# Extending the dual of the Petersen graph

Thomas Meixner

## 1 Introduction

We are interested in geometries with a lot of symmetry. To be precise, we consider geometries $\Gamma$ with the following properties:

(a) every flag is contained in a flag of maximal cardinality (called chamber),
(b) $\Gamma$ is residually connected,
(c) a subgroup $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the set of chambers of $\Gamma$.

The cardinality of the chambers (which we assume to be finite) is called the rank of $\Gamma$. So what we are looking at are residually connected chamber-transitive chamber geometries $\Gamma$ of finite rank. And we will use the word geometry from now on in this sense.

By hypotheses (a) and (c), if $\{i, j\}$ is a 2-element subset of the type set $I$ of $\Gamma$, then there are flags $F$ of co-type $\{i, j\}$, such that $\mathrm{res}(F)$ is a rank 2 geometry over the type set $\{i, j\}$, and all such $\{i, j\}$-residues are isomorphic. Hence, such a geometry $\Gamma$ can be described by a diagram, in which the types (or even the exact isomorphism types) of all rank-2-residues are listed; this is often done in a graphical way. A graph $D$ is drawn having vertices $i \in I$, one vertex for every type of objects in a hypothetical geometry belonging to the diagram, and the way two vertices $i, j \in I$ are connected in the diagram reflects the type $D_{ij}$ of some rank 2 geometry.

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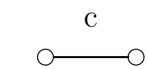
A geometry $\Gamma$ is said to belong to (or to have) a diagram $D$, if the types of its objects can be indexed by the elements $i \in I$ such that all $\{i, j\}$- residues of $\Gamma$ are of type $D_{ij}$. The types $D_{ij}$ can hereby stand for a whole class of rank 2 geometries, such as projective planes, but can also denote just a single rank 2 geometry.

Since we are mainly interested in finite geometries and finite groups (maybe possessing an infinite universal cover, though), we assume that all rank 2 residues are finite.

Quite naturally, one is led to try and classify geometries $\Gamma$ in terms of the diagram they have, maybe together with the group $G$ acting. The following question arises.

**For which diagrams $\mathcal{D}$ is it interesting to have a classification of all (simply connected) geometries belonging to $\mathcal{D}$?**

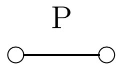
Most interesting are, of course, the Coxeter diagrams - here all rank 2-residues are assumed to be finite generalized polygons - because here buildings arise and one gets geometries for the finite simple groups of Lie type and rank at least 2, and for some sporadic groups.

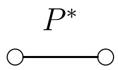
To get more geometries, and in particular also more sporadic groups into the picture, the range of rank 2 residues allowed in the diagrams was widened by Buekenhout ([B1]). In particular, he introduced the 'circle geometries' into the world of generalized polygons, where a circle geometry, with rank 2 diagram 

is nothing else but the geometry of vertices and edges of a finite complete graph. In fact circle geometries are 'line-thin' analogues of projective planes, as they are precisely the linear spaces with thin lines. In the literature, many more rank 2 residues are discussed - for instance from inspection of the sporadic simple groups ([RS],[B3]). For a long time, it was not quite clear, however, what class of diagrams one should consider as the natural extension of the class of Coxeter diagrams. The set of rank 2 geometries allowed as rank 2 residues should be large enough such that the diagrams considered provide descriptions of many (all) known interesting geometries of higher rank, but again should be small enough to make classification theorems possible. Recall that these give characterizations of the corresponding groups as well, which are also interesting.

Only recently, work by Buekenhout and Van Maldeghem suggests what could be the 'natural generalization' of the class of flag-transitive generalized polygons (see [BV]).

Usually, in the diagrams considered, the rank 2 residues have at least three numerical parameters connected with them (which coincide in the case of Coxeter diagrams): the point-diameter $d_p$, the line diameter $d_l$ and the girth $g$.
And the class of rank 2 geometries with a given triple of parameters $(g, d_p, d_l)$ is called a $(g, d_p, d_l)$-gon (see [B2], [BV]). Such $(g, d_p, d_l)$-gons generalize polygons in a moderate and, seemingly, interesting way, if $g \leq d_p$, $d_l \leq g + 1$ (see [BV]).

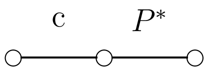
As an interesting example, we take the Petersen geometry $P$ with points (resp. lines) the set of vertices (edges) of the Petersen graph, and natural incidence. (To see that the automorphism group of the Petersen graph is isomorphic to $\Sigma_5$, identify vertices of it with 2-sets of $\{1, 2, 3, 4, 5\}$; then two vertices are adjacent if and only if the corresponding 2-sets are disjoint.)
This rank 2 geometry $P$ appears in a lot of geometries for sporadic groups (see [IS]), and also comes up in the list of interesting $(g, d_p, d_l)$-gons in [BV] Theorem 1. Here, $d_p = 5$, $d_l = 6$, and $g = 5$. Hence this geometry belongs to the class of $(5, 5, 6)$-gons; as an individual, it is usually denoted by the diagram 

($P$). The 'dual' of it, the geometry on the same set of objects with the same incidence relation, but the roles of points and lines interchanged, is then given the diagram 

($P^*$).

Geometries, whose rank 2 residues are $(g, d_p, d_l)$-gons, we call Buekenhout-Tits-geometries (BT-geometries), and their diagrams BT-diagrams. In their graphical description, to the edge representing a rank 2 residue, which is some $(g, d_p, d_l)$-gon, will be attached the three parameters $g, d_p, d_l$. As usual, to the node $i$ of the diagram there will be attached the *local parameter*, which describes, how many chambers lie on a flag of co-type $i$.
Sometimes, as with the Petersen geometry above, rank 2 residues are prescribed as indivuduals, then clearly there is no need to note the local parameters.

In this paper, we will treat the diagram




which is called (c.P*). Here, the local parameters of the circle geometry are determined by the local parameters in the Petersen geometry, hence are 1 and 2.

If interest is focussed on the girth, one gets a 'derived' diagram of the 'Coxeter type' again: a set of numbers $m_{ij}$'s, one for each rank 2 residue; this is the minimal circuit diagram. Of course, just like in Coxeter diagrams, we can define sphericity for minimal circuit diagrams. So far it is known that for rank 3 geometries with non-spherical minimal circuit diagram the simply connected ones are infinite ([Ro],[GS]). It is not clear for what class of diagrams (properly containing the class of Coxeter diagrams) the converse holds.

In this paper, we classify simply connected (c.P*)-geometries: see the theorem below.

The interest in the particular BT-diagram (c.P*) comes from the fact that it is 'close' to being a Coxeter diagram, that its minimal circuit diagram is spherical, but, as will be shown, simply connected geometries having this diagram are infinite. It is maybe worth noting that there are finite simply connected geometries with

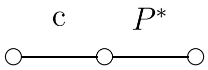
diagram (c*.P) (see [Me],[IS]). Hence for a possible definition of sphericity of BT-diagrams not only the set of all rank 2 residues must be used, but also the way they are amalgamated.
By ‘close’, we mean that all parameters (also the local ones) of the rank 2 geometries in this diagram differ only slightly from the ones in some generalized polygon.
Rank 2 residues in (c.P*)-geometries are indeed close to generalized polygons of order 2: the circle geometry on 4 points can be viewed as an affine plane of order 2, and the (dual) Petersen geometry can be viewed as an affine part of the $Sp_4(2)$-quadrangle.

**Theorem.** Let $\Gamma$ be a connected geometry with diagram (c.P*) and flag-transitive automorphism group $\Gamma$. Then $\Gamma$ is a quotient of the universal 2-cover of one of the two examples $\Delta_1$ and $\Delta_2$ below. These universal 2-covers are not isomorphic, and they are both infinite.

The two examples mentioned in the theorem are given in the following. They will be given in a slightly different setting once more as examples (8) in section 2.

**Examples:**

(1) Let $V$ be an elementary abelian 2-group of order $2^4$, viewed as a 2-dimensional $GF(4)$-space. Consider $X = \Sigma_5$ as a subgroup of $\Gamma L_2(4)$ acting naturally on $V$. Let $W$ be an affine space with translation group $V$, and consider the semi-direct product $G = V.X$ as a subgroup of $\mathrm{Aut}(W)$. Then $G$ acts 2-transitively on points of $W$ and some Sylow 2-subgroup $S$ of $X$ equals the stabilizer of two points (the pointwise stabilizer in $G$ of a line of $W$) in $W$; moreover these two are the only fixed points of $S$ on $W$. The group $G$ has precisely three classes of involutions, one of which contains the transpositions of $X$. Let $t$ be one such transposition of $X$. Then $t$ has precisely four fixed points on $W$, which form a plane $\pi$ in $W$, whose pointwise stabilizer in $G$ is again $\langle t\rangle$. The setwise stabilizer $T$ of this line $l$ in $G$ is isomorphic to $Z_2 \times \Sigma_4$ and acts 4-transitively on points of $l$. Consider the geometry $\Delta_1$, whose points are the points of $W$, whose lines are the lines of $W$ and whose planes are the planes in the $G$-orbit of $\pi$. Take over incidence from $W$. Then $\Delta_1$ is a geometry, on which $G = V.X$ acts flag-transitively, and $\Delta_1$ has diagram




The residues of planes and lines have obviously the indicated types. Consider the residue of the point $p$ fixed by $X$. Then $X$ is the stabilizer of $p$ in $G$, and if $\pi'$ is a plane in $\mathrm{res}(p)$, then the pointwise stabilizer $\langle t'\rangle$ of $\pi'$ is a transposition in $X$ and the setwise stabilizer of $\pi'$ in $X = G_p$ is $C_X(t')$. Let $L$ be a line in $\mathrm{res}(p)$. Then the stabilizer in $G_p$ of $L$ is a Sylow 2-subgroup $S$ of $X$, and $L$ is contained in $\pi'$ if and only if $t'$ is contained in $S$. Now it is clear that $\mathrm{res}(p)$ is isomorphic to the dual of the Petersen geometry.

(2) Consider the set $\Psi = \{1, 2, 3, 4, 5, 6\}$. Let points of the geometry $\Delta_2$ be the elements of $\Psi$, planes the 4-subsets of $\Psi$, and lines the pairs $(l, \pi)$ of a 2-set $l$ of $\Psi$

and a 4-set $\pi$ of $\Psi$ containing $l$. Let incidence be defined in the natural way. Then $\Delta_2$ has the diagram $(c.P^*)$ and $\Sigma_6$ acts flag-transitively on $\Delta_2$.
Note that the geometry $\Delta_2$ fails to satisfy condition (LL), whereas (LL) is satisfied in the geometry $\Delta_1$.

## 2 The proof of the theorem.

Let in this whole section $\Gamma$ be a $(c.P^*)$ geometry with chamber transitive group of automorphisms $G$. For any vertex $y$ in $\Gamma$, let $G_y$ be the stabilizer of $y$ in $G$, and let $K_y$ denote the kernel of the group $G_y$ acting on $\mathrm{res}(y)$. Let $c = \{p, l, x\}$ be a chamber, where $p$ is a point, $l$ a line and $x$ a hyperline (plane). Then we denote $B = G_c, X_0 = G_{lx}, X_1 = G_{px}$ and $X_2 = G_{pl}$.

We determine the structure of the groups $B, X_0, X_1$, and $X_2$, and the way they are amalgamated, in a number of steps.

**(1)** $G_p/K_p = \mathrm{Alt}(5)$ or $\Sigma_5$.
*Proof.* The automorphism group of the Petersen graph is isomorphic to $\Sigma_5$, flag-transitivity implies that a group whose order is divisible by 3 and 5 is induced by $G_p$ on $\mathrm{res}(p)$, hence the statement follows.

We refer to the two possibilities as to the cases $(A)$ and $(\Sigma)$. It will turn out, that the geometries arising in the two cases coincide, i.e. for every simply connected $(c.P^*)$-geometry on $\mathrm{res}(p)$ there is $\Sigma_5$ induced by the stabilizer of $p$ in the full automorphism group, and also there is a chamber transitive subgroup inducing only $\mathrm{Alt}(5)$ on it.

**(2)** $G_{pl}/K_p = Z_2 \times Z_2$, $G_{px}/K_p = \Sigma_3$, and $B/K_p = Z_2$ in case $(A)$, while $G_{pl}/K_p = D_8$, $G_{px}/K_p = \Sigma_3 \times Z_2$ and $B/K_p = Z_2 \times Z_2$ in case $(\Sigma)$.
*Proof.* This follows from (1).

**(3)** $G_x/K_x = \Sigma_4$, $B/K_x = Z_2$, $G_{lx}/K_x = Z_2 \times Z_2$ and $G_{px}/K_x = \Sigma_3$.
*Proof.* By (2), $G_{px}/K_p = \Sigma_3$ resp. $\Sigma_3 \times Z_2$ in the case $(A)$ resp. $(\Sigma)$. In both cases, $|G_{px} : B| = 3$. Since $K_x \leq B$, $G_{px}$ induces $\Sigma_3$ on $\mathrm{res}(p) \cap \mathrm{res}(x)$ in both cases. Now the statement follows.

**(4)** $K_p = 1$.
*Proof.* Assume case $(A)$. Then $K_x$ is a normal subgroup of $G_{px}$ contained in $B$, hence $K_x \leq K_p$. Now $K_p = K_x$ is a subgroup of $B$ invariant under $G_p$ and $G_x$. But by connectedness of $\Gamma$ these two parabolics generate $G$, hence $K_p$ is $G$-invariant and therefore $K_p = 1$.
Assume therefore we are in case $(\Sigma)$. Let $p$ and $a$ be the two points in the residue of $l$. Then $K_a K_p$ is normal in $G_l$ and also contained in $B$. Moreover, $K_p \leq K_x$ and $K_a \leq K_x$ by (3). But $B/K_p$ has order 4. Hence one of the following holds:

- $K_a K_p = K_p$, then $K_p = K_a$ is invariant under $G_p$ and $G_l$, hence is a normal subgroup of $G$ contained in $B$, and therefore equals 1.

- $K_a K_p / K_p$ is a nontrivial normal subgroup of $G_{pl}/K_p$ contained in $K_x/K_p$, which is of order 2 by (2) and (3). Hence $K_a K_p = K_x$ is invariant under $\langle G_l, G_x \rangle = G$, and $K_a K_p = 1$, a contradiction.

We have now determined the structures of $B, X_1$ and $X_2$ completely in both cases. But the structure of $X_0$ and $\langle X_0, X_1 \rangle$ and $\langle X_0, X_2 \rangle$ is still to be described. At least, we have the following.

**(5)** $G_l = G_{pl}.G_{lx}$.
*Proof.* This follows since res$(l)$ is a generalized digon.

Let us introduce the chamber system $C$ of $\Gamma$. Its chambers are the chambers (maximal flags) of $\Gamma$ and two chambers are $i$-adjacent, if they are contained in a residue of co-type $i$. Clearly, the flag-transitive group $G$ acts chamber transitively on $C$.

We treat the cases $(A)$ and $(\Sigma)$ now separately and turn to the case $(A)$ first.

**(6) Lemma.** Assume we are in case $(A)$. Then we find elements $a, b, c, d$ of order 2 in $G$ such that $G = \langle a, b, c, d \rangle$ and one of the following sets of relations hold:

$(A1)$ $(ab)^3 = (bc)^5 = (bac)^5 = (ac)^2 = (bd)^3 = (ad)^2 = a(cd)^2 = 1$.

$(A2)$ $(ab)^3 = (bc)^5 = (bac)^5 = (ac)^2 = (bd)^3 = (ad)^2 = (cd)^2 = 1$.

Furthermore, $C$ is isomorphic to $C(G; \langle a \rangle; \langle a, b \rangle, \langle a, c \rangle, \langle a, d \rangle)$ in both cases.

*Proof.* Let $B = \langle a \rangle$. Then $a$ is an involution by (2) and (4). Moreover $G_{px} = \Sigma_3$, hence we can pick another involution $b$ in $G_{px}$ different from $a$ and get the relation $(ab)^3 = 1$. Again by (2) and (4), $G_{lx} = Z_2 \times Z_2$, and we pick an involution $c \in G_{pl}$ different from $a$. Clearly $(ac)^2 = 1$. Now in the group $G_p = \mathrm{Alt}(5)$ we verify that the relations $(bc)^5 = (bac)^5 = 1$ hold.
Since $G_x = \Sigma_4$, and $G_{px} = \Sigma_3$, we see that $a$ corresponds to some transposition in $G_x = \Sigma_4$. Also, $G_{lx} = Z_2 \times Z_2$ contains exactly two transpositions of $G_x = \Sigma_4$. Hence we may pick the second transposition different from $a$ in $G_{lx}$, and call it $d$. Clearly, the relations $(ad)^2 = (bd)^3 = 1$ hold.
Now $G_l = G_{pl}.G_{lx}$ and $B = G_{pl} \cap G_{lx}$ by (5), and therefore $(cd)^2 = [c, d]$ is contained in $B = \langle a \rangle$. This gives two possibilities for the last relation, and (6) is proved.

Let us turn to the case $(\Sigma)$ now.

**(7)** Assume we are in case $(\Sigma)$. Then we can find involutions $a, b, c, w$ and $t$ in $G$ such that $G = \langle a, b, c, w, t \rangle$ and one of the following sets of relations hold:

(I) $(ab)^2 = (ac)^3 = (at)^2 = (bc)^3 = [b, w]^{bt} = (bt)^2 = [t, w]^{bt} = (ct)^2 = (cw)^6 = w^{bcb} w^{cw} bcw = [a, w] = 1$.

(II) $(ab)^2 = (ac)^3 = (at)^2 = (bc)^3 = [b,w]^{bt} = (bt)^2 = [t,w]^{bt} = (ct)^2 = (cw)^6 = w^{bcb}w^{cw}bcw = [a,w]^{bt} = 1$.

Note that these two sets of relation differ only in the relation $[a,w] = 1$ or $bt$. Furthermore, $C$ is isomorphic to $C(G; \langle b,t\rangle; \langle b,t,a\rangle, \langle b,t,c\rangle, \langle b,t,w\rangle)$ in both cases.
*Proof.* We recall, how $G_p = \Sigma_5$ acts on res$(p)$, which isomorphic to the Petersen graph. Here hyperlines correspond to 2-sets of $\{1,2,3,4,5\}$, whereas lines can be identified with the 2-sets of disjoint 2-sets of $\{1,2,3,4,5\}$. Hence we may assume that $x$ corresponds to $\{4,5\}$, and $l$ corresponds to $\{\{1,2\},\{4,5\}\}$, and so $B = \langle(1,2),(4,5)\rangle$, $G_{pl} = \langle B,(1,4)(2,5)\rangle$ and $G_{px} = \langle B,(2,3)\rangle$ in the natural representation of $G_p = \Sigma_5$. Pick elements $b, c, w$ and $t$ corresponding to $(1,2)$, $(2,3)$, $(1,4)(2,5)$ and $(4,5)$ respectively.
Then they are all involutions, and we have the relations $(bc)^3 = (bt)^2 = (ct)^2 = [b,w]^{bt} = 1$. To make it clear: $B = \langle b,t\rangle$, $G_{pl} = \langle b,w,t\rangle$, $G_{px} = \langle b,c,t\rangle$.
Clearly also $[t,w] = bt$, and one can easily verify in $\Sigma_5$ that $(cw)^6 = w^{bcb}w^{cw}bcw = 1$.
Clearly, $t$ lies in the center of $G_{px}$, and since $G_x/K_x = \Sigma_4$, $|K_x| = 2$, we have $\langle t\rangle = K_x$. Moreover, by the action of $G_x$ on res$(x)$, $G_{lx}/K_x$ is the centralizer of $B/K_x$ in $G_x/K_x = \Sigma_4$. Hence we can take some conjugate $a$ of $b$ in $G_x$, which together with $B$ generates $G_{lx}$ and satisfies modulo $K_x$ the relations $(ac)^3 = (ab)^2 = 1$. Clearly $[a,t] = 1$ holds.
In $G$ this implies relations $a^2 = 1$, and $(ab)^2, (ac)^3 \in \langle t\rangle$.
By (5), we get $[a,w] \in B$. Hence we have 16 possibilities for the triples of elements $((ab)^2, (ac)^3, [a,w])$.
Coset enumeration using CAYLEY gives that only four sets of relations do not force the group to collapse:
if we put $(ab)^2 = t^i, (ac)^3 = t^j, [a,w] = b^u t^v$, these are the cases:

$$(i,j,u,v) \in \{(2,2,2,2),(2,2,1,1),(2,1,2,2),(2,1,1,1)\}.$$

As we can see, $(ab)^2 = 1$ holds in any case, and by replacing $a$ by $at$, we can identify the cases $(i,j,u,v) = (2,2,2,2)$ and $(i,j,u,v) = (2,1,1,1)$, and the cases $(i,j,u,v) = (2,1,2,2), (i,j,u,v) = (2,2,1,1)$.
Hence we end up with the two possible sets of relations:

(I): $a^2 = b^2 = c^2 = w^2 = t^2 = (ab)^2 = (ac)^3 = (at)^2 = (cb)^3 = (ct)^2 = (bt)^2 = b^w t = (cw)^6 = w^{bcb}w^{cw}bcw = [a,w] = 1 >$

(II): $a^2 = b^2 = c^2 = w^2 = t^2 = (ab)^2 = (ac)^3 = (at)^2 = (cb)^3 = (ct)^2 = (bt)^2 = bwt = (cw)^6 = w^{bcb}w^{cw}bcw = [a,w]bt = 1 >$

This proves (7).

The first case lives in some group of type $2^4\Sigma_5$, the second case lives in the group $\Sigma_6$, as we shall see in the following example. In fact, the geometries (chamber systems) in example (8) are the same as the ones given in section 1.

**(8) Examples.**

(i) Consider in $G = \Sigma_6$ the following elements:
$a = (1,2)$, $c = (2,3)$, $b = (3,4)$, $w = (3,5)(4,6)$, $t = (5,6)$.
Then obviously, $G = \langle a, b, c, w, t \rangle$, and it is easily verified that the relations (I) hold between these elements.

(ii) Consider in $G = \Sigma_{16}$ the following elements:
$a = (1,2)(5,6)(7,9)(8,12)(10,11)(13,14)$,
$b = (3,4)(7,10)(8,13)(9,11)(12,14)(15,16)$,
$c = (2,3)(5,7)(6,8)(9,12)(11,15)(14,16)$,
$w = (3,5)(4,6)(7,11)(8,9)(10,14)(12,13)$,
$t = (5,6)(7,8)(9,12)(10,13)(11,14)(15,16)$.

Then $N = \langle ab, (ab)^c, (ab)^{cw}, (ab)^{cwc} \rangle$ is an elementary abelian normal subgroup of order 16 of $G$, and it is easily seen, that $b, c, c^{wt}, t$ generate a subgroup isomorphic $\Sigma_5$, containing $w$.
Hence $G$ is isomorphic to $2^4\Sigma_5$, and the relations (II) are easily verified.

Let us consider the following groups.

$$G_I = \langle a, b, c, w, t : a^2 = b^2 = c^2 = w^2 = t^2 = (ab)^2 = (ac)^3 = (at)^2 = (cb)^3 = (ct)^2 = (bt)^2 = b^w t = (cw)^6 = w^{bcb} w^{cw} bcw = [a, w] = 1 \rangle,$$

$$G_{II} = \langle a, b, c, w, t : a^2 = b^2 = c^2 = w^2 = t^2 = (ab)^2 = (ac)^3 = (at)^2 = (cb)^3 = (ct)^2 = (bt)^2 = b^w t = (cw)^6 = w^{bcb} w^{cw} bcw = [a, w]^{bt} = 1 \rangle.$$

And denote by $A_I$ (resp. $A_{II}$) the subgroup of $G_I$ (resp. $G_{II}$) generated by $at, bt, ct$ and $w$.

**(9) Lemma.** The following holds:

(a) The chamber systems $C_I = C(G_I; \langle b, t \rangle; \langle b, t, a \rangle, \langle b, c, t \rangle, \langle b, w, t \rangle)$, $C_{II} = C(G_{II}; \langle b, t \rangle; \langle b, t, a \rangle, \langle b, c, t \rangle, \langle b, w, t \rangle)$ have diagram (c.P*).

(b) The group $A_I$ (respectively $A_{II}$) is a chamber transitive subgroup of $G_I$ (respectively $G_{II}$).

(c) The chamber systems $C_I$ and $C_{II}$ are simply 2-connected.

(d) The groups $A_I$ and $A_{II}$ have presentations (A1) and (A2) respectively, and the chamber systems $C_I$ and $C_{II}$ are isomorphic to the chamber systems described in (6).

(e) $C_I$ and $C_{II}$ are not isomorphic.

*Proof.* It is left to the reader to verify that the given relations not only hold in the parabolic subgroups of the so presented groups, but also define the corresponding parabolic subgroup. And, by (8), since there is no collapsing in the finite quotients $2^4\Sigma_5$ and $\Sigma_6$, there is no collapsing in the groups $G_I$ and $G_{II}$ either, and the parabolics in $G_I$ and $G_{II}$ look as follows:

$\langle b,c,w,t\rangle = \Sigma_5$, $\langle a,b,c,t\rangle = \Sigma_4 \times Z_2$, $\langle a,b,w,t\rangle$ is a group of order 16, $\langle b,c,t\rangle = \Sigma_3 \times Z_2$, $\langle a,b,t\rangle = Z_2 \times Z_2 \times Z_2$, and $\langle w,b,t\rangle$ is dihedral of order 8, $\langle b,t\rangle = Z_2 \times Z_2$. Hence (a) follows.

Obviously, the groups $\langle bt,ct,w\rangle$, $\langle at,bt,ct\rangle$ and $\langle at,bt,w\rangle$ are isomorphic to Alt(5), $\Sigma_4$ and a group of order 8 respectively, and act transitively on the corresponding residues.
This implies (b).

The automorphism group $G_I$ (resp. $G_{II}$) of the chamber system $C_I$ (resp. $C_{II}$) lifts to a group of automorphisms of the universal 2-cover of the chamber system $C_I$ (resp. $C_{II}$). This group has to is generated by the generators of $G_I$ (resp. $G_{II}$), and satisfies the defining relations. Hence it equals $G_I$ (resp. $G_{II}$). Obviously, (c) follows.

Now we know the simply 2-connected (c.P*) chamber systems $C_I$ (resp. $C_{II}$) with transitive group of automorphisms $A_I$ (resp. $A_{II}$), such that Alt(5) is induced on the corresponding residues. By (6), $A_I$ (resp. $A_{II}$) are quotients of the groups presented by (A2) or (A1). These quotients must not be proper quotients, as $C_I$ and $C_{II}$ are simply 2-connected. (d) follows.

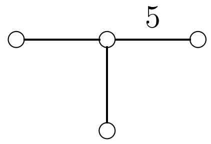
By (7), the automorphism group of the chamber system of any (c.P*) geometry has 'point' stabilizer isomorphic to Alt(5) or $\Sigma_5$, hence the Frattini argument tells that $G_I$ is the full automorphism group of $C_I$ ($G_{II}$ is the full automorphism group of $C_{II}$ respectively). Since line stabilizers are not isomorphic in the groups $G_I$ respectively $G_{II}$, the two chamber systems can not be isomorphic to each other. The same argument applies to the chamber-transitive subgroups of index 2, $A_I$, $A_{II}$ of the groups $G_I$, $G_{II}$ respectively. Moreover, the isomorphism type of the chamber system it acts on is determined by the structure of the line stabilizer. Hence the group presented by (A1) is a subgroup of $G_I$, while the group presented by (A2) is a subgroup of $G_{II}$. This implies (e).

**(10) Lemma.** The groups $G_I$ and $G_{II}$ (and equivalently the chamber systems $C_I$ and $C_{II}$) are infinite.

*Proof.* First, we prove that $G_I$ is infinite. For that purpose, it is enough to find a subgroup $U$ of $G_I$ which is infinite; we will in fact find a suitable subgroup $U$ of $G_I$ and show that $U/U'$ is infinite. This is a task that can most efficiently be verified by the 'abelian quotients'-algorithm which is implemented in the group theoretical programming language CAYLEY - as long as the index $|G_I : U|$ is not too big.
We first derive generators for the (normal) subgroup $N$ of $G_I$ such that $G_I/N = \Sigma_6$: $N = \langle (x_{12} \times x_{25})^3, (x_{12} \times x_{26})^3, (x_{13} \times x_{25})^2, (x_{13} \times x_{26})^2, (x_{14} \times x_{25})^2, (x_{14} \times x_{26})^3, (x_{15} \times x_{23})^2 \rangle$, where the elements $x_{ij}$ are involutions (the conjugates $a, a^c, a^{cb}$, $a^{cw}, c, c^w, c^{wt}$) that project onto the transposition $(i,j)$ in $\Sigma_6 = G_I/N$, $((ij) = (12),(13),(14),(15),(23),(25),(26)$ ).
Still, $|G_I : N| = 720$ is rather big for the abelian quotients algorithm. But for the subgroup $U = \langle a,b,t,N\rangle$ with index 90 in $G_I$, the algorithm works well, and shows

that the invariants of $U/U'$ are $(2,2,2,0)$, which immediately tells that $U/U'$ and hence $U$ and hence $G_I$ are infinite.

For the proof that $C_{II}$ is infinite we turn to the chamber transitive subgroup of $A_{II}$ of $G_{II}$, which has the rather nice presentation (A2).
Clearly, the group $A_{II}$ with the presentation (A2) is the quotient of the Coxeter group $W(s_0, s_1, s_2, s_3)$ of type




modulo the relation that factors out the central involution in the parabolic subgroup $\langle s_0, s_1, s_2\rangle$ of type $H_3$ (compare also remark (13)). We will construct an infinite group having four involutory generators that satisfy these relations.

Consider the free $Z$-module $V$ with basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. And consider the linear transformations of $V$:

$$a := (e_1, e_2)(e_5, e_6) \;\; b := (e_2, e_3)(e_4, e_5) \;\; c := (e_3, e_4)(e_5, e_6) \;\; d := (e_1, -e_2)(e_5, -e_6).$$

Then the elements $a, b, c, d$ (are involutions and) satisfy the relations of the above Coxeter diagram.
Moreover, the group $\langle a, b, c\rangle$ is isomorphic to $A_5$, and hence $(bac)^5 = 1$. Note, however, that the relation $(bdc)^5 = 1$ does not hold: $\langle b, d, c\rangle$ is isomorphic to $Z_2 \times A_5$.
Then the vector $v := e_1 - e_2$ (and hence the translation $T_v$ of $V$) is centralized by $c$ and $d$, and is sent onto its negative by $a$. Furthermore, the equation $v - v^b + v^{ba} = 0$ holds.
Now it is easily checked that the elements $A = a.T_v$, $B = b$, $C = c$, $D = d$ still satisfy the relations of the diagram together with $(BAC)^5 = 1$ : if not, the group $\langle A, B, C\rangle$ would be a group of type $Z_2 \times A_5$ as well, and the central involution would be a translation, which is clearly impossible. And the subgroup $G := \langle A, B, C, D\rangle$ of the affine group on $V$ is infinite: $G$ contains the group $\langle B, C, D\rangle$ which is isomorphic to $Z_2 \times A_5$, and hence has a unique fixed point on $V$, namely the zero vector. As $A$ does not fix the zero vector, $G$ has no fixed point on $V$, not even on $V$ when tensored with the reals. But a finite subgroup of $\mathrm{Aff}(6, R)$ must have a fixed point.

The above group is an infinite quotient of $X$, the Coxeter group $W(s_0, s_1, s_4, s_3)$ modulo the relation $(s_1 s_0 s_2)^5 = 1$. Hence $X$ itself is infinite. Note that there are obvious examples (of affine types) where the corresponding quotients are finite.

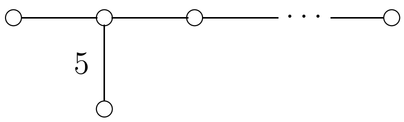
**(11) Lemma.** Let $C$ be a chamber system of type (c.P*) with flag-transitive group of automorphisms $G$. Then the geometry $G(C)$, which equals the group geometry on cosets of the stabilizers $G_p, G_l$ and $G_x$ of the rank 2 residues of $C$ containing a given chamber, has type (c.P*) and $G$ acts flag-transitively on it.
*Proof.* We know that $G_p$ induces at least $A_5$ on res$(p)$, and $G_x$ induces $\Sigma_4$ on res$(x)$. Further, $G_l$ is flag-transitive on the residue of $l$, which contains just four elements. The lemma follows from application of either [MT] or [As]. We decided to apply a result by Aschbacher ([As]).

Obviously, $G$ can not be written as a product of two of the stabilizers, as $G'_p$ does not fix all points. And $G_{lx} = G_l \cap G_x$ has at most two orbits on $G_l/G_{pl}$ and $G_x/G_{px}$. Now [As] gives the result.

**(12) Proof of the theorem.**

By (4), we are in case $(A)$ or $(\Sigma)$. Hence, by (6), (7) and (9), the chamber system $C(\Gamma)$ is isomorphic to $C_I$ or $C_{II}$. By (11), $G$ is isomorphic to $\Gamma(C_I)$ or $\Gamma(C_{II})$, and by (10), these are infinite.
By (9), $G$ is isomorphic to the universal 2-cover of one of the two examples of (8).

**(13) Remark.** Consider the Coxeter $W$ group with diagram




in its reflection representation on the real vector space of dimension $n+1$. The generating involutions $s_i$ are reflections in basis vectors $e_i$ with respect to the symmetric bilinear form $B$, which has Gram matrix $M = (-\cos(\pi/m(i,j))_{i,j}$ on the basis $E = \{e_0, \cdots, e_n\}$. Note that in $2M$, entries are $0, -1, 2$ and $\varphi$, where $\varphi$ is the negative real solution of $X^2 + X - 1 = 0$. ($-\varphi - 1$ is the positive one.) Let $R$ be the subring of the reals generated by 1 and $\varphi$. Clearly, if $I$ is the ideal of $R$ generated by the prime 2, then $R/I$ is a field $k$ with 4 elements.
Then the group $W$ leaves invariant the $R$-module $L$ generated by $E$, and acts on the $k$-space $V := L/I.L$ which carries the symmetric bilinear form $\langle \cdot, \cdot \rangle$ induced from $2M$. Identify the basis $E$ with its image in $V$.
The radical $V_0$ of this space is spanned by the vector $e_0 + \varphi.e_1$, if $n$ is even (resp. by $e_0 + \varphi.e_1$ and $e_1 + e_3 + \cdots + e_n$, if $n$ is odd).
Hence $V/V_0$ is a nondegenerate symplectic space of dimension $n$ (resp. $n-1$) and obviously $W$ induces $Sp_n(k)$ (resp. $Sp_{n-1}(k)$) on $V/V_0$.
The relation $(s_2 s_1 s_0)^5 = 1$ holds in $X := W/C_W(V)$, as clearly for $n = 2$ just $Sp_2(4) = A_5$ is induced on $V$.

Consider in $X$ the 'parabolic subgroups' $X_i = \langle s_j, j \text{ different from } i \rangle$, $i = 0, 1, \cdots, n$. Then the group geometry on cosets of $X_0, X_2, X_3, \cdots, X_n$ is a geometry of type $(c^{n-1}.P^*)$.

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Math. Institut der JLU Gießen
Arndtstraße 2
35392 Gießen
BR Deutschland
e-mail : Thomas.Meixner@math.uni-giessen.de