q g,$$

whenever $f \geq g,\ f,g \in G(Q)$. In fact, since $f - g \in G(Q)$ is a positive measure, Lemma 3 (iii) gives $\int_Q f - \int_Q g = F(f-g)(b,d) = \mu_{(f-g)}(Q) \geq 0$. We also have other usual relations between the G-integral and the ordering. For example, we have the following result:

Corollary 1 *If $f,g,h \in D'(Q)$, $f \leq g \leq h$ and if f and h are G-integrable, then g is also G-integrable.*

Proof. Without loss of generality, we can assume that $f = 0$, i.e., g, h are positive measures on Q and $g \leq h$. Letting μ_g, μ_h be the Borel measures corresponding to g and h, we have $0 \leq \mu_g \leq \mu_h$. Lemma 3 (iii) implies that μ_h is a bounded measure on Q and $F(h)(x,y) = \mu_h((a,x)\times(c,y))$, $(x,y) \in \overline{Q}$. Then μ_g is also bounded. Putting $F(x,y) = \mu_g((a,x)\times(c,y))$, $(x,y) \in \overline{Q}$, one has from Lemma 3 (ii) that $\partial F = g$. We check that $F \in C(\overline{Q})$. Let $(x,y) \in Q$. If F is not continuous at (x,y) then we can find a sequence $\{(x_n,y_n)\}$ converging to (x,y) such that $\inf_n |F(x_n,y_n) - F(x,y)| > 0$. Divide Q into four subrectangles:

$$\begin{cases} Q_1 = (a,x]\times(c,y], & Q_2 = (x,b)\times(c,y), \\ Q_3 = [x,b)\times[y,d), & Q_4 = (a,x)\times(y,d). \end{cases}$$

Passing to a subsequence if necessary, we can assume that $(x_n,y_n) \in Q_i$ for some $i \in \{1,2,3,4\}$. Assume $i = 1$. Passing once more to a subsequence, we can assume that $x_n \nearrow x$, $y_n \nearrow y$. Hence

$$\mu_g((a,x_n)\times(c,y_n)) \to \mu_g\left(\bigcup_{n\geq 1}(a,x_n)\times(c,y_n)\right) = \mu_g((a,x)\times(c,y)),$$

contradicting the choice of (x_n, y_n). Assume now $i = 2$. As above we can also assume that $x_n \searrow x$, $y_n \nearrow y$. Simple calculations show that

$$\begin{aligned} |F(x,y) - F(x_n, y_n)| &\leq \mu_g((a,x) \times [y_n, y)) + \mu_g([x_n, x) \times (c, y_n)) \\ &\leq \mu_h((a,x) \times [y_n, y)) + \mu_h([x_n, x) \times (c, y_n)) \\ &= F(h)(x,y) - F(h)(x, y_n) - F(h)(x_n, y) + F(h)(x_n, y_n). \end{aligned}$$

By the continuity of $F(h)$, the right hand side of this inequality tends to 0. We obtain again a contradiction. The cases $i = 3, 4$ are treated in the same way. We have proved that F is continuous at $(x, y) \in Q$. The continuity of F on ∂Q is established similarly. ■

1.3 Some further properties

The usual restriction and extension properties of the G-integral is given in the following theorem.

Theorem 2 *(i) Let $f \in G(Q)$ and $Q' = (a', b') \times (c', d') \subset Q$. Then $f|_{Q'} \in G(Q')$ and*

$$F(f|_{Q'})(x,y) = F(f)(x,y) - F(f)(a',y) - F(f)(x,c') + F(f)(a',c'), \forall (x,y) \in \overline{Q'}.$$

(ii) For $a \leq m \leq b$ and $c \leq n \leq d$, let

$$Q_1 = (a,m) \times (c,n), Q_2 = (m,b) \times (c,n), Q_3 = (a,m) \times (n,d), Q_4 = (m,b) \times (n,d).$$

Then for each $(f_1, f_2, f_3, f_4) \in \prod_{i=1}^{4} G(Q_i)$, there exists a unique $f \in G(Q)$ such that $f|_{Q_i} = f_i$, $1 \leq i \leq 4$. Moreover,

$$\int_Q f = \sum_{i=1}^{4} \int_{Q_i} f_i.$$

(iii) Let $a < m < m' < b$ and $f \in D'(Q)$. Then $f \in G(Q)$ if and only if $f|_{(a,m') \times (c,d)} \in G((a,m') \times (c,d))$ and $f|_{(m,b) \times (c,d)} \in G((m,b) \times (c,d))$.

If $Q_1 = (a, m_1) \times (c, n_1)$, $Q_2 = (m_2, b) \times (c, n_2)$, $Q_3 = (a, m_3) \times (n_3, d)$, $Q_4 = (m_4, b) \times (n_4, d)$ are subrectangles of Q such that $\bigcap_{i=1}^{4} Q_i \neq \emptyset$, then for every $f \in D'(Q)$, $f \in G(Q)$ if and only if $f|_{Q_i}$ for all $i \in \{1, 2, 3, 4\}$.

We conclude this part with the important fact that the G-integral represents an extension of the Lebesgue integral.

Theorem 3 *If we identify $f \in L^1(Q)$ with the distribution*

$$f : g \mapsto (L) \int_Q fg, \; g \in D(Q),$$

where $(L) \int$ is the Lebesgue integral, then $f \in G(Q)$ and $\int_Q f = (L) \int_Q f$. Moreover, $G(Q)$ is the completion of $L^1(Q)$ (or $C(\overline{Q})$) with respect to the norm

$$\|f\| = \sup \left\{ \left| (L) \int_a^x \int_c^y f \right| : (x,y) \in Q \right\}.$$

Proof. Let $f \in L^1(Q)$, and let

$$F(x,y) = (L)\int_a^x \int_c^y f,\ (x,y) \in Q.$$

Then $F \in \hat{C}(Q)$. For $g \in D(Q)$, by Fubini's theorem and the integration by parts formula, we have

$$\begin{aligned}
(L)\int_Q fg &= -\int_a^b \left[\int_c^d \left(\int_c^y f(x,t)dt\right) \partial_2 g(x,y)dy\right] dx \\
&= -\int_a^b \left[\int_c^d \partial_1 F(x,y)\partial_2 g(x,y)dx\right] dy \\
&= \int_c^d \left[\int_a^b F(x,y)\partial_{12} g(x,y)dx\right] dy \\
&= (L)\int_Q F\partial g.
\end{aligned}$$

We therefore conclude that $f = \partial F$ in $D'(Q)$. Thus $f \in G(Q)$ and $F = F(f)$. In particular,

$$(L)\int_Q f = F(b,d) = F(f)(b,d) = \int_Q f,$$

and $L^1(Q) \subset G(Q)$. Now, let $f \in G(Q)$. Choose a sequence $\{F_n\} \subset C^2(Q)$ such that $F_n \to F(f)$ uniformly on $\overline{Q}$. Since $F_n(a,\cdot) \to 0$ uniformly on $[c,d]$, $F_n(\cdot,c) \to 0$ uniformly on $[a,b]$, by replacing $F_n(x,y)$ by $F_n(x,y) - F_n(a,y) - F_n(x,c) + F_n(a,c)$, we can assume that $F_n(a,\cdot) = 0$, $F_n(\cdot,c) = 0$. Thus $f_n = \partial F_n \in C(\overline{Q}) \subset G(Q)$ and $F(f_n) = F_n$, $\forall n$. We have

$$\|f_n - f\| = \|F(f) - F_n\|_\infty \to 0,$$

proving the density of $C(\overline{Q})$ (and thus of $L^1(Q)$) in $G(Q)$. ∎

2 Fubini theorems for G-integrable distributions

In this section, we consider some Fubini type theorems for the G-integral. We next apply these results to some initial value problems for the two-dimensional wave equation with nonsmooth initial data.

2.1 Fubini theorems

We first make some remarks on traces of integrals of G-integrable distributions. For $f \in C(\overline{Q})$ and $x \in [a,b]$, the function

$$\int_a^x f(s,\cdot)ds : [c,d] \to \mathbb{R},\ y \mapsto \int_a^x f(s,y)ds$$

clearly satisfies

$$\int_a^x f(s,\cdot)ds = [F(f)(x,\cdot)]' \text{ on } [c,d].$$

Generalizing to the case $f \in G(Q)$, we define, for $f \in G(Q)$, $x \in [a,b]$, $y \in [c,d]$:

$$\int_a^x f(s,\cdot)ds = [F(f)(x,\cdot)]' \text{ in } D'(c,d), \quad \int_c^y f(\cdot,t)dt = [F(f)(\cdot,t)]' \text{ in } D'(a,b).$$

It is clear that

$$\int_a^x f(s,\cdot)ds \in G(c,d), \quad \int_c^y f(\cdot,t)dt \in G(a,b),$$

where $G(a,b)$ and $G(c,d)$ are respectively the spaces of G-integrable distributions on (a,b) and (c,d), i.e.

$$G(a,b) = \{g' \in D'(a,b) : g \in C[a,b]\},$$

where g' is the distributional derivative of g, etc.

The consistency of the above definition can also be seen by remarking that if $f \in G(Q)$, $\{f_n\} \subset C(\overline{Q})$, $f_n \to f$ in $G(Q)$, then $\int_a^x f_n(s,\cdot)ds \to \int_a^x f(s,\cdot)ds$ in $G(c,d)$, and $\int_c^y f_n(\cdot,t)dt \to \int_c^y f(\cdot,t)dt$ in $G(a,b), x \in [a,b]$, $y \in [c,d]$. This means that the mapping $G(Q) \to G(c,d)$, $f \mapsto \int_a^x f(s,\cdot)ds$ is the unique extension of the mapping

$$C(\overline{Q}) \to G(c,d), \ f \mapsto \int_a^x f(s,\cdot)ds.$$

We are now in a position to prove a simple Fubini type theorem for $G(Q)$.

Theorem 4 *For $f \in G(Q)$,*

$$\int_Q f = \int_a^b \left(\int_c^d f(\cdot,t)dt \right) = \int_c^d \left(\int_a^b f(s,\cdot)ds \right).$$

Proof. We first remark that the above repeated integrals exist in the sense of the one-dimensional G-integral. Since $F(f)(\cdot,d)(a) = F(f)(a,d) = 0$, one has

$$F(f)(\cdot,d) = F\left(\int_c^d f(\cdot,t)dt \right) \text{ and } \int_a^b \left(\int_c^d f(\cdot,t)dt \right) = F(f)(b,d) = \int_Q f.$$

This proves the first equality. The second is proved in a similar way. ∎

We next derive another form of Fubini's theorem for some subclasses of $G(Q)$. A fundamental property of those classes is that one can define traces of their elements on segments parallel to the sides of Q. We put

$$G_i(Q) = \{\partial_i F : F \in C(\overline{Q})\}, i = 1,2,$$

and

$$\begin{aligned} G_1^*(Q) = \{&\partial_1 F : F \in L^1(Q),\ F(\cdot,y) \in C([a,b]),\ \text{a.e. } y \in [c,d], \\ &\text{and } \exists g = g(F) \in L^1(c,d) \text{ such that } |F(x,\cdot)| \le g,\ \forall x \in [a,b]\}, \end{aligned}$$

and

$$\begin{aligned} G_2^*(Q) = \{&\partial_2 F : F \in L^1(Q),\ F(x,\cdot) \in C([c,d]),\ \text{a.e. } x \in [a,b] \\ &\text{and } \exists g = g(F) \in L^1(a,b) \text{ such that } |F(\cdot,y)| \le g,\ \forall y \in [c,d]\}. \end{aligned}$$

Some elementary properties of these classes are given in the following proposition.

Proposition 1 *(i)* $G_i(Q) \subset G_i^*(Q),\ i = 1, 2.$
(ii) $L^1(Q) \subset G_1^*(Q) \cap G_2^*(Q)$, *and* $G_1(Q) \cup G_2(Q) \subset G(Q)$.
(iii) If $f \in G_i(Q)$ *then* $\partial f_j \in G(Q)(j \in \{1, 2\}$ *and* $\{i\} = \{1, 2\} \setminus \{j\})$

For the definition of traces, we need the following lemma.

Lemma 4 *If* $F \in L^1(Q)$ *and if* $\partial_1 F = 0$ *in* $D'(Q)$, *then there exists* $g \in L^1(c, d)$ *such that* $F(x, y) = g(y)$ *a.e.* $x \in (a, b),\ y \in (c, d)$.

The proof of the lemma relies on arguments similar to those used in the proof of Lemma 1, and is thus omitted. Now suppose $f \in G_1^*(Q)$ and that $F, F_1 \in L^1(Q)$ are as in the definition of $G_1^*(Q)$, i.e. $f = \partial_1 F = \partial_1 F_1$. It follows from Lemma 4 that $[F(\cdot, y)]' = [F_1(\cdot, y)]'$ in $D'(a, b)$, which proves the consistency of the following definition.

Definition 2 *Let* $f \in G_1^*(Q)$. *We put, for almost all* $y \in [c, d]$, $f(\cdot, y) = [F(\cdot, y)]'$ *in* $D'(a, b)$, *where* $F \in L^1(Q)$, $\partial_1 F = f$ *as in the definition of* $G_1^*(Q)$.

We remark that this definition generalizes the one usually given for traces of continuous functions. Indeed, suppose $f \in C(\overline{Q})$. Consider

$$F(x, y) = \int_a^x f(s, y)ds, (x, y) \in \overline{Q}.$$

Since $f(x, y) = \dfrac{dF(x, y)}{dx}$, $(x, y) \in \overline{Q}$, one has $\partial_1 F = f$ in $D'(Q)$, and $f(\cdot, y) = [F(\cdot, y)]'$ in $D'(a, b)$. We have a similar definition for $f(x, \cdot)$ if $f \in G_2^*(Q)$. We see that if $f \in G_1^*(Q)$ then $f(\cdot, y) \in G(a, b)$ for a.e. $y \in [c, d]$. Hence the integral $\int_a^b f(\cdot, y)$ exists as a one-dimensional G-integral. Summarizing we have the following theorem.

Theorem 5 *If* $f \in G(Q) \cap G_1^*(Q)$, *then the function*

$$y \mapsto \int_a^b f(\cdot, y),\ y \in [c, d]$$

is Lebesgue integrable on $[c, d]$, *and* $\int_Q f = \int_a^b (\int_c^d f(\cdot, y))dy$. *Hence, for all* $f \in G(Q) \cap G_1^*(Q) \cap G_2^*(Q)$, *we have*

$$\int_Q f = \int_c^d \left(\int_a^b f(\cdot, y) \right) dy = \int_a^b \left(\int_c^d f(x, \cdot) \right) dx$$

Proof. Let $F \in L^1(Q)$ be such that $\partial_1 F = f$ and that F satisfies the conditions in the definition of $G_1^*(Q)$. A direct proof shows that for a.e. $y \in [a, b]$,

$$\int_a^b f(\cdot, y) = F(b, y) - F(a, y).$$

Since $|F(x, \cdot)| \leq g, x \in [a, b]$ with $g \in L^1(c, d)$, the mapping $G(x, y) = \int_c^y F(x, t)dt$ is defined for every $(x, y) \in \overline{Q}$. Let $(x_n, y_n) \to (x, y)$ in $\overline{Q}$. We have $\chi_{(c, y_n)}(t) \to$

$\chi_{(c,y)}(t),\ t \in [c,d]\backslash\{y\}$, and by the continuity of $F(\cdot,t)$ on $[a,b]$, $F(x_n,t) \to F(x,t)$, a.e. $t \in [c,d]$. Hence

$$F(x_n,t)\chi_{(c,y_n)}(t) \to F(x,t)\chi_{(c,y)}(t)$$

for a.e. $t \in [c,d]$. Moreover, $|F(x_n,t)\chi_{(c,y_n)}(t)| \le |F(x_n,t)| \le g(t)$, $\forall n \in \mathbb{N}$, a.e. t$\in [c,d]$. By the dominated convergence theorem

$$G(x_n,y_n) = \int_c^d F(x_n,t)\chi_{(c,y_n)}(t)dt \to \int_c^d F(x,t)(c,y)(t)dt = G(x,y)$$

as $(n \to \infty)$. We have proved that $G \in C(\overline{Q})$. On the other hand, since $F(x,\cdot) \in L^1(c,d)$, we have $\dfrac{\partial G(x,y)}{\partial y} = F(x,y)$ for a.e. $y \in [c,d]$. Integrating by parts, we have $\partial_2 G = F$ in $D'(Q)$. Hence $\partial G = \partial_1 F = f$ in $D'(Q)$. Thus $G \in I(f)$. By definition, $\displaystyle\int_Q f = G(b,d) - G(a,d)$. Moreover,

$$\int_c^d \left| \int_a^b f(\cdot,y) \right| = \int_c^d |F(b,y) - F(a,y)|dy \le 2\int_c^d g(y)dy < \infty,$$

i.e. the function $y \to \displaystyle\int_a^b f(\cdot,y)$ is in $L^1(c,d)$. Furthermore,

$$\int_c^d \left(\int_a^b f(\cdot,y) \right) dy = \int_c^d F(b,y)dy - \int_c^d F(a,y)dy = G(b,d) - G(a,d).$$

We have thus proved the first equality of Theorem 5. The second inequality may be proved similarly. ∎

Remark 1 From Proposition 1, we see that the first equality holds for all $f \in G_1(Q)$ and the second holds for $f \in G_1(Q) \cap G_2(Q)$. In view of Theorem 2 and Proposition 4, we see that Theorem 5 is valid for all $F \in L^1(Q)$. It is thus a generalization of the classical Fubini theorem for Lebesgue integrable functions on Q.

2.2 An application to differential equations

We now apply G-integration to an elementary "initial value" problem for the wave equation. Using the G-integral we may consider initial value problems with initial data which must not necesarily be smooth , and we seek solutions in the class $G(\Omega)$.

Consider the following simple problem in the unit square $\Omega = (0,1)^2$:

$$\left\{ \begin{array}{rcl} u_{xy} := \partial_{12}u &=& f \text{ in } \Omega, \\ u(x,x) &=& h(x), \\ u_y(x,x) := \partial_2 u(x,x) &=& g(x) \text{ for } x \in (0,1). \end{array} \right. \tag{4}$$

We assume that g and h are continuous on $[0,1]$, $f \in G_1(\Omega)$, and we are to find solutions u of (4) in the class

$$A = \{u \in C(\overline{\Omega}) : \partial_2 u \in C(\overline{\Omega})\}.$$

In view of Theorem 5 (Fubini's Theorem for G-integrable distributions) and Proposition 1, we have for all $(x,y) \in \overline{\Omega}$,

$$\begin{aligned} F(x,y) := F(f)(x,y) &= \int_{[0,x]\times[0,y]} f \\ &= \int_0^y \left(\int_0^x f(\xi,\eta) d\xi \right) d\eta. \end{aligned} \tag{5}$$

Moreover, as in Proposition 1, we have by the definition of $G_1(\Omega)$ that

$$\partial_2 F(x,y) = \int_0^x f(\xi,y) d\xi, \tag{6}$$

and $\partial_2 F(x,y)$ is continuous on $\overline{\Omega}$. The integral in (6) is understood as a one dimensional G-integral. Now, since

$$\partial_{12} u = \partial_{12} F = f \ \text{ in } \ D'(\Omega),$$

it follows from Lemma 1 that

$$u(x,y) = F(x,y) + H(x) + K(y), \forall x,y \in [0,1], \tag{7}$$

where K and H are continuous on $[0,1]$. Now, we have in the distributional sense that

$$\partial_2 u(x,y) = \partial_2 F(x,y) + K'(y). \tag{8}$$

Since $\partial_2 F$ and $\partial_2 u$ are continuous, we also have that K' is also continuous on $[0,1]$. From (4), (7), and (8), one obtains

$$F(x,x) + H(x) + K(x) = h(x),$$

$$\partial_2 F(x,x) + K'(x) = g(x).$$

Without loss of generality, one can choose K such that $K(0) = 0$. The above equation gives

$$\begin{aligned} K(x) &= \int_0^x [g(\xi) - \partial_2 F(\xi,\xi)] d\xi \\ &= \int_0^x [g(\xi) - \int_0^\xi f(t,\xi) dt] d\xi. \end{aligned}$$

It follows that

$$\begin{aligned} H(x) &= h(x) - F(x,x) - K(x) \\ &= h(x) - \int_0^x \left(\int_0^x f \right) - \int_0^x \left[g(\xi) - \int_0^\xi f(t,\xi) dt \right] d\xi. \end{aligned}$$

Therefore, there exists a unique solution $u \in A$ of (4); moreover, u is given by the following integral

$$\begin{aligned} u(x,y) &= \int_0^x \int_0^y f + h(x) - \int_0^x \int_0^x f - \int_0^x \left[g(\xi) - \int_0^\xi f(t,\xi) dt \right] d\xi \\ &\quad + \int_0^x g - \int_0^x \int_0^\xi f(t,\xi) dt d\xi. \end{aligned} \tag{9}$$

We note that (4) is now equivalent to the integral equation (9). This equivalence permits us to investigate the existence of solutions of semilinear equations of the form:

$$\begin{cases} u_{xy} &= f(u) \text{ in } \Omega, \\ u(x,x) &= h(u)(x,x), \\ u_y(x,x) &= g(u)(x,x) \text{ for } x \in (0,1). \end{cases} \tag{10}$$

Here h, g are mappings from A to $C(\overline{\Omega})$, and f maps A into $G_1(\Omega)$. Using (9), we can convert (10) into a nonlinear integral equation that may be treated by usual fixed point methods.

We also note that wave equations with nonsmooth (distributional) data have been studied extensively since the appearance of distributions (cf. [Ho] and the references therein, see also [W] for a nice and elementary presentation). In those approaches, the initial conditions are usually included in the source distribution; hence, it is not clear what one means by the initial conditions, related to the classical solutions. Our approach here is, in some sense, in between of the classical and the distributional ones. We relax some smoothness assumptions on the source functions and the initial conditions, but still keep the usual meaning of traces and initial conditions. Moreover, the differential equations can be now written as integral equations.

3 Green's theorem for G-integrable distributions

3.1 Green's theorem on rectangles

In this section, the boundary $\Gamma = \partial Q = \{a,b\} \times [c,d] \cup [a,b] \times \{c,d\}$ is assumed oriented in the usual (counter-clockwise) direction. Let $pdx + qdy$ be a differential form in Q, where $(p,q) \in D'(Q) \times D'(Q)$ is a (distributional) vector field. If the traces of p and q on the sides of Q can be defined and if the integrals

$$\int_{[a,b]\times\{c\}} p|_{[a,b]\times\{c\}} = \int_a^b p(\cdot,c), \quad \int_{[a,b]\times\{d\}} p|_{[a,b]\times\{d\}} = \int_a^b p(\cdot,c),$$

$$\int_{\{a\}\times[c,d]} q|_{a\times[c,d]} = \int_c^d q(a,\cdot), \quad \int_{\{b\}\times[c,d]} q|_{\{b\}\times[c,d]} = \int_c^d q(b,\cdot)$$

exist in some sense, then we can define the integral of the form $pdx + qdy$ on in the usual way, by

$$\int_\Gamma pdx + qdy = \int_a^b p(\cdot,c) - \int_a^b p(\cdot,d) + \int_c^d q(b,\cdot) - \int_c^d q(a,\cdot).$$

We have the following form of Green's theorem for $G(Q)$.

Theorem 6 *Suppose the vector field $(p,q) \in G_1(Q) \times G_2(Q)$. Then: (i) The traces $p(\cdot,c)$, $p(\cdot,d)$ (resp. $q(a,\cdot)$, $q(b,\cdot)$) are (one-dimensional) G-integrable distributions on $[a,b]$ (resp. $[c,d]$).*

(ii) $\partial_1 q, \partial_2 p \in G(Q)$ and we have Green's formula

$$\int_\Gamma pdx + qdy = \int_Q (\partial_1 q - \partial_2 p). \tag{11}$$

Proof. (i) and the first part of (ii) follow from Proposition 1. To prove the second part of (ii), we remark that since $p \in G_1(Q)$, there exist $F \in C(\overline{Q})$ such that $p = \partial_1 F$ in $D'(Q)$. We have $\partial_2 p = \partial F$ in $D'(Q)$, i.e., $F \in I(\partial_2 p)$. By Lemma 2, one has

$$\int_Q \partial_2 p = F(b,d) - F(b,c) - F(a,d) + F(a,c).$$

On the other hand, since $p(\cdot, y) = [F(\cdot, y)]'$ in $D'(a,b)(y \in [c,d])$, we have $\int_a^b p(\cdot, c) = F(b,c) - F(a,c)$ and $\int_a^b p(\cdot, d) = F(b,d) - F(a,d)$. Thus $\int_Q \partial_2 p = p(\cdot, d) - p(\cdot, c)$. Similarly, $\int_Q \partial_1 q = q(b, \cdot) - q(a, \cdot)$. Combining these equalities, we obtain (ii). ■

As an immediate consequence of this theorem, we have the following Green's formula for continuous vector fields.

Corollary 2 *Let $(p,q) \in C(\overline{Q}) \times C(\overline{Q})$ be a continuous vector field on $\overline{Q}$. Then $\partial_1 q - \partial_2 p \in G(Q)$ and we have (11).*

Proof. Let $P \in C(\overline{Q})$. Since $\frac{d}{dx}\left(\int_a^x P(\xi, x) d\xi\right) = P(x,y)$ in $\overline{Q}$,

$$\partial_1 \left(\int_a^x P(\xi, x) d\xi\right) = P \ \text{ in } \ D'(Q).$$

The continuity of P implies that of the function $(x,y) \mapsto \left(\int_a^x P(\xi, x) d\xi\right)$ on $\overline{Q}$. Thus $P \in G_1(Q)$. Similarly $P \in G_2(Q)$. A similar proof shows that $Q \in G_1(Q) \cap G_2(Q)$. We also note that in this case, the integral in the left hand side of (11) is an usual Riemann integral. ■

3.2 Integration on elementary sets

As usual, we mean by an elementary set in $\overline{Q}$, a finite union of (closed) subrectangles of $\overline{Q}$. Using basic properties of G-integral, as presented in Section 1 and classical processes, one can prove that G-integrals on elementary sets are in fact independent of their partitions into subrectangles. Therefore, we can extend our integration on rectangles to one on elementary sets. The results presented in Sections 1, 2, 3, 4 above and 5 below can thus be generalized in a natural way. For example, we have:

Theorem 7 *Suppose $(p,q) \in G_1(Q) \times G_2(Q)$ and that E is an elementary set in $\overline{Q}$. Then*

$$\int_{\partial E} p dx + q dy = \int_E (\partial_1 q - \partial_2 p).$$

Remark 2 One of the principal aims of the integration theories presented in [M1], [KMP], or [CD] has been to derive Green type theorems with weakened conditions on the smoothness of the vector field (p,q). In these papers, the continuous differentiability of (p,q) was replaced by its continuity and pointwise differentiability or asymptotic differentiability. Here, as is shown by the corollary of Theorem 6, differentiability assumptions on (p,q) may be removed if we deal with G-integrable distributions.

4 Convergence theorems

We examine conditions on sequences $\{f_n\} \subset G(Q)$ in order that the convergence of f_n to f (in some sense) with $f \in G(Q)$, implies that $\int_Q f_n \to \int_Q f$. We recall that a sequence $\{F_n\}$ on Q is said to be locally uniformly bounded in Q if for each $x \in Q$, there exists a neighborhood $U_x \subset Q$ of x such that $\sup\{|F_n(y)| : y \in U_x, n \in \mathbb{N}\} < \infty$. It is clear that if $\{F_n\}$ is locally uniformly bounded in Q, then it is uniformly bounded on compact subsets of Q as well. We have the following convergence theorem.

Theorem 8 *Let $\{f_n\}$ be a sequence in $G(Q)$ such that*

(i) The sequence of primitives $\{F(f_n)\}$ is locally uniformly bounded in Q.

(ii) $\{F(f_n)\}$ converges pointwise on $\overline{Q}$ to a continuous function on $\overline{Q}$.

Then $\{f_n\}$ converges distributionally to a G-integrable distribution f and moreover, $\int_Q f_n \to \int_Q f$ as $n \to \infty$.

This theorem admits the following variant.

Theorem 9 *Let $\{f_n\}$ be a sequence in $G(Q)$ such that (i) and (ii) above hold and that $f_n \to f$ in $D'(Q)$. Then $f \in G(Q)$ and $\int_Q f_n \to \int_Q f$ as $n \to \infty$.*

Proof of Theorem 8. Let $F = \lim F_n$ on $\overline{Q}$. By (ii), F is continuous. Also, $F \in \hat{C}(Q)$ since all F_n have this property. Let $\phi \in D(Q)$. Since $F(f_n)\phi \to F\phi$ everywhere on Q and

$$\sup_Q |F(f_n)\phi| \le \|\phi\|_\infty \sup\{|F(f_n)(x)| : n \in \mathbb{N}, x \in \operatorname{supp}\phi\} < \infty,$$

by the local uniform boundedness of $\{F(f_n)\}$. Hence $\int_Q F(f_n)\phi \to \int_Q F\phi$. We have checked that $F(f_n) \to F$ in $D'(Q)$. Consequently, $f_n \to f := \partial F$ in $D'(Q)$. We have $f \in G(Q)$ and $F = F(f)$. It follows that $\int_Q f_n = F(f_n)(b,d) \to F(f)(b,d) = \int_Q f$. ∎

We now derive from these theorems some familiar consequences.

Corollary 3 *Let $\{f_n\}$ be a sequence in $G(Q)$ such that $f_n \to f$ in $D'(Q)$ and that $\{F(f_n)\}$ is equicontinuous on $\overline{Q}$. Then $f_n \to f$ in $G(Q)$ and $\int_Q f_n \to \int_Q f$ as $n \to \infty$.*

Proof. The equicontinuity of $\{F(f_n)\}$ implies that $\{F(f_n)\}$ is uniformly bounded on $\overline{Q}$. According to Ascoli's theorem, $\{F(f_n)\}_n$ is relatively compact in $C(\overline{Q})$; let F to be one of its limit points. We have $F \in \hat{C}(Q)$ and for some subsequence $\{f_{n_k}\} \subset \{f_n\}$,

$$F(f_{n_k}) \to F \text{ uniformly on } \overline{Q},$$
$$f_{n_k} \to \partial F \text{ in } D'(Q).$$

Thus $f = \partial F$ and $F = F(f)$, i.e., $F(f)$ is the unique limit point of $\{F(f_n)\}$. Consequently, $F(f_n) \to F(f)$ uniformly on $\overline{Q}$ and our proof is complete. ∎

Corollary 4 *(Monotone convergence theorem for G-integral)*
Let $\{f_n\}$ *be a sequences in* $G(Q)$ *such that* $f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots$, *and that* $\int_Q f_n \to a$ *as* $n \to \infty$. *Then* $f_n \to f \in G(Q)$ *in* $G(Q)$ *and* $\int_Q f = a$.

Proof. Let $m, n \in \mathbb{N}, m \leq n$. For $(x,y) \in \overline{Q}$, since $f_m - f_n \geq 0$,

$$0 \leq \int_{(a,x)\times(c,y)} (f_n - f_m) = [F(f_n) - F(f_m)](x,y) \leq \int_Q f_n - \int_Q f_m.$$

It follows that $\|F(f_n) - F(f_m)\|_\infty \leq \left|\int_Q f_n - \int_Q f_m\right|$ for all $m, n \in \mathbb{N}$. Since $\left\{\int_q f_n\right\}$ is a Cauchy sequence in $\mathbb{R}$, $\{F(f_n)\}$ is a Cauchy sequence in $C(\overline{Q})$, which completes our proof. ■

Corollary 5 *(Dominated convergence theorem for G-integral)*
Let $\{f_n\}$ *be a sequence in* $G(Q)$ *such that* $f_n \to f$ *in* $D'(Q)$. *Suppose there exist* $g, h \in G(Q)$ *satisfying* $g \leq f_n \leq h$, $\forall n \in \mathbb{N}$. *Then* $f \in G(Q)$ *and* $lim_{n\to\infty} \int_Q f_n = \int_Q f$.

Proof. We check that $\{F(f_n)\}$ is equicontinuous on $\overline{Q}$. Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\begin{cases} |F(g)(x,y) - F(g)(x',y')| < \epsilon/4 \\ |F(h)(x,y) - F(h)(x',y')| < \epsilon/4, \end{cases}$$

for all $(x,y), (x',y') \in \overline{Q}$ such that $|(x,y) - (x',y')| < \delta$. For $(x,y), (x',y')$ satisfying this condition, put $x_1 = \min(x,x')$, $x_2 = \max(x,x')$, $y_1 = \min(y,y')$, $y_2 = \max(y,y')$. Simple calculations show that

$$\begin{aligned} & |F(f_n)(x,y) - F(f_n)(x',y')| \\ \leq\ & |F(f_n)(x,y_2) - F(f_n)(x,y_1) - F(f_n)(a,y_2) + F(f_n)(a,y_1)| \\ & + |F(f_n)(x_2,y') - F(f_n)(x_1,y') - F(f_n)(x_2,c) + F(f_n)(x_1,c)| \\ =\ & \left|\int_{(a,x)\times(y_1,y_2)} f_n\right| + \left|\int_{(x_1,x_2)\times(c,y')} f_n\right|. \end{aligned}$$

On the other hand, since $\int_{Q'} g \leq \int_{Q'} f_n \leq \int_{Q'} h$ for all subrectangles $Q' \subset Q$, we have $\left|\int_{Q'} f_n\right| \leq \left|\int_{Q'} g\right| + \left|\int_{Q'} h\right|$. The right hand side of the above inequality is therefore bounded by

$$\begin{aligned} & \left|\int_{(a,x)\times(y_1,y_2)} g\right| + \left|\int_{(a,x)\times(y_1,y_2)} h\right| + \left|\int_{(x_1,x_2)\times(c,y')} g\right| + \left|\int_{(x_1,x_2)\times(c,y')} h\right| \\ =\ & |F(g)(x,y_2) - F(g)(x,y_1)| + |F(h)(x,y_2) - F(h)(x,y_1)| \\ & + |F(g)(x_2,y') - F(g)(x_1,y')| + |F(h)(x_2,y') - F(h)(x_1,y')| \\ =\ & |F(g)(x,y) - F(g)(x,y')| + |F(h)(x,y) - F(h)(x,y')| \\ & + |F(g)(x,y') - F(g)(x',y')| + |F(h)(x,y') - F(h)(x',y')| \\ \leq\ & \epsilon. \end{aligned}$$

We have proved the equicontinuity of $\{F(f_n)\}$ over $\overline{Q}$, which together with Corollary 3, completes the proof. ■

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