1-Type Submanifolds of Non-Euclidean Complex Space Forms

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1 Introduction

The purpose of this paper is to prove a classification result for submanifolds of complex space forms which are immersed into a suitable Euclidean spaces of complex matrices in such way that the immersion of each submanifold is of 1-type, i.e. up to a translation, all components of the immersion vector are eigenfunctions of the Laplacian from a single eigenspace.

Throughout this paper, by a complex space form $\mathbb{C}Q^m(4c)$, $c = \pm 1$ $(m \geq 2)$, we mean a complete, simply connected model space form i.e. either the complex projective space $\mathbb{C}P^m(4)$ of constant holomorphic sectional curvature 4, or the complex hyperbolic space $\mathbb{C}H^m(-4)$ of holomorphic sectional curvature -4.

It is well known that any bounded domain in $\mathbb{C}^m$ can be given a Kähler metric, the so called Bergman metric. Accordingly, the complex hyperbolic space $\mathbb{C}H^m(-4)$ can be realized as the open unit ball in $\mathbb{C}^m$ with Bergman metric $g = -\sum_{\alpha,\beta=1}^m \partial_\alpha \bar{\partial}_\beta \ln(1 - |z|^2)\, dz_\alpha \otimes d\bar{z}_\beta$, where $z = (z_1, ..., z_m) \in \mathbb{C}^m$ (see [13, vol.II, p.162]). There is also another (equivalent) definition of $\mathbb{C}H^m(-4)$ particularly suitable for a study of submanifolds of that space. Namely, recall that the complex projective space $\mathbb{C}P^m(4)$ can be defined by means of the Hopf fibration $\pi : S^{2m+1} \to \mathbb{C}P^m(4)$, which is also a Riemannian submersion with totally geodesic fibers. This approach enables us to embed $\mathbb{C}P^m(4)$ isometrically into a suitable Euclidean space R^N of Hermitian matrices by the map $\phi(p) = z\bar{z}^t$, where $p \in \mathbb{C}P^m$ and $z \in \pi^{-1}(p) \subset S^{2m+1} \subset \mathbb{C}^{m+1}$ is regarded as a column vector. This

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embedding turns out to be the first standard embedding of $\mathbb{C}P^m(4)$ and has parallel second fundamental form (see e.g. [3]). By analogy, $\mathbb{C}H^m(-4)$ is obtained by a fibration from the anti - de Sitter space, $\pi : H_1^{2m+1} \to \mathbb{C}H^m(-4)$, and by identifying a complex line with the projection operator onto it one gets an isometric embedding ϕ into some pseudo-Euclidean space R^N_K [10]. Therefore, for any submanifold $x : M^n \to \mathbb{C}Q^m$ there is an associated immersion $\tilde{x} = \phi \circ x : M^n \to R^N_{(K)}$ into a (pseudo) Euclidean space which immerses M^n as a spacelike submanifold of $R^N_{(K)}$. By agreement, parentheticized characters appear only in the hyperbolic case. On the other hand, for a submanifold of a pseudo-Euclidean space there is a theory of finite type immersions of B.Y. Chen [3], [4], whereby a submanifold M is said to be of finite type if its position vector is globally decomposable into a finite sum of vector eigenfunctions of the Laplacian on M. In particular, $\tilde{x}$ is said to be of 1-type if $\tilde{x} = \tilde{x}_0 + \tilde{x}_t$ where $\tilde{x}_0 = \text{const}$, $\Delta\tilde{x}_t = \lambda\tilde{x}_t$, $\tilde{x}_t \neq \text{const}$. Here, Δ is the Laplacian of the induced metric on M acting on vector valued functions componentwise. 1-Type submanifolds of $\mathbb{C}Q^m$ have been studied by several authors. Ros [17] had classified CR-minimal submanifolds of $\mathbb{C}P^m$ which are of 1-type via ϕ, and the present author generalized these results by assuming that a submanifold has only parallel mean curvature vector, or that it is a CR - submanifold [6]. In addition, 1-type real hypersurfaces of $\mathbb{C}Q^m$ were studied in [15], [10]. In this paper we solve the classification problem of 1-type submanifolds of $\mathbb{C}Q^m$ (up to the identification of minimal totally real ones of half the dimension), immersed into an appropriate set of complex matrices via ϕ. Namely, we have

The Main Theorem *Let $x : M^n \to \mathbb{C}Q^m$ $(n,\, m \geq 2)$ be an isometric immersion of a smooth connected Riemannian n-manifold into a non-Euclidean complex space form of complex dimension m. Then $\tilde{x}$ is of 1-type if and only if one of the following cases occurs:*

(i) n is even, and M^n is locally a complex space form $\mathbb{C}Q^{n/2}$ isometrically immersed as a totally geodesic complex submanifold of $\mathbb{C}Q^m$.

(ii) M^n is immersed as a totally real minimal submanifold of a complex totally geodesic $\mathbb{C}Q^n \subset \mathbb{C}Q^m$.

(iii) n is odd, and M^n is locally congruent to a geodesic hypersphere

$$\pi(S^1(\sqrt{1/(n+3)}) \times S^n(\sqrt{(n+2)/(n+3)}))$$

of radius $\rho = \cot^{-1}(1/\sqrt{n+2}) \in (0, \pi/2)$ of a canonically embedded complex projective space $\mathbb{C}P^{(n+1)/2} \subset \mathbb{C}P^m$. This case happens only for submanifolds of the complex projective space.

Note that we do not make any a priori assumption on a 1-type submanifold.

As is well known [3], [4], a 1-type submanifold of $\mathbb{C}Q^m$ is minimal in certain hyperquadric of an appropriately defined (pseudo) Euclidean space of Hermitian matrices, thus, the study of 1-type submanifolds is a contribution to the theory of minimal submanifolds of such hypersurfaces. The reason we do not study 1-type submanifolds of the complex Euclidean space $\mathbb{C}^m \cong \mathbb{R}^{2m}$ is that it is well known by a result of Takahashi's [20] that such submanifolds are either minimal in $\mathbb{R}^{2m}$ or minimal in some hypersphere. On the other hand the immersion composed of products of coordinate functions in $\mathbb{R}^{2m}$ (as the standard embedding of $\mathbb{C}Q^m$ is accomplished by products of coordinates in $\mathbb{C}^{m+1}$) is of 1-type only for spheres centered at the origin; see [7] for details.

The study of compact finite type submanifolds, and in particular those submanifolds of compact rank-1 symmetric spaces which are of low type via standard immersions, has proven valuable in obtaining some information on spectra of the Laplace-Beltrami operators of these submanifolds. By showing that a certain compact submanifold has type 1, 2, etc., one can usually extract one, two, etc. eigenvalues of the Laplacian from the equations involved. Also, this method yields some interesting eigenvalue inequalities and produces sharp upper bound on λ_1 for certain classes of submanifolds. We refer to the works [3], [8], [17], [18], and also [11] where those extrinsic bounds were reached at by using a related technique.

2 Preparation

Let us briefly recall the definition of $\mathbb{C}Q^m$ via submersion and the construction of map ϕ. Let Ψ be the standard Hermitian form in $\mathbb{C}^{m+1}$ i.e. $\Psi(z,w) = c\bar{z}_0 w_0 + \sum_{k=1}^m \bar{z}_k w_k$, where $z = (z_k), w = (w_k) \in \mathbb{C}^{m+1}$ and $c = \pm 1$ corresponding to the two cases. Then $g := Re\Psi$ is a (pseudo) Euclidean metric on $\mathbb{R}^{2m+2} \cong \mathbb{C}^{m+1}$ which is clearly S^1 invariant. The projective space $\mathbb{C}P^m$ is the set of complex lines $[z]$ through the origin and $z \in \mathbb{C}^{m+1}$, appropriately equipped with a manifold structure, and $\mathbb{C}H^m$ is the set of timelike complex lines (i.e. those $\mathbb{C}$-lines on which g is negative definite). To define $\mathbb{C}Q^m$ via submersion, let $N^{2m+1} := \{z \in \mathbb{C}^{m+1} | \Psi(z,z) = c\}$. When $c = 1$ then $N^{2m+1} = S^{2m+1}$ is the ordinary hypersphere and when $c = -1$, N^{2m+1} is a complete Lorentzian hypersurface H_1^{2m+1}, the so-called anti - de Sitter space. The circle group $S^1 = \{e^{i\theta}\}$ naturally acts on N^{2m+1}. The orbit through w is a circle $e^{i\theta}w = (\cos\theta)\, w + (\sin\theta)\, iw$ centered at the origin and lying in the plane $w \wedge iw$ with the vector iw tangent to the fiber. The orbit space N^{2m+1}/S^1 defines the complex space form $\mathbb{C}Q^m$ and we also have an associated fibration $\pi : N^{2m+1} \to \mathbb{C}Q^m(4c)$. Consequently, $\mathbb{C}Q^m$ naturally inherits the complex structure J and Riemannian metric g of constant holomorphic sectional curvature $4c$ via this fibration.

By identifying a complex line $L = [z]$ (a J invariant plane in $\mathbb{C}^{m+1}$) with the operator of the orthogonal projection P onto L we can obtain the embedding of $\mathbb{C}Q^m$ into the set of complex matrices $M_{m+1}(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^{m+1})$. Namely, the orthogonal projection onto a (timelike) complex line L with respect to the Hermitian form Ψ is given by

$$P(v) = c\Psi(z,v)\, z = cg(z,v)\, z + cg(iz,v)\, iz = A\, v,$$

where $z \in L$ is unit, $v \in \mathbb{C}^{m+1}$ and A is the matrix

$$A = \begin{bmatrix} |z_0|^2 & cz_0\bar{z}_1 & \dots & cz_0\bar{z}_m \\ z_1\bar{z}_0 & c|z_1|^2 & \dots & cz_1\bar{z}_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m\bar{z}_0 & cz_m\bar{z}_1 & \dots & c|z_m|^2 \end{bmatrix}.$$

It is easy to see that P satisfies the following properties: (i) P is $\mathbb{C}$-linear ; (ii) $P^2 = P$; (iii) $\Psi(Pv,w) = \Psi(v,Pw)$; (iv) $tr_{\mathbb{C}}P = 1$. Conversely, an endomorphism P satisfying $(i) - (iv)$ is the orthogonal projection onto a $\mathbb{C}$-line $L = \{v|\, Pv = v\}$.

The condition (iii) says that the Ψ-adjoint of P equals P, i.e $P^* = P$, where $P^* = \bar{P}^t$ in the projective case and $P^* = G\bar{P}^tG$, $G = \mathrm{diag}(-1, I_m)$, in the hyperbolic case. Let $H^{(1)}(m+1) := \{A \in M_{m+1} | A^* = A\}$. This space is an $(m+1)^2$-dimensional real subspace of complex matrices M_{m+1} which becomes a (pseudo) Euclidean space $R^N_{(K)}$, where $N = (m+1)^2$, $K = m^2+1$, once it is equipped with the metric $\tilde{g}(A,B) = \frac{c}{2} tr(AB)$. Note that in the hyperbolic case this metric is indefinite, of index m^2+1. The identification of a complex line $[z]$, $g(z,z) = c$, with the projection operator P onto that line gives rise to an isometric embedding $\phi : \mathbb{C}Q^m \to H^{(1)}(m+1)$, $\phi([z])(v) = c\,\Psi(z,v)\,z$ with the image

$$\phi(\mathbb{C}Q^m) = \{P \in M_{m+1} | P^2 = P,\ P^* = P \text{ and } trP = 1\},$$

lying fully in the hyperplane $\{trP = 1\}$ as a spacelike submanifold (having a timelike normal bundle in the hyperbolic case). In the projective case this embedding is simply given by $\phi([z]) = z\bar{z}^t$. The embedding ϕ is equivariant with respect to the action of the Ψ-unitary group $U^{(1)}(m+1) = \{A \in M_{m+1} | AA^* = I\}$, and that fact enables us to do the computations locally, at a suitably chosen point. That point, called the “origin”, is taken to be the matrix $P_0 = \mathrm{diag}\,(1, 0, ..., 0) \in H^{(1)}(m+1)$.

We shall hereafter identify $\mathbb{C}Q^m$ and its ϕ-image. The tangent and normal space of $\phi(\mathbb{C}Q^m)$ at a point P are given below together with representative tangent and normal vectors at the origin:

$$T_P(\mathbb{C}Q^m) = \{X \in H^{(1)}(m+1) |\, XP + PX = X\}; \quad X = \begin{pmatrix} 0 & c\bar{u}^t \\ u & 0 \end{pmatrix}, \quad u \in \mathbb{C}^m \quad (1)$$

$$T^\perp_P(\mathbb{C}Q^m) = \{Z \in H^{(1)}(m+1) |\, ZP = PZ\}; \quad Z = \begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix}, \quad a \in \mathbb{R},\ D \in H(m). \quad (2)$$

The second fundamental form σ of the embedding ϕ is parallel, and for tangent vectors X, Y and a normal vector Z, the expressions for σ and the shape operator $\bar{A}$ are as follows:

$$\sigma(X,Y) = (XY + YX)(I - 2P), \qquad \bar{A}_Z X = (XZ - ZX)(I - 2P) \qquad (3)$$

The complex structure J satisfies

$$JX = i\,(I - 2P)X, \qquad \sigma(JX, JY) = \sigma(X,Y). \qquad (4)$$

One easily checks that

$$\langle \sigma(X,Y), I\rangle = 0, \qquad \langle \sigma(X,Y), P\rangle = -\langle X, Y\rangle, \qquad (5)$$

where $\langle A, B\rangle = \frac{c}{2}\, tr\,(AB)$. For more details on the embedding ϕ and its properties see [17], [3], [10] and references there. The following formulas of Ros are also well known [18]:

$$\langle \sigma(X,Y), \sigma(V,W)\rangle = c\,[\,2\langle X,Y\rangle\langle V,W\rangle + \langle X,V\rangle\langle Y,W\rangle + \langle X,W\rangle\langle Y,V\rangle + \langle JX,V\rangle\langle JY,W\rangle + \langle JX,W\rangle\langle JY,V\rangle], \qquad (6)$$

so that the shape operator of ϕ in the direction $\sigma(X,Y)$ is

$$\bar{A}_{\sigma(X,Y)}V = c\,[\,2\langle X,Y\rangle V + \langle Y,V\rangle X + \langle X,V\rangle Y + \langle JY,V\rangle JX + \langle JX,V\rangle JY]. \quad (7)$$

Lemma 1. *The following is an orthonormal basis of the normal bundle of $\phi(\mathbb{C}Q^m)$ at P:*

$$\{\sqrt{2}P,\ \frac{1}{\sqrt{2}}[\sigma(e_i,e_i)+2cP],\ \sigma(e_i,e_j),\ \sigma(e_i,e_{j^*})\},$$

where $1 \le i,\, j \le m,\ i<j,\ \ e_{i^*} = Je_i,$ and $\{e_i, e_{i^*}\}$ is a chosen J-basis. Consequently, $I = (m+1)P + (c/2)\sum_{i=1}^{m} \sigma(e_i,e_i)$.

Proof. Proof follows directly by a dimension count and by using (4) - (6). Note that dim $H^{(1)}(m+1) = (m+1)^2$, dim $\mathbb{C}Q^m = 2m$, so that the dimension of the normal space of the immersion ϕ is m^2+1. ∎

3 Demonstration

Suppose now that $x : M^n \longrightarrow \mathbb{C}Q^m$ is an isometric immersion of a connected Riemannian manifold into a complex space form. Let $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ denote the set of all (local) smooth sections (i.e vector fields) of the tangent and normal bundle of M respectively. We consider a local adapted frame of orthonormal vector fields $e_1,...,e_n,e_{n+1},...,e_{2m}$ tangent to $\mathbb{C}Q^m$ where the first n vectors are tangent to M and the remaining ones are normal to M. In general, index i will range from 1 to n and index r from $n+1$ to $2m$, so that e_i, e_r represent basis vectors which are tangent to M and normal to M respectively. Let $\bar{\nabla}, \bar{A}_Z, \bar{\nabla}^{\perp}$ denote the Levi-Civita connection on $\mathbb{C}Q^m$, the Weingarten map in the direction Z, and the connection in the normal bundle of ϕ; $\nabla, A_{\xi}, \nabla^{\perp}$ denote the induced connection on M, the Weingarten map in the direction ξ and the connection in the normal bundle of the immersion x, and let $h, \tilde{h},\ H,\ \tilde{H}$ denote the second fundamental forms and the mean curvature vectors of the immersions x and $\tilde{x} = \phi\circ x$ respectively, so that $H = (1/n)\sum_i h(e_i,e_i)$. All immersions are assumed smooth and all manifolds are connected smooth Riemannian manifolds of dimension ≥ 2. J will denote the orthogonal almost complex structure on $\mathbb{C}Q^m$. A submanifold M is called a CR-submanifold if the tangent bundle TM splits into an orthogonal direct sum of two differentiable distributions $TM = \mathcal{D}\oplus\mathcal{D}^{\perp}$, such that $J\mathcal{D}\subset\mathcal{D}$ and $J\mathcal{D}^{\perp}\subset T^{\perp}M$. If $\mathcal{D}^{\perp}=\emptyset$ a submanifold is said to be complex, and if $\mathcal{D}=\emptyset$, M is called a totally real submanifold. For a vector field V tangent to $\mathbb{C}Q^m$ along M we denote by V_T and V_N its components which are tangent and normal to M respectively. For a tangent vector field $X\in\Gamma(TM)$ and a normal vector field $\xi\in\Gamma(T^{\perp}M)$ we define operators $S,\ F,\ s, f$ by $SX = (JX)_T,\ FX = (JX)_N,\ s\xi = (J\xi)_T,\ f\xi = (J\xi)_N$.

Let M^n be a 1-type submanifold of $R^N_{(K)} = H^{(1)}(m+1)$ via ϕ, i.e. $\tilde{x} = \phi\circ x = \tilde{x}_0 + \tilde{x}_t$ with $\Delta\tilde{x}_t = \lambda\tilde{x}_t$ and $\tilde{x}_0 =$ const. Then by using the same argument as in [6] , from the 1-type condition by eliminating $\tilde{x}_t$ and differentiating with respect to a tangential vector field X, we get the following by comparing parts tangent and normal to $\mathbb{C}Q^m$:

$$nA_H X - n\nabla^{\perp}_X H + 2c(n+1)X - 2cJ(JX)_T = \lambda X, \quad \text{and} \tag{8}$$

$$\sum_r \sigma(B_r X, e_r) = 0, \text{ where } B_r = (trA_r)I + 2A_r. \tag{9}$$

Because of the equivariancy of ϕ and formulas (1)-(3), it follows that $\bar{A}_Z = 0$ if and only if $Z = aI$, so that the condition (9) is equivalent to $\sum_r \bar{A}_{\sigma(B_r X, e_r)} Y = \sum_r \bar{A}_{\sigma(B_r X, e_r)} \xi = 0$, which, by using (7) and (8) leads to the following characterization:

Lemma 2. *If $\tilde{x}$ is of 1-type then for every $X, Y \in \Gamma(TM)$ and every $\xi \in \Gamma(T^{\perp}M)$ we have*

(10) $A_H X = bX + \frac{2c}{n} S^2 X$, *where* $b = \frac{1}{n}[\lambda - 2c(n+1)]$

(11) $\nabla^{\perp}_X H = -\frac{2c}{n} F(SX)$

(12) $$n\langle X, Y\rangle H + 2h(X,Y) + n\langle JH, Y\rangle JX - n\langle X, SY\rangle JH - 2JA_{FY}X - 2Jh(X, SY) = 0$$

(13) $n\langle H, \xi\rangle X + 2A_\xi X + n\langle JX, \xi\rangle JH + n\langle JH, \xi\rangle JX - 2JA_{f\xi}X - 2Jh(X, s\xi) = 0.$

Conversely, if (10)-(13) are satisfied and $\lambda \neq 0$ is a constant, then $\tilde{x}$ is of 1-type.

Proof. This lemma is essentially proved in [6, Lemma 2] where formulas (10) and (11) of [6] when summation is carried out lead to formulas (12) and (13), and formula (8) gives (10) and (11) when tangential and normal parts are separated. Conversely, if (10)-(13) hold then the equivalent formulas (8) and (9) hold, and for $\lambda \neq 0$, define $\tilde{x}_0 := \tilde{x} - (1/\lambda)\Delta\tilde{x}$. Then by taking an arbitrary $X \in \Gamma(TM)$, since $\tilde{x}$ is the position vector of M in $H^{(1)}(m+1)$ we have $\tilde{\nabla}_X \tilde{x} = X$ and

$$\begin{aligned}
\tilde{\nabla}_X \tilde{x}_0 &= X + (1/\lambda)\tilde{\nabla}_X[\, nH + \sum_i \sigma(e_i, e_i)\,] \\
&= X + (1/\lambda)[\, n\sigma(X,H) - n\,A_H X + n\,\nabla^{\perp}_X H \\
&\qquad - \sum_i \bar{A}_{\sigma(e_i,e_i)} X + 2\sum_i \sigma(h(X,e_i), e_i)] \\
&= X + (1/\lambda)[\sum_r \sigma(B_r X, e_r) - n\,A_X H + \nabla^{\perp}_H X - 2c(n+1)X + 2c\,JSX] \\
&= X + (1/\lambda)(-\lambda X) = 0
\end{aligned}$$

by (7), (8) and (9) (cf. [6, p. 284]). Hence $\tilde{x}_0$ is a constant vector and $\Delta\tilde{x}_t = \lambda\tilde{x}_t$ where $\tilde{x}_t = \tilde{x} - \tilde{x}_0$, proving that $\tilde{x}$ is a 1-type immersion. ∎

We are ready now to prove the main theorem :

Proof of the Main Theorem. If for a local normal frame $\{e_r\}$, $r = n+1, ..., 2m$, we choose e_{n+1} to be parallel to the mean curvature vector H, then $H = \alpha e_{n+1}$ determines the mean curvature α of the immersion which also equals $\alpha = (1/n)\,\mathrm{tr}\, A_{n+1}$. Because of a different choice of direction of the normal and absence of orientability in general, α is determined up to a sign, but $\alpha^2 = \langle H, H\rangle$ is well defined throughout. Let us show first that the mean curvature α is constant and that JH is tangent to M^n. Noting that $\sum_i h(e_i, Se_i) = 0$ since h is symmetric and S is skew-symmetric, let $X = Y = e_i$ in (12) and add on i to get

$$(n^2 + 2n)H + n\sum_i \langle JH, e_i\rangle Je_i - 2\sum_i JA_{Fe_i}e_i = 0,$$

from which it follows that JH is tangent to M. In fact $JH = -2/n(n+1)\sum_i A_{Fe_i}e_i$. Then by (11) we have

$$nX\langle H, H\rangle = 2n\langle \nabla^\perp_X H, H\rangle = -4c\langle JSX, H\rangle = 4c\langle JX, JH\rangle = 0,$$

which implies that the mean curvature α is constant. We shall eventually show that M must be a CR submanifold (in fact, complex, totally real, or a hypersurface) of $\mathbb{C}Q^m$. There are two possibilities regarding α : (1°) $\alpha = 0$ or (2°) $\alpha \neq 0$. If $\alpha = 0$ then from (8) it follows that $2c(JX)_T = [\lambda - 2c(n+1)]JX$. Therefore, if $\lambda = 2c(n+1)$ then $(JX)_T = 0$ for every X and consequently M is totally real. On the other hand if $\lambda \neq 2c(n+1)$ then $(JX)_N = 0$, so that M is a J-invariant (complex) submanifold. Case $\alpha \neq 0$ is more interesting. Assume that the constant mean curvature is nonzero so that JH is a nonzero tangent vector. For each point $p \in M$ denote by $\mathcal{D}^\perp_p$ the maximal subspace of T_pM having the property that $J\mathcal{D}^\perp_p \subset T^\perp_p M$, and let $\mathcal{D}_p$ be the orthogonal complement of $\mathcal{D}^\perp_p$ in T_pM and $\mathcal{L}_p$ the orthogonal complement of $J\mathcal{D}^\perp_p$ in $T^\perp_p M$. Then we have the following orthogonal splitting:

$$T_p\mathbb{C}Q^m = \mathcal{D}_p \oplus \mathcal{D}^\perp_p \oplus J\mathcal{D}^\perp_p \oplus \mathcal{L}_p,$$

where $J\mathcal{D}_p \subset \mathcal{D}_p \oplus \mathcal{L}_p$. We are going to show that actually $J\mathcal{D}_p = \mathcal{D}_p$. Since $JH \in \mathcal{D}^\perp$, $\mathcal{D}^\perp_p$ is nonempty, and since $S(\mathcal{D}^\perp_p) = 0$ we have $0 = \langle S(\mathcal{D}^\perp_p), \mathcal{D}_p\rangle = -\langle \mathcal{D}^\perp_p, S(\mathcal{D}_p)\rangle$, which shows $S(\mathcal{D}_p) \subset \mathcal{D}_p$. Moreover, $S(\mathcal{D}_p) = \mathcal{D}_p$, for otherwise, there exists a nonzero vector $Y \in \mathcal{D}_p$ such that $Y \perp S(\mathcal{D}_p)$. Then $\langle SY, \mathcal{D}^\perp_p\rangle = 0$ and $\langle SY, \mathcal{D}_p\rangle = -\langle Y, S(\mathcal{D}_p)\rangle = 0$, and hence $SY = 0 \Rightarrow Y \in \mathcal{D}^\perp_p$, which contradicts the maximality of $\mathcal{D}^\perp_p$. Next we will show that $l = \dim \mathcal{D}_p$ and $k = \dim \mathcal{D}^\perp_p$ do not depend on p and hence they are constant. Putting $\xi = H$ in (13) we get

$$n\alpha^2 X + 2A_H X = n\langle X, JH\rangle JH + 2Jh(X, JH).$$

Thus, $n\alpha^2 X + 2A_H X \in \mathcal{D}^\perp_p$ for every $X \in \Gamma(TM)$. In particular for $X \in \mathcal{D}_p$ from (10) we get

$$n\alpha^2 X + 2A_H X = (n\alpha^2 + 2b)X + (4c/n)S^2 X \in \mathcal{D}^\perp_p.$$

However, both X and S^2X are in $\mathcal{D}_p$, and therefore both sides of the equation above are zero, hence $A_H X = -(n\alpha^2/2)\,X$ for $X \in \mathcal{D}_p$. From (10) it is also clear that $A_H X = bX$ for $X \in \mathcal{D}^\perp_p$. Therefore, with respect to the splitting $T_pM =$

$\mathcal{D}_p \oplus \mathcal{D}_p^{\perp}$, A_H has the block form $\begin{pmatrix} \frac{-n\alpha^2}{2} I_l & 0 \\ 0 & bI_k \end{pmatrix}$, where I's are the identity matrices of indicated order. Thus

$$n\alpha^2 = \mathrm{tr} A_H = -n\alpha^2 l/2 + bk = \mathrm{const} \tag{14}$$

and $l + k = n$, therefore dimensions l, k do not depend on a particular point. Differentiability of the distributions $\mathcal{D}, \mathcal{D}^{\perp}$ follows by an argument as in [13, vol II, p.38]. Let Ω be the fundamental Kähler 2-form on $\mathbb{C}Q^m$ i.e. $\Omega(X, Y) = \langle X, JY \rangle$. Take arbitrary vector fields $X, Y \in \mathcal{D}^{\perp}$ and $Z \in \mathcal{D}$. Then since $JX, JY \in T^{\perp}M$ and Ω is closed we get

$$\begin{aligned} 0 &= 3\, d\Omega(X, Y, Z) \\ &= X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) \\ &\quad - \Omega([X, Y], Z) - \Omega([Y, Z], X) - \Omega([Z, X], Y) \\ &= -\langle [X, Y], JZ \rangle = -\langle [X, Y], SZ \rangle, \end{aligned}$$

and thus $[X, Y] = \nabla_X Y - \nabla_Y X \in \mathcal{D}^{\perp}$. Since $\mathbb{C}Q^m$ is Kähler it follows that for $X, Y \in \mathcal{D}^{\perp}$, $\quad J\bar{\nabla}_X Y = \bar{\nabla}_X (JY)$ i.e.

$$J\nabla_X Y + Jh(X, Y) = -A_{JY} X + \nabla_X^{\perp}(JY).$$

Consequently,

$$J[X, Y] = (A_{JX} Y - A_{JY} X) + \nabla_X^{\perp}(JY) - \nabla_Y^{\perp}(JX). \tag{15}$$

Since $[X, Y] \in \mathcal{D}^{\perp}$, the tangent part in this equation vanishes, i.e. $A_{JX} Y - A_{JY} X = 0$. In particular, when $Y = JH$ then

$$A_H X + A_{JX}(JH) = 0, \quad \text{for every} \quad X \in \mathcal{D}^{\perp}. \tag{16}$$

Suppose now that $\dim \mathcal{D}^{\perp} \geq 2$ and there exists a unit vector $X \in \mathcal{D}^{\perp}$ such that $X \perp JH$. By putting $Y = JH$ in formula (12) and taking the inner product $\langle -, JX \rangle$ we get

$$\begin{aligned} 0 &= 2\langle h(X, JH), JX \rangle + n\alpha^2 + 2\langle A_H X, X \rangle \\ &= 2\langle A_{JX}(JH), X \rangle + n\alpha^2 + 2\langle A_H X, X \rangle = n\alpha^2 \end{aligned}$$

by (16). Since we assumed $\alpha \neq 0$ this is a contradiction, which means that $k = \dim \mathcal{D}^{\perp} = 1$. From the block form of A_H we get by setting $X \in \mathcal{D}$, $\xi = H$ in (13) that $h(X, JH) = 0$ for $X \in \mathcal{D}$, and by setting $X = Y = JH$ in (12) we have $h(JH, JH) = bH$, where $b = n(n+1)\alpha^2/2 \neq 0$ is a constant. Since JH is tangent to M and autoparallel [6, Th.2], and $\dim\, \mathcal{D}^{\perp} = 1$, it follows that $\nabla_{JH} Y$ and $\nabla_Y (JH)$ are both in $\mathcal{D}$ whenever $Y \in \mathcal{D}$. Finally, we use (11) and the Codazzi equation $(\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) = (\bar{R}(X, Y)Z)_N$ with $X = Z = JH$, $Y \in \mathcal{D}$, to conclude that $\nabla^{\perp} H = 0$ [6, Lemma 3]. Then (11) shows that $F(SX) = 0$ for every tangent vector X, and since $S(\mathcal{D}) = \mathcal{D}$ it follows that $J\mathcal{D} = \mathcal{D}$, i.e. $\mathcal{D}$ is a holomorphic subbundle. Consequently, $\mathcal{L}$ is holomorphic as well. Since $e_r \perp H$

for $e_r \in \mathcal{L}$, by putting $\xi = e_r$ in the formula (13) we get $A_{r^*} = -JA_r$ where $e_{r^*} := Je_r \in \mathcal{L}$. On the other hand, the Kähler condition gives

$$\bar{\nabla}_X e_{r^*} = J\bar{\nabla}_X e_r \;\Rightarrow\; -A_{r^*}X + \nabla^{\perp}_X e_{r^*} = -JA_rX + J\nabla^{\perp}_X e_r.$$

Since $\langle H, e_r\rangle = 0$ and H is parallel, both $\nabla^{\perp}_X e_r,\ J\nabla^{\perp}_X e_r \in \mathcal{L}$. Then the last equation gives $A_{r^*}X = JA_rX$ and therefore $A_{r^*} = A_r = 0$ for $e_r \in \mathcal{L}$. This implies that the first normal space $N_1 := \mathrm{Span}_R\{\mathrm{Image}\, h\}$ equals $J\mathcal{D}^{\perp} = \mathbb{R}\{H\}$ at each point. Since N_1 is parallel, the reduction of codimension theorem [5] (see also [12], [16]) says that M must be a real hypersurface of a complex totally geodesic $\mathbb{C}Q^{(n+1)/2}$ in $\mathbb{C}Q^m$. By a result of [10] there is no real hypersurface of a complex hyperbolic space which is of 1-type. On the other hand, a result of [15] says that a 1-type real hypersurface of $\mathbb{C}P^m$ is locally congruent to a geodesic hypersphere as given in (iii) of the Main Theorem. As a matter of fact, if ξ is the unit normal of a hypersurface of $\mathbb{C}Q^{(n+1)/2}$ and $A = A_\xi$ the shape operator, then from $\dim \mathcal{D}^{\perp} = 1$, the block form of A_H, its trace $\mathrm{tr}\, A_H$, (14) and (10) we get $\alpha^2 = 4c/n^2(n+2)$. Thus, when $c = -1$ we see that $\alpha^2 < 0$ which is a contradiction. In the projective case we get $\alpha = -2/n\sqrt{n+2}$ and $\lambda = 2(n+1)(n+3)/(n+2)$ and then the formulas (10)-(13) reduce to

$$A(J\xi) = -\frac{n+1}{\sqrt{n+2}}\,J\xi, \qquad AX = \frac{1}{\sqrt{n+2}}\,X, \quad \text{for } X \in \mathcal{D}.$$

Therefore the hypersurface has two constant principal curvatures $\mu_1 = -\frac{n+1}{\sqrt{n+2}}$ of multiplicity 1 and $\mu_2 = \frac{1}{\sqrt{n+2}}$ of multiplicity $n-1$. According to the results of Takagi [19] and Cecil-Ryan [2], this hypersurface is an open portion of the geodesic hypersphere $\pi\,(S^1(\sqrt{1/(n+3)}) \times S^n(\sqrt{(n+2)/(n+3)}))$ of radius ρ with respective curvatures $2\cot(2\rho)$ and $\cot\rho$, $\rho \in (0, \pi/2)$, where in our case $\cot\rho = 1/\sqrt{n+2}$. On the other hand this hypersurface is indeed of 1-type by Lemma 2.

Returning back to the case $\alpha = 0$, we saw that then M^n must be either a complex submanifold or a totally real submanifold of $\mathbb{C}Q^m$. If M is minimal and totally real in $\mathbb{C}Q^m$ then $T^{\perp}M = J(TM) \oplus \mathcal{L}$. From (13) we have that for $\eta \in \mathcal{L}$, $A_\eta X = JA_{J\eta}X = 0$ and thus $\mathrm{Im}\, h \subset J(TM)$. For $\xi \in J(TM)$, again by (13) $A_\xi X = Jh(X, J\xi)$, and the Kähler condition

$$\bar{\nabla}_X(J\xi) = J\bar{\nabla}_X\xi \;\Leftrightarrow\; \nabla_X(J\xi) + h(X, J\xi) = -JA_\xi X + J\nabla^{\perp}_X\xi$$

then gives $\nabla^{\perp}_X\xi = -J\nabla_X(J\xi) \in J(TM)$. Thus $J(TM)$ is a parallel subbundle and $TM \oplus J(TM)$ is J invariant. By the reduction theorems of [5], [16] and [12], there exists a complex totally geodesic $\mathbb{C}Q^n \subset \mathbb{C}Q^m$ such that $T_p(\mathbb{C}Q^n) = T_pM \oplus J(T_pM)$ for every p and M^n is totally real submanifold of $\mathbb{C}Q^n$. Now we check that such submanifolds are indeed of 1-type. Let $\tilde{H}$ denote the mean curvature vector of M^n in $H^{(1)}(n+1) \subset H^{(1)}(m+1)$. From Lemma 1 we have $\sum_i \sigma(e_i, e_i) = -2c(n+1)\tilde{x} + 2c\,I$, so we get

$$\Delta(\tilde{x} - I/(n+1)) = \Delta\tilde{x} = -n\tilde{H} = -\sum_{i=1}^{n}\sigma(e_i, e_i) = 2c(n+1)(\tilde{x} - I/(n+1)).$$

Thus, if we denote $\tilde{x}_t = \tilde{x} - I/(n+1)$, it follows $\Delta\tilde{x}_t = 2c(n+1)\tilde{x}_t$ and $\tilde{x}$ is therefore a 1-type immersion.

If M is complex, from (12) we get $h(X,Y) = Jh(X,JY)$. But from $\bar{\nabla}_X JY = J\bar{\nabla}_X Y$ it follows $h(X,Y) = -Jh(X,JY)$. Thus $h = 0$, i.e. submanifold is totally geodesic and therefore an open portion of $\mathbb{C}Q^{n/2}$ (n even). But it is well known that such submanifolds are of 1-type in the corresponding spaces $H^{(1)}(n/2+1) \subset H^{(1)}(m+1)$. See [17], [10], [3], or apply the same kind of reasoning as used above for totally real submanifolds. ∎

Corollary 1. *Let M^n be a complete connected Riemannian manifold and $x : M^n \longrightarrow \mathbb{C}Q^m$ an isometric immersion into a non-Euclidean complex space form. Then $\tilde{x}$ is of 1-type if and only if*
(i) n is even , M^n is congruent to a complex space form $\mathbb{C}Q^{n/2}$ and x embeds M^n as a complex totally geodesic $\mathbb{C}Q^{n/2} \subset \mathbb{C}Q^m$.
(ii) M^n is immersed as a totally real minimal submanifold of a complex totally geodesic $\mathbb{C}Q^n \subset \mathbb{C}Q^m$.
(iii) n is odd and M^n is embedded by x as a geodesic hypersphere of radius $\rho = \tan^{-1}\sqrt{n+2}$ of a complex, totally geodesic $\mathbb{C}P^{(n+1)/2} \subset \mathbb{C}P^m$.

Proof. We need to clarify only parts (i) and (iii). If M^n is complete, the proof of the main theorem shows that $x(M^n)$ is either $\mathbb{C}Q^{n/2} \subset \mathbb{C}Q^m$ of case (i), or a geodesic (distance) hypersphere of case (iii). According to [13, vol.I, Th. IV.4.6] and [9, Exercise 2.108] M^n is a covering space of $x(M^n)$. It is well known that both $\mathbb{C}P^n$ and $\mathbb{C}H^n$ are simply connected, and the same is true for a geodesic hypersphere in $\mathbb{C}P^n$ which has the diffeomorphic type of S^{2n-1}. As a matter of fact, geodesic hyperspheres are models of the so called Berger spheres, i.e. odd dimensional spheres with metric scaled along the fibers of the Hopf fibration [21]. Therefore M^n is isometric to $x(M^n)$ and x is an embedding in cases (i) and (iii). ∎

Totally real minimal submanifolds $M^n \subset \mathbb{C}Q^n$ of case (ii) play the role of Lagrangian submanifolds and they seem to be ample in number [6, Remark 1]. At present time there is no exhaustive classification of such submanifolds. Also, I would like to point out that the hypersurface of case (iii) was mislabeled as $M^C_{0,(n-1)/2}$ in one place in [6, Th.1]. The notation $M^C_{p,q}$ is usually reserved for a minimal hypersurface of $\mathbb{C}P^m$ obtained by the Hopf projection of generalized Clifford surfaces $S^{2p+1}(\sqrt{\frac{2p+1}{2m}}) \times S^{2q+1}(\sqrt{\frac{2q+1}{2m}}) \subset S^{2m+1}$, $p+q = m-1$, where the spheres lie in complex subspaces of $\mathbb{C}^{m+1}$ [14]. Thus, $M^C_{0,(n-1)/2}$ would denote the minimal geodesic hypersphere $\pi(S^1(\sqrt{\frac{1}{n+1}}) \times S^n(\sqrt{\frac{n}{n+1}})) \subset \mathbb{C}P^{(n+1)/2}$ of radius $\rho = \cot^{-1}(1/\sqrt{n})$, which is different than a nonminimal hypersphere in (iii).

From the considerations above it follows that the values $\lambda = 2c(n+2)$, $\lambda = 2c(n+1)$ and $\lambda = 2(n+1)(n+3)/(n+2)$ are the corresponding eigenvalues of the Laplacian for three cases (i)-(iii) of the Theorem. Thus, if Spec(M^n) denotes the spectrum of the Laplacian on a compact manifold M^n, all eigenvalues are nonnegative and we have

Corollary 2. *If M^n is the geodesic hypersphere of case (iii) then $2(n+1)(n+3)/(n+2) \in$ Spec(M^n); if M^n is a compact totally real minimal submanifold of $\mathbb{C}P^n$ then $2(n+1) \in$ Spec(M^n), and there exists no compact totally real minimal submanifold M^n of $\mathbb{C}H^n$.*

We remark that for the geodesic hypersphere of case (iii), the eigenvalue listed above is the first nonzero eigenvalue λ_1 since for a geodesic hypersphere of radius ρ in $\mathbb{C}P^{(n+1)/2}$,

$$\lambda_1 = \min\left(\frac{1}{\cos^2\rho} + \frac{n}{\sin^2\rho},\ \frac{2(n+1)}{\sin^2\rho}\right)$$

according to results of [1] and [11]. Moreover, the radius $\rho = \tan^{-1}\sqrt{n+2}$ is the largest radius for which a geodesic hypersphere around a point in $\mathbb{C}P^{(n+1)/2}$ (n odd) is stable in the sense of [1].

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Bibliography

[1] J.L. Barbosa, M. Do Carmo and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 123–138;

[2] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499;

[3] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, 1984;

[4] B. Y. Chen, *Finite-type pseudo-Riemannian submanifolds*, Tamkang J. of Math. **17** (1986), 137–151;

[5] M. Dajczer and L. Rodriguez, *On isometric immersions into complex space forms*, Math. Ann. **299** (1994), 223–230;

[6] I. Dimitrić, *1-Type submanifolds of the complex projective space*, Kodai Math. J. **14** (1991), 281–295;

[7] I. Dimitrić, *Quadric representation of a submanifold*, Proc. Amer. Math. Soc. **114** (1992), 201–210;

[8] I. Dimitrić, *Quadric representation of a submanifold and spectral geometry*, Proc. Symp. Pure Math. **54:3** (1993), 155–167;

[9] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry*, Springer-Verlag, 1987;

[10] O. Garay and A. Romero, *An isometric embedding of the complex hyperbolic space in a pseudo-Euclidean space and its application to the study of real hypersurfaces*, Tsukuba J. Math. **14** (1990), 293–313;

[11] E. Heintze, *Extrinsic upper bounds for λ_1*, Math. Ann. **280** (1988), 389–402;

[12] S.I. Kawamoto, *Codimension reduction for real submanifolds of a complex hyperbolic space*, Rev. Mat. Univ. Complut. Madrid **7** (1994), 119–128;

[13] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,*, vols I&II, Wiley - Interscience, 1963, 1969;

[14] H.B. Lawson, Jr., *Rigidity theorems in rank-1 symmetric spaces*, J. Differential Geometry **4** (1970), 349–357;

[15] A. Martinez and A. Ros, *On real hypersurfaces of finite type of CP^m*, Kodai Math. J. **7** (1984), 304–316;

[16] M. Okumura, *Codimension reduction problem for real submanifold of complex projective space*, Coll. Math. Soc. János Bolyai 56. Differential Geometry, Eger (1989), North Holland, Amsterdam (1992), 573–585;

[17] A. Ros, *Spectral geometry of CR-minimal submanifolds in the complex projective space*, Kodai Math. J. **6** (1983), 88–99;

[18] A. Ros, *On spectral geometry of Kaehler submanifolds*, J. Math. Soc. Japan **36** (1984), 433–447;

[19] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures*, J. Math. Soc. Japan **27** (1975), 43–53;

[20] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385;

[21] A. Weinstein, *Distance spheres in complex projective spaces*, Proc. Amer. Math. Soc. **39** (1973), 649-650;

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