Simultaneous Generation of Koecher and Almkvist-Granville's Apéry-Like Formulae

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CONTENTS

We prove a very general identity, conjectured by Henri Cohen, involving the generating function of the family $(\zeta(2r+4s+3))_{r,s\geq 0}$: it unifies two identities, proved by Koecher in 1980 and Almkvist and Granville in 1999, for the generating functions of $(\zeta(2r+3))_{r\geq 0}$ and $(\zeta(4s+3))_{s\geq 0}$, respectively. As a consequence, we obtain that, for any integer $j \geq 0$, there exists at least $[j/2]+1$ Apéry-like formulae for $\zeta(2j+3)$.

1. INTRODUCTION

In proving that $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$ is irrational, Apéry [Apéry 79] noted that

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3}. \tag{1–1}$$

Although the series on the right-hand side converges much faster than the defining series for $\zeta(3)$, Formula (1–1) is not essential in Apéry's proof since truncations of this series are not diophantine approximations to $\zeta(3)$. On the other hand, it is very likely that (1–1) was a source of inspiration for Apéry[1] and many authors have looked for similar identities, in the hope that they might give some idea of how to prove the irrationality of $\zeta(2s+1) = \sum_{k=1}^{\infty} 1/k^{2s+1}$ for any integer $s \geq 2$; see for example [Borwein and Bradley 97, Cohen 81, Koecher 80, Leshchiner 81, van der Poorten 80]. This problem is far from being solved,[2] but many beautiful Apéry-like formulae have been proved. In fact, two apparently unrelated families of such formulae for $\zeta(2s+3)$ and $\zeta(4s+3)$ have emerged, both of which are more easily explained

[1] See [Cohen 78, van der Poorten 79] for a detailed explanation of Apéry's original method.

[2] We now know that infinitely many of the values $\zeta(2s+1)$ $(s \geq 1)$ are $\mathbb{Q}$-linearly independent [Ball and Rivoal 01, Rivoal 00] and that at least one amongst $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational [Zudilin 04].

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with the help of the generating functions

$$\sum_{s=0}^{\infty} \zeta(2s+3)\, a^{2s} = \sum_{n=1}^{\infty} \frac{1}{n(n^2-a^2)}$$

and

$$\sum_{s=0}^{\infty} \zeta(4s+3)\, b^{4s} = \sum_{n=1}^{\infty} \frac{n}{n^4-b^4}.$$

(The series on the left-hand sides of the equal signs converge only for $|a| < 1$ and $|b| < 1$, whereas the right-hand sides converge on much larger domains.) Koecher [Koecher 80] (and independently Leshchiner [Leshchiner 81] in an expanded form) proved that

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2-a^2)} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3}\frac{5k^2-a^2}{k^2-a^2}\prod_{n=1}^{k-1}\left(1-\frac{a^2}{n^2}\right), \tag{1–2}$$

for any complex number a such that $|a| < 1$, and, more recently, Almkvist and Granville [Almkvist and Granville 99] proved another identity, first conjectured by Borwein and Bradley [Borwein and Bradley 97]:

$$\sum_{n=1}^{\infty} \frac{n}{n^4-b^4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}}\frac{5k}{k^4-b^4}\prod_{n=1}^{k-1}\left(\frac{n^4+4b^4}{n^4-b^4}\right), \tag{1–3}$$

for any complex number b such that $|b| < 1$. For $a = b = 0$, these identities reduce to (1–1), but otherwise produce different identities for the values of the zeta function at odd integers. For example, Borwein and Bradley note that (1–2) implies

$$\zeta(7) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^7} - 2\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^5j^2} + \frac{5}{2}\sum_{k>j>i\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^2i^2}$$

while (1–3) implies

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^7} + \frac{25}{2}\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^4}.$$

The purpose of this article is to prove the following very general generating function identity, which was conjectured by H. Cohen on the basis of computations in Pari.

Theorem 1.1. *Let a and b be complex numbers such that $|a|^2 + |b|^4 < 1$. Then*

$$\sum_{n=1}^{\infty} \frac{n}{n^4-a^2n^2-b^4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k}\frac{5k^2-a^2}{k^4-a^2k^2-b^4}\prod_{n=1}^{k-1}\left(\frac{(n^2-a^2)^2+4b^4}{n^4-a^2n^2-b^4}\right). \tag{1–4}$$

We remark that Identity (1–4) unifies (1–2) (case $b = 0$) and (1–3) (case $a = 0$); consequently, it should yield new Apéry-like formulae. This is indeed true since

$$\sum_{n=1}^{\infty} \frac{n}{n^4-a^2n^2-b^4} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \binom{r+s}{r}\zeta(2r+4s+3)\, a^{2r}\, b^{4s},$$

and since the number of representations of an integer $j \geq 0$ as $j = r + 2s$ with integers $r, s \geq 0$ is $[j/2] + 1$. Hence, (1–4) produces $[j/2] + 1$ different identities for $\zeta(2j+3)$ for any integer $j \geq 0$, obtained by differentiating the right-hand side of (1–4) r, respectively s, times with respect to a^2, respectively b^4, with $j = r + 2s$, and then by letting $a = b = 0$.

For $0 \leq j \leq 2$, one of r and s is 0 and we only obtain identities resulting from (1–2) or (1–3). This is also the case for $j = 3$, $(r, s) = (3, 0)$. The first apparently new identity is for $j = 3$, $(r, s) = (1, 1)$:

$$\begin{aligned}\zeta(9) = {} & \frac{9}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}k^9} + 5\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^5j^4} \\ & + 5\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^6} - \frac{5}{4}\sum_{k>j\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^7j^2} \\ & - \frac{25}{4}\sum_{k>j>i\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^4i^2} - \frac{25}{4}\sum_{k>j>i\geq 1} \frac{(-1)^{k+1}}{\binom{2k}{k}k^3j^2i^4}.\end{aligned}$$

To prove Theorem 1.1, we will use Borwein and Bradley's method in which the proof of (1–4) was reduced in several steps to the proof of a finite combinatorial identity (the last step in [Borwein and Bradley 97] is due to Wenchang Chu), which was finally proved by Almkvist and Granville. In our case, we will show that Theorem 1.1 follows from the identity

$$\sum_{k=1}^{n} \frac{2}{k^2-a^2} \cdot \frac{\prod_{j=1}^{n-1}(k^2+(j-k)^2-a^2)(k^2+(j+k)^2-a^2)}{\prod_{j=1,\, j\neq k}^{n}(k^2-j^2)(k^2+j^2-a^2)} = \frac{1}{n^2-a^2}\binom{2n}{n}$$

for any integer $n \geq 1$, which we will then prove as corollary of the following result.

Theorem 1.2. *Let $g(X) \in \mathbb{C}[X]$ be of degree at most 2. For any integer $n \geq 1$ and any complex numbers a and t, with $a \notin \{\pm 1, \pm 2, \dots, \pm n\}$, we have that*

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n}{n-k} \frac{4k^2}{k^2-a^2} \cdot \Bigg(\prod_{\substack{0 \leq j < n-k \\ \text{or } n<j<n+k}} \big(t(k^2-a^2)+g(j)\big) - \prod_{\substack{0 \leq j < n-k \\ \text{or } n<j<n+k}} g(j) \Bigg) = 0. \tag{1–5}$$

For the special case $a = 0$, we obtain the key identity proved in [Almkvist and Granville 99].

2. FIRST STEP

We transform the right-hand side of (1–4) by a partial fraction decomposition, with respect to b^4:

$$\frac{1}{k^4 - a^2k^2 - b^4} \prod_{n=1}^{k-1} \frac{(n^2-a^2)^2 + 4b^4}{n^4 - a^2n^2 - b^4} = \sum_{n=1}^{k} \frac{C_{n,k}(a)}{n^4 - a^2n^2 - b^4}, \tag{2–1}$$

where

$$C_{n,k}(a) = \frac{\prod_{j=1}^{k-1}(n^2+(j-n)^2-a^2)(n^2+(j+n)^2-a^2)}{\prod_{j=1,\, j\neq n}^{k}(j^2-n^2)(j^2+n^2-a^2)}. \tag{2–2}$$

Inserting (2–1) in the right-hand side of (1–4) and inverting the summations, we see that it will be enough to show that (and in fact, this is equivalent)

$$\sum_{n=1}^{\infty} \frac{n}{n^4-a^2n^2-b^4} = \sum_{n=1}^{\infty} \frac{1}{n^4-a^2n^2-b^4} \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}} \frac{5k^2-a^2}{2k} C_{n,k}(a).$$

Clearly, it is enough to show that, for any integer $n \geq 1$ and any complex a with $|a| < 1$,

$$\sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{\binom{2k}{k}} \frac{5k^2-a^2}{2k} C_{n,k}(a) = n. \tag{2–3}$$

From now on, and unless otherwise specified, we assume that $|a| < 1$.

3. SECOND STEP

We define $t_n(k)$ to be the summand of the series in (2–3) and δ to be $\sqrt{n^2-a^2}$ (for any fixed branch of the logarithm). We observe that $t_n(k)$ can be extended to a meromorphic function of the complex variable k:

$$t_n(k) = \frac{(-1)^n e^{i\pi k} n^2 \Gamma(1 \pm i\delta)(5k^2-a^2)}{\Gamma(1-n \pm i\delta)\Gamma(n \pm i\delta))\, k\, \Gamma(2k+1)} \cdot \frac{\Gamma(k+1)^2 \Gamma(k \pm n \pm i\delta)}{\Gamma(k+1 \pm n)\Gamma(k+1 \pm i\delta)}, \tag{3–1}$$

where $\Gamma(x \pm y \pm z)$ is defined to be $\Gamma(x+y+z)$ $\Gamma(x+y-z)\,\Gamma(x-y+z)\,\Gamma(x-y-z)$, etc.

We note that, as a result of the factor $\Gamma(k+1-n)$ in the denominator of (3–1), we have $t_n(k) = 0$ for $k = 1, \dots, n-1$. Furthemore, simple computations give that $t_n(0) = a^2n/(2n^2-a^2)$ and, for $k \in \{1, \dots, n\}$,

$$t_n(-k) = -\frac{n^3(n^2-a^2)}{2n^2-a^2} \binom{2k}{k} \cdot \frac{5k^2-a^2}{(n^2+(k-n)^2-a^2)(n^2+(k+n)^2-a^2)} \cdot \prod_{j=1}^{k-1} \frac{(n^2-j^2)(j^2+n^2-a^2)}{(n^2+(j-n)^2-a^2)(n^2+(j+n)^2-a^2)}. \tag{3–2}$$

We are now ready to prove our second step.

Proposition 3.1. *For any given $n \geq 1$, Equation (2–3) is equivalent to the following finite combinatorial identity:*

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{5k^2-a^2}{(n^2+(k-n)^2-a^2)(n^2+(k+n)^2-a^2)} \cdot \prod_{j=1}^{k-1} \frac{(n^2-j^2)(j^2+n^2-a^2)}{(n^2+(j-n)^2-a^2)(n^2+(j+n)^2-a^2)} = \frac{2}{n^2-a^2}. \tag{3–3}$$

Remark 3.2. Given any integer $n \geq 1$, if (3–3) is true for $|a| < 1$, it is true for any complex number a such that a^2 can not be written $a^2 = n^2 + m^2$ with an integer $m \in \{0, \pm 1, \dots, \pm n\}$.

Proof of Proposition 3.1: We will prove below that

$$\sum_{k=-n}^{+\infty} t_n(k) = 0. \tag{3–4}$$

Equation (3–4) can be written

$$\sum_{k=1}^{n} t_n(-k) = -t_n(0) - \sum_{k=1}^{n-1} t_n(k) - \sum_{k=n}^{\infty} t_n(k)$$
$$= -\frac{a^2 n}{2n^2 - a^2} - \sum_{k=n}^{\infty} t_n(k)$$

and $\sum_{k=n}^{\infty} t_n(k) = n$ is clearly equivalent to

$$\sum_{k=1}^{n} t_n(-k) = -\frac{2n^3}{2n^2 - a^2},$$

which, given (3–2), is exactly (3–3).

We now prove (3–4), and for that we closely follow Borwein and Bradley, whose method is based on Gosper's hypergeometric summation algorithm (see [Graham et al. 94, pages 225–227] for details). We note that

$$\frac{t_n(k+1)}{t_n(k)} = -\frac{1}{2}\frac{5(k+1)^2 - a^2}{5k^2 - a^2}\frac{k}{2k+1} \cdot \frac{(k \pm n \pm i\delta)}{(k+1 \pm n)(k+1 \pm i\delta)}$$
$$= \frac{p_n(k+1) q_n(k)}{p_n(k) r_n(k+1)},$$

is a rational function of k, with $q_n(k) = (k - n \pm i\delta)$, $r_n(k) = -2(2k-1)(k+n)$ and

$$p_n(k) = (5k^2 - a^2) \prod_{j=1}^{n-1} (k-j)(k+j \pm i\delta).$$

Since q_n and r_n do not have roots differing by integers,[3] Gosper's algorithm ensures that there exists a polynomial s_n of degree at most $\deg(p_n) - \deg(q_n - r_n) = 3n - 3$ such that $p_n(k) = s_n(k+1) q_n(k) - r_n(k) s_n(k)$. We now define

$$T_n(k) = \frac{r_n(k) s_n(k) t_n(k)}{p_n(k)},$$

which satisfies $T_n(k+1) - T_n(k) = t_n(k)$. Since $t_n(-n)$ is finite and $p_n(-n) \neq 0 = r_n(-n)$, we have $T_n(-n) = 0$. Hence, for any $k \geq 1 - n$, $T_n(k) = \sum_{j=-n}^{k-1} t_n(k)$. Since $\deg(r_n s_n) = \deg(p_n)$, we have $T_n(k) = O(t_n(k))$ as $k \to +\infty$, which implies that $T_n(k)$ tends to 0 as $k \to +\infty$. It follows that (3–4) holds. □

4. THIRD STEP

Here, we generalise the last reduction step of [Borwein and Bradley 97] (due to Wenchang Chu).

[3] Since $|a| < 1$ and $n \geq 1$, $i\delta$ can't be an integer.

Proposition 4.1. *Equation* (3–3) *for every integer* $n \geq 1$ *is equivalent to the following identity for every integer* $n \geq 1$*:*

$$\sum_{k=1}^{n} \frac{2}{k^2 - a^2} \cdot \frac{\prod_{j=1}^{n-1}(k^2 + (j-k)^2 - a^2)(k^2 + (j+k)^2 - a^2)}{\prod_{j=1,\, j\neq k}^{n}(k^2 - j^2)(k^2 + j^2 - a^2)}$$
$$= \frac{1}{n^2 - a^2}\binom{2n}{n}. \qquad (4–1)$$

Remark 4.2. The simplification (4–2) below shows that, given any integer $n \geq 1$, if (4–1) is true for $|a| < 1$, it is true for any complex number a such that $a \notin \{\pm 1, \ldots, \pm n\}$. Furthermore, it can also be written as

$$2\sum_{k=1}^{n} \frac{C_{k,n}(a)}{k^2 - a^2} = \frac{(-1)^{n+1}}{n^2 - a^2}\binom{2n}{n},$$

where $C_{k,n}(a)$ is defined in (2–2).

Proof of Proposition 4.1: We use Krattenthaler's inversion formula [Krattenhaler 96]:

$$f(n) = \sum_{k=r}^{n} \frac{a_n d_n + b_n c_n}{d_k} \frac{\varphi(c_k/d_k; n)}{\psi_k(-c_k/d_k; n+1)}\, g(k)$$
$$\text{iff} \quad g(n) = \sum_{k=r}^{n} \frac{\psi(-c_n/d_n; k)}{\varphi(c_n/d_n; k+1)}\, f(k),$$

where

$$\varphi(x;k) = \prod_{j=0}^{k-1}(a_j + x b_j),$$
$$\psi(x;k) = \prod_{j=0}^{k-1}(c_j + x d_j), \quad \text{and}$$
$$\psi_m(x;k) = \prod_{\substack{j=0\\ j\neq m}}^{k-1}(c_j + x d_j).$$

Applied to (3–3), it yields the result with the choices $r = 1$, $a_j = (j^2 - a^2)^2$, $b_j = 4$, $c_j = j^4 - a^2 j^2$, $d_j = 1$,

$$f(k) = (-1)^k (5k^2 - a^2)\binom{2k}{k},$$

and

$$g(k) = \frac{2}{k^2 - a^2}\frac{4k^4 - 4a^2k^2 + (a^2 - 1)^2}{k^4 - a^2k^2}.$$

□

Using the same trick as Almkvist and Granville, it is easy to write (4–1) in a more convenient form, that we will prove below: for any $n \geq 1$,

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{4k^2}{k^2-a^2}\prod_{\substack{0\leq j<n-k\\ \text{or } n<j<n+k}}(k^2+j^2-a^2) = \frac{(2n)!}{n^2-a^2}\binom{2n}{n}. \quad (4\text{–}2)$$

5. THE FINAL STEP

Note that (4–2) is simply Theorem 1.2 with $g(X) = X^2$ and $t = 1$: indeed, the first product in the left-hand side of (1–5) corresponds exactly to the left-hand side of (4–2) and (since only the nth summand is nonzero)

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{4k^2}{k^2-a^2}\prod_{\substack{0\leq j<n-k\\ \text{or } n<j<n+k}}g(j) = \frac{n^2}{n^2-a^2}\prod_{n<j<2n}j^2 = \frac{4n^2}{n^2-a^2}\frac{(2n-1)!^2}{n!^2} = \frac{(2n)!}{n^2-a^2}\binom{2n}{n}.$$

Hence Theorem 1.1 follows from Theorem 1.2.

Proof of Theorem 1.2: So far, we have been very lucky in that every step of [Borwein and Bradley 97] generalises without problems to this more general setting. But here, the general Theorem 1′ in [Almkvist and Granville 99] is apparently not strong enough to prove (4–2). Fortunately, we can adapt the method used there for our purpose. For any $k \geq 1$, we define the polynomial of degree $n-1$

$$F_k(X) = \prod_{\substack{0\leq j<n-k\\ \text{or } n<j<n+k}}(X - g(j)).$$

Proposition 1 in [Almkvist and Granville 99] establishes the existence of polynomials $Q_r(X)$ of degree $d_r \leq r$ such that

$$F_k(X) - F_k(0) = \sum_{r=0}^{n-2} Q_r(k^2-a^2)\, X^{n-1-r}. \quad (5\text{–}1)$$

The important point for us is the fact that since $F_k(X) - F_k(0)$ vanishes at $X = 0$, then the sum in (5–1) terminates at $n-2$. (In fact, $Q_r(X) = c_r(X + a^2)$ with the polynomials c_r given in [Almkvist and Granville 99].) We write $Q_r(X) = \sum_{i=0}^{d_r} q_{r,i}\, X^i$. Equation (1–5) can be expressed as

$$(-1)^{n-1}\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{4k^2}{k^2-a^2}\cdot\big(F_k(-t(k^2-a^2)) - F_k(0)\big) = (-1)^{n-1}\sum_{r=0}^{n-2}\sum_{i=0}^{d_r}(-t)^{n-1-r}q_{r,i} \cdot\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{4k^2}{k^2-a^2}(k^2-a^2)^{i+n-1-r}. \quad (5\text{–}2)$$

Since $i \geq 0$ and $r \leq n-2$, we have

$$\frac{4k^2}{k^2-a^2}(k^2-a^2)^{i+n-1-r} = P(k^2),$$

where $P(X) = 4X(X-a^2)^{n+i-r-2}$ is a polynomial of degree $i+n-r-1 \leq d_r+n-r-1 \leq n-1$ such that $P(0) = 0$. Lemma 1 in [Almkvist and Granville 99], which reads

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}k^{2\ell} = 0 \quad (5\text{–}3)$$

for any $1 \leq \ell \leq n-1$, then gives that

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}\frac{4k^2}{k^2-a^2}(k^2-a^2)^{i+n-1-r} = \sum_{k=1}^{n}(-1)^{n-k}\binom{2n}{n-k}P(k^2) = 0.$$

This proves that the left-hand side of (5–2) is 0 for all t and the proof of Theorem 1.2 is complete. □

We conclude this section with the following remark. Almkvist and Granville proved (5–3) by expressing its left-hand side as the 2ℓth Taylor coefficient of the function $e^{-nz}(e^z-1)^{2n}$. Another proof is as follows: define $S(z) = z^\ell/z(z-1^2)\cdots(z-n^2)$ for any integers $\ell \geq 0$ and $n \geq 0$. Then, by the residue theorem, for any closed direct contour Γ enclosing the poles of S, we have

$$-\mathrm{Res}_\infty(S) = \frac{1}{2i\pi}\int_\Gamma S(z)\,\mathrm{d}z = \sum_{k=0}^{n}\mathrm{Res}_{k^2}(S) = 2\sum_{k=0}^{n}(-1)^{n-k}\frac{k^{2\ell}}{(n-k)!(n+k)!}.$$

If we assume that $\ell \leq n-1$, then $\mathrm{Res}_\infty(S) = 0$ and if $\ell \geq 1$, then (5–3) follows after multiplication by $(2n)!/2$.

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