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PARTIAL AVERAGING FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH SUPREMUM

D. BAINOV, YU. DOMSHLAK, AND S. MILUSHEVA

Abstract. Partial averaging for impulsive differential equations with supremum is justified. The proposed averaging schemes allow one to simplify considerably the equations considered.

1. Introduction

For mathematical simulation in various fields of science and technology impulsive differential equations with supremum are successfully used. The investigation of these equations is rather difficult due to the discontinuous character of their solutions and the presence of the supremum of the unknown function, hence the need for approximate methods for solving them.

In the present paper three schemes for partial averaging of impulsive differential equations with supremum are considered and justified.

2. Preliminary Notes

We note, for the sake of definiteness, that by the value of a piecewise continuous function at a point of discontinuity we mean the limit from the left of the function (provided it exists) at this point. By the symbol $\sum\limits_{0<\tau_k<T}$ we denote summation over all values of k for which the inequality $0 < \tau_k < T$ is satisfied, and by the symbol $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$.

In the proof of the main result we shall use the following two lemmas.

Lemma 1. *Let $u(t)$ be a piecewise continuous function for $t \geq a$ with points of discontinuity of the first kind $\tau_k > a$, $k = 1, 2, \dots,$ for which $\tau_k < \tau_{k+1}$ for $k = 1, 2, \dots,$ and $\lim\limits_{k\to\infty} \tau_k = \infty$. Then, if for $t \geq a$ the*

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inequality

$$u(t) \leq c_1 + c_2 \int\limits_a^t u(\tau)\,d\tau + c_3 \sum_{a<\tau_k<t} u(\tau_k)$$

holds, where c_2 *is a positive constant and* c_1, c_3 *are nonnegative constants, then for* $t \geq a$ *the estimate*

$$u(t) \leq c_1(1+c_3)^{i(a,t)} \exp\{c_2(t-a)\}$$

is valid, where $i(a,t)$ *is the number of points* τ_k *which belong to the interval* $[a,t)$*.*

Lemma 1 is proved by means of the Gronwall–Bellman inequality and by induction.

Lemma 1 is a particular case of the theorems on integral inequalities obtained in [1] and [2].

Lemma 2. *Let the sequence* $\tau_1, \dots, \tau_k \dots$ *be such that for any* $k \in \mathbb{N}$ *the inequality* $\tau_k - \tau_{k-1} \geq \theta$ *holds, where* θ *is a positive constant and* $\tau_0 = 0$*. Then for any* $T \geq \theta$ *and* $t_0 \geq 0$ *the inequality*

$$\sum_{t_0 \leq \tau_k < t_0+T} (\tau_k - t_0) < \frac{3T^2}{2\theta}$$

is valid.

Proof. Let $n \geq 2$ and $\tau_j < t_0 \leq \tau_{j+1} < \cdots < \tau_{j+n} < t_0 + T$. Then for $T \geq \theta$ we have

$$\sum_{t_0 \leq \tau_k < t_0+T} (\tau_k - t_0) = \sum_{k=j+1}^{j+n-1} (\tau_k - t_0) + (\tau_{j+n} - t_0) \leq$$

$$\leq \frac{1}{\theta} \sum_{k=j+1}^{j+n-1} (\tau_k - t_0)(\tau_{k+1} - \tau_k) + (\tau_{j+n} - t_0) \leq$$

$$\leq \frac{1}{\theta} \int\limits_{\tau_{j+1}}^{\tau_{j+n}} (t - t_0)\,dt + (\tau_{j+n} - t_0) < \frac{T^2}{2\theta} + T \leq \frac{3T^2}{2\theta}.$$

Let $\tau_j < t_0 \leq \tau_{j+1} < t_0 + T \leq \tau_{j+2}$. Then for $T \geq \theta$ we have

$$\sum_{t_0 \leq \tau_k < t_0+T} (\tau_k - t_0) = \tau_{j+1} - t_0 < T \leq \frac{3T^2}{2\theta}. \qquad \square$$

3. Main Results

Let the impulsive system of differential equations with supremum have the form

$$\begin{aligned} \dot{z}(t) &= \varepsilon Z(t, z(t), \overline{z}(t), \widetilde{z}(t)), \quad t > 0, \quad t \neq \tau_k, \\ z(t) &= \varkappa(t), \quad \dot{z}(t) = \dot{\varkappa}(t), \quad -h \le t \le 0, \\ \Delta z(t) &\equiv z(t+0) - z(t-0) = \varepsilon L_k(z(t-0)), \quad t = \tau_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1}$$

where $z(t) = (z_1(t), \dots, z_n(t))$, h is a positive constant,

$\overline{z}(t) = (\overline{z}_1(t), \dots, \overline{z}_n(t)), \quad \widetilde{z}(t) = (\widetilde{z}_1(t), \dots, \widetilde{z}_n(t)),$
$z(t+0) = (z_1(t+0), \dots, z_n(t+0)), \quad z(t-0) = (z_1(t-0), \dots, z_n(t-0)),$
$\overline{z}_i(t) = \sup\{z_i(s) : s \in [t-h, t]\}, \quad \widetilde{z}_i(t) = \sup\{\dot{z}_i(s) : s \in [t-h, t]\},$
$z_i(t+0) = \lim\limits_{s \to t,\ s>t} z_i(s), \quad z_i(t-0) = \lim\limits_{s \to t,\ s<t} z_i(s), \quad i = 1, \dots, n,$
$\varkappa(t) = (\varkappa_1(t), \dots, \varkappa_n(t))$ is an initial function,
$0 = \tau_0 < \tau_1 < \dots < \tau_k < \cdots$, $\lim\limits_{k\to\infty} \tau_k = \infty$ and $\varepsilon > 0$ is a small parameter.

Let the functions $\overline{Z}(t,z)$ and $\overline{L}_k(z)$ exist for which

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big[Z(t,z,z,0) - \overline{Z}(t,z)\big]\, dt = 0 \tag{2}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0<\tau_k<T} \big[L_k(z) - \overline{L}_k(z)\big]\, dt = 0, \quad k \in \mathbb{N}. \tag{3}$$

Then with the initial-value problem (1) we associate the averaged impulsive system of differential equations

$$\begin{aligned} &\dot{\zeta}(t) = \varepsilon \overline{Z}(t, \zeta(t)), \quad t > 0, \quad t \neq \tau_k, \\ &\Delta\zeta(t) \equiv \zeta(t+0) - \zeta(t-0) = \varepsilon \overline{L}_k(\zeta(t-0)), \ \ t = \tau_k, \ \ k \in \mathbb{N}, \\ &\zeta(0) = \varkappa(0). \end{aligned} \tag{4}$$

We shall prove a theorem on nearness of the solutions of the initial-value problems (1) and (4).

Theorem 1. *Let the following conditions hold:*

1. *The functions $Z(t,z,u,v)$, $\overline{Z}(t,z)$, $L_k(z)$, and $\overline{L}_k(z)$, $k \in \mathbb{N}$, are continuous in the respective projections of the domain $\{t \ge 0,\ z, u \in G,\ v \in H\}$, where G and H are open domains in $\mathbb{R}^n$. The functions $\varkappa(t)$ and $\dot{\varkappa}(t)$ are continuous in the interval $[-h, 0]$, $h = const > 0$, $\varkappa_i(t)$ and $\dot{\varkappa}_i(t)$*

$(i = 1, \dots, n)$ have a finite number of extremums in the interval $[-h, 0]$, and $\varkappa(t) \in G$, $\dot{\varkappa}(t) \in H$ for $t \in [-h, 0]$.

2. *There exist positive constants λ and M such that for all $t \geq 0$, $z, u, z', u' \in G$, $v, v' \in H$, $k \in \mathbb{N}$, the following inequalities are valid:*

$$\|Z(t, z, u, v)\| + \|\overline{Z}(t, z)\| + \|L_k(z)\| + \|\overline{L}_k(z)\| \leq M,$$
$$\|Z(t, z, u, v) - Z(t, z', u', v')\| \leq \lambda\big(\|z - z'\| + \|u - u'\| + \|v - v'\|\big),$$
$$\|\overline{Z}(t, z) - \overline{Z}(t, z')\| + \|L_k(z) - L_k(z')\| + \|\overline{L}_k(z) - \overline{L}_k(z')\| \leq \lambda\|z - z'\|$$

and for $t \in [-h, 0]$ the estimate $\|\varkappa(t)\| + \|\dot{\varkappa}(t)\| \leq M$ is valid.

3. *For each $z \in G$ there exist the limits (2) and (3).*

4. *There exists a positive constant θ such that for $k \in \mathbb{N}$ the inequality $\tau_k - \tau_{k-1} \geq \theta$ holds, where $\tau_0 = 0$.*

5. *For any $\varepsilon \in (0, \varepsilon^*]$, $\varepsilon^* = const > 0$, the initial-value problem (1) has a unique solution $z(t)$ which is defined for $t \geq 0$; $z_i(t)$ and $\dot{z}_i(t)$ $(i = 1, \dots, n)$ have a finite number of extremums in each interval of length h; $z(t)$ and $\dot{z}(t)$ satisfy respectively the conditions $z(0+0) = \varkappa(0)$ and $\dot{z}(0+0) = \dot{\varkappa}(0)$.*

6. *For any $\varepsilon \in (0, \varepsilon^*]$ the initial-value problem (4) has a unique solution $\zeta(t)$ which is defined for $t \geq 0$ and for $t \geq 0$ belongs to the compact $Q \subset G$ together with some ρ-neighborhood of it ($\rho = const > 0$).*

Then for any $\eta > 0$ and $L > 0$ there exists $\varepsilon_0 \in (0, \varepsilon^*)$ $(\varepsilon_0 = \varepsilon_0(\eta, L))$ such that for $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L\varepsilon^{-1}]$ the inequality $\|z(t) - \zeta(t)\| < \eta$ holds.

Proof. From the conditions of Theorem 1 it follows that for the compact $Q \subset G$ there exists a continuous function $\alpha(T)$ which monotonically tends to zero as $T \to \infty$ and is such that for $z \in Q$ the inequalities

$$\Big\|\int_0^T \big[Z(t, z, z, 0) - \overline{Z}(t, z)\big]dt\Big\| \leq T\alpha(T) \tag{5}$$

and

$$\Big\|\sum_{0<\tau_k<T} \big[L_k(z) - \overline{L}_k(z)\big]\Big\| \leq T\alpha(T) \tag{6}$$

hold.

For $t \in T_\varepsilon = [0, L\varepsilon^{-1}]$ we have

$$z(t) = z(0) + \varepsilon \int_0^t Z(\tau, z(\tau), \overline{z}(\tau), \widetilde{z}(\tau))d\tau + \varepsilon \sum_{0<\tau_k<t} L_k(z(\tau_k)), \tag{7}$$

$$\zeta(t) = \zeta(0) + \varepsilon \int\limits_0^t \overline{Z}(\tau, \zeta(\tau))d\tau + \varepsilon \sum_{0<\tau_k<t} \overline{L}_k(\zeta(\tau_k)), \tag{8}$$

where $z(t) = \varkappa(t)$ and $\dot{z}(t) = \dot{\varkappa}(t)$ for $t \in [-h, 0]$.

Subtracting (8) from (7) for $t \geq 0$ we obtain

$$\begin{aligned}\|z(t) - \zeta(t)\| \leq \varepsilon \int\limits_0^t \|Z(\tau, z(\tau), \overline{z}(\tau), \widetilde{z}(\tau)) - Z(\tau, \zeta(\tau), \zeta(\tau), 0)\| d\tau + \\ + \varepsilon \sum_{0<\tau_k<t} \|L_k(z(\tau_k)) - L_k(\zeta(\tau_k))\| + \\ + \varepsilon \Big\| \int\limits_0^t \big[Z(\tau, \zeta(\tau), \zeta(\tau), 0) - \overline{Z}(\tau, \zeta(\tau))\big] d\tau \Big\| + \\ + \varepsilon \Big\| \sum_{0<\tau_k<t} \big[L_k(\zeta(\tau_k)) - \overline{L}_k(\zeta(\tau_k))\big] \Big\|.\end{aligned} \tag{9}$$

Denote successively by $\beta(t)$, $\gamma(t)$, $\delta(t)$, and $\sigma(t)$ the summands on the right-hand side of (9).

Without loss of generality, for the estimation of the function $\beta(t)$ we shall assume that $h < t$, $z_i(\tau_k) = \max(z_i(\tau_k - 0), z_i(\tau_k + 0))$, and $\dot{z}_i(\tau_k) = \max(\dot{z}_i(\tau_k - 0), \dot{z}_i(\tau_k + 0))$ for $i = 1, \ldots, n$ and $k \in \mathbb{N}$. From the last assumption it follows that the supremums in $\overline{z}_i(t)$ and $\widetilde{z}_i(t)$ $(i = 1, \ldots, n)$ are achieved.

Denote by $\overline{s}_i(t)$ (resp., $\widetilde{s}_i(t)$) the leftmost point of the interval $[t - h, t]$, $t \geq 0$, at which $z_i(s)$ (resp., $\dot{z}_i(s)$) takes its greatest value in this interval. Then $\overline{z}_i(t) = z_i(\overline{s}_i(t))$ and $\widetilde{z}_i(t) = \dot{z}_i(\widetilde{s}_i(t))$ $(i = 1, \ldots, n)$.

Using the conditions of Theorem 1, we obtain

$$\begin{aligned}\beta(t) = \varepsilon \int\limits_0^t \|Z(\tau, z(\tau), \overline{z}(\tau), \widetilde{z}(\tau)) - Z(\tau, \zeta(\tau), \zeta(\tau), 0)\| d\tau \leq \\ \leq 2\varepsilon\lambda \int\limits_0^t \|z(\tau) - \zeta(\tau)\| d\tau + \varepsilon\lambda \int\limits_0^t \big[\|\overline{z}(\tau) - z(\tau)\| + \|\widetilde{z}(\tau)\|\big] d\tau.\end{aligned} \tag{10}$$

Denote by $\beta_0(t)$ the second summand on the right-hand side of (10). We obtain the following estimate:

$$\beta_0(t) = \varepsilon\lambda \int\limits_0^t \big[\|\overline{z}(\tau) - z(\tau)\| + \|\widetilde{z}(\tau)\|\big] d\tau \leq$$

$$\le \varepsilon\lambda \int_0^h \big[\|\overline{z}(\tau)\| + \|z(\tau)\| + \|\widetilde{z}(\tau)\|\big] d\tau +$$

$$+ \varepsilon\lambda \int_h^t \big[\|\overline{z}(\tau) - z(\tau)\| + \|\widetilde{z}(\tau)\|\big] d\tau \le$$

$$\le \varepsilon\lambda \int_0^h \big[\|\varkappa(\tau - h)\| + \|\dot{\varkappa}(\tau - h)\| + 2\|z(\tau)\| + \|\dot{z}(\tau)\|\big] d\tau +$$

$$+ \varepsilon\lambda \int_h^t \Big[\sum_{i=1}^n \big(z_i(\overline{s}_i(\tau)) - z_i(\tau)\big)^2\Big]^{1/2} d\tau +$$

$$+ \varepsilon\lambda \int_h^t \Big[\sum_{i=1}^n \dot{z}_i^2(\widetilde{s}_i(\tau))\Big]^{1/2} d\tau \le$$

$$\le \varepsilon\lambda h\Big[3 + \varepsilon(1+h) + \varepsilon\,\frac{h}{\theta}\Big] M + \varepsilon^2\lambda(t-h)M\sqrt{n} +$$

$$+ \varepsilon\lambda \int_h^t \Big[\sum_{i=1}^n \Big(\varepsilon \int_{\overline{s}_i(\tau)}^{\tau} Z_i(\ell, z(\ell), \overline{z}(\ell), \widetilde{z}(\ell)) d\ell +$$

$$+ \varepsilon \sum_{\overline{s}_i(\tau) < \tau_k < \tau} L_{ki}(z(\tau_k))\Big)^2\Big]^{1/2} d\tau.$$

Setting $A = \lambda h[3 + \varepsilon(1+h) + \varepsilon\frac{h}{\theta}]M + \lambda(L - \varepsilon h)M\sqrt{n}$ and applying Minkowski's inequality, we obtain for $t \in T_\varepsilon$

$$\beta_0(t) \le \varepsilon A + \varepsilon^2\lambda \int_h^t \Big\{\Big[\sum_{i=1}^n \Big|\int_{\overline{s}_i(\tau)}^{\tau} Z_i(\ell, z(\ell), \overline{z}(\ell), \widetilde{z}(\ell)) d\ell\Big|^2\Big]^{1/2} +$$

$$+ \Big[\sum_{i=1}^n \Big|\sum_{\overline{s}_i(\tau) < \tau_k < \tau} L_{ki}(z(\tau_k))\Big|^2\Big]^{1/2}\Big\} d\tau \le$$

$$\le \varepsilon A + \varepsilon^2\lambda \int_h^t \Big\{\Big[\sum_{i=1}^n h^2 M^2\Big]^{1/2} + \Big[\sum_{i=1}^n \Big(\frac{h}{\theta}\Big)^2 M^2\Big]^{1/2}\Big\} d\tau \le$$

$$\le \varepsilon A + \varepsilon\lambda(L - \varepsilon h)\frac{1+\theta}{\theta}\, hM\sqrt{n}\,.$$

For $\gamma(t)$ and $t \in T_\varepsilon$ by the conditions of Theorem 1 we get the estimate

$$\gamma(t) = \varepsilon \sum_{0<\tau_k<t} \|L_k(z(\tau_k)) - L_k(\zeta(\tau_k))\| \le \varepsilon\lambda \sum_{0<\tau_k<t} \|z(\tau_k) - \zeta(\tau_k)\|.$$

In order to obtain estimates of the functions $\delta(t)$ for $t \in T_\varepsilon$ we partition the interval T_ε into q equal parts by the points $t_i = \frac{iL}{\varepsilon q}$, $i = 0, 1, \dots, q$.

Let t be an arbitrary chosen and fixed number from the interval T_ε, and let $t \in (t_s, t_{s+1}]$, where $0 \le s \le q-1$. Then, using the conditions of Theorem 1 and inequality (5), we obtain

$$\begin{aligned}
\delta(t) \le \varepsilon\Big\| \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} \big[&Z(\tau,\zeta(\tau),\zeta(\tau),0) - \overline{Z}(\tau,\zeta(\tau)) - \\
&- Z(\tau,\zeta(t_i),\zeta(t_i),0) + \overline{Z}(\tau,\zeta(t_i))\big]d\tau\Big\| + \\
&+ \varepsilon\Big\| \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} \big[Z(\tau,\zeta(t_i),\zeta(t_i),0) - \overline{Z}(\tau,\zeta(t_i))\big]d\tau\Big\| + \\
&+ \varepsilon \int_{t_s}^{t} \big[\|Z(\tau,\zeta(\tau),\zeta(\tau),0)\| + \overline{Z}(\tau,\zeta(\tau))\|\big]d\tau \le \\
&\le \varepsilon \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} \big[\|Z(\tau,\zeta(\tau),\zeta(\tau),0) - Z(\tau,\zeta(t_i),\zeta(t_i),0)\| + \\
&+\|\overline{Z}(\tau,\zeta(\tau)) - \overline{Z}(\tau,\zeta(t_i))\|\big]d\tau + \varepsilon \sum_{i=0}^{s-1} \Big\| \int_0^{t_{i+1}} \big[Z(\tau,\zeta(t_i),\zeta(t_i),0) - \\
&- \overline{Z}(\tau,\zeta(t_i))\big]d\tau\Big\| + \varepsilon \sum_{i=1}^{s-1} \Big\| \int_0^{t_i} \big[Z(\tau,\zeta(t_i),\zeta(t_i),0) - \overline{Z}(\tau,\zeta(t_i))\big]d\tau\Big\| + \\
&+ \frac{ML}{q} \le 3\varepsilon\lambda \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} \|\zeta(\tau) - \zeta(t_i)\|d\tau + \varepsilon \sum_{i=0}^{s-1} t_{i+1}\alpha(t_{i+1}) + \\
&+ \varepsilon \sum_{i=1}^{s-1} t_i\alpha(t_i) + \frac{ML}{q} \le 3\varepsilon^2\lambda\, \frac{1+\theta}{\theta}\, M \sum_{i=0}^{s-1} \int_{t_i}^{t_{i+1}} (\tau - t_i)d\tau + \\
&+ 2\varepsilon \sum_{i=1}^{s} t_i\alpha(t_i) + \frac{ML}{q} \le 3\varepsilon^2\lambda\, \frac{1+\theta}{2\theta}\, M \sum_{i=0}^{s-1} \Big(\frac{L}{\varepsilon q}\Big)^2 +
\end{aligned}$$

$$+\frac{2s^2L}{q}\,\alpha\Big(\frac{L}{\varepsilon q}\Big)+\frac{ML}{q}=\frac{(3\lambda s(1+\theta)L+2\theta q)ML}{2\theta q^2}+\frac{2s^2L}{q}\,\alpha\Big(\frac{L}{\varepsilon q}\Big).$$

We pass to the estimation of $\sigma(t)$. Using the conditions of Theorem 1 and inequality (6) for $t\in T_\varepsilon$, we obtain

$$\sigma(t)=\varepsilon\Big\|\sum_{0<\tau_k<t}\big[L_k(\zeta(\tau_k))-\overline{L}_k(\zeta(\tau_k))\big]\Big\|\le$$
$$\le\varepsilon t\alpha(t)\le\sup_{0\le\tau\le L}\tau\alpha\Big(\frac{\tau}{\varepsilon}\Big)=\delta_1(\varepsilon),\quad \delta_1(\varepsilon)\to 0,\ \ \varepsilon\to 0.$$

From (9) and the estimates obtained for the functions $\beta(t)$, $\gamma(t)$, $\delta(t)$, and $\sigma(t)$ it follows that for $t\in T_\varepsilon$ the inequality

$$\|z(t)-\zeta(t)\|\le b(q)+c(q,\varepsilon)+2\varepsilon\lambda\int_0^t\|z(\tau)-\zeta(\tau)\|d\tau+$$
$$+\varepsilon\lambda\sum_{0<\tau_k<t}\|z(\tau_k)-\zeta(\tau_k)\| \tag{11}$$

holds, where

$$b(q)=\frac{(3\lambda(1+\theta)L+2\theta)ML}{2\theta q},$$
$$c(q,\varepsilon)=\varepsilon A+\varepsilon\lambda(L-\varepsilon h)\,\frac{1+\theta}{\theta}\,hM\sqrt{n}+2qL\alpha\Big(\frac{L}{\varepsilon q}\Big)+\delta_1(\varepsilon).$$

Choose a sufficiently large $q_0\in\mathbb{N}$ so that the inequality

$$b(q_0)\le\frac12\exp\Big\{-\lambda L\Big(2+\frac1\theta\Big)\Big\}\min(\eta,\rho)$$

holds.

Consider inequality (11) for $q=q_0$, $\varepsilon\in(0,\varepsilon^*]$, $t\in T_\varepsilon$ and apply to it Lemma 1. Thus we obtain

$$\|z(t)-\zeta(t)\|\le\big(b(q_0)+c(q_0,\varepsilon)\big)\exp\{2\varepsilon\lambda t\}(1+\varepsilon\lambda)^{i(0,t)}\le$$
$$\le\big(b(q_0)+c(q_0,\varepsilon)\big)\exp\Big\{2\varepsilon\lambda t+\frac{t}{\theta}\,\ln(1+\varepsilon\lambda)\Big\}\le$$
$$\le\big(b(q_0)+c(q_0,\varepsilon)\big)\exp\Big\{\lambda L\Big(2+\frac1\theta\Big)\Big\}.$$

Choose a sufficiently small $\varepsilon_0\in(0,\varepsilon^*]$ so that for $\varepsilon\in(0,\varepsilon_0]$ we have

$$c(q_0,\varepsilon)<\frac12\exp\Big\{-\lambda L\Big(2+\frac1\theta\Big)\Big\}\min(\eta,\rho).$$

Then for any $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 = \varepsilon_0(\eta, L)$) and $t \in T_\varepsilon$ the inequality

$$\|z(t) - \zeta(t)\| < \min(\eta, \rho)$$

holds.

Hence, for $\varepsilon \in (0, \varepsilon_0]$ and $t \in T_\varepsilon$, $z(t)$ belongs to the domain G, and the estimate $\|z(t) - \zeta(t)\| < \eta$ is valid. $\square$

Consider the initial-value problem (1), where

$$\begin{gathered} z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad Z(t, z(t), \overline{z}(t), \widetilde{z}(t)) = \begin{pmatrix} X(t, z(t), \overline{z}(t), \widetilde{z}(t)) \\ Y(t, z(t), \overline{z}(t), \widetilde{z}(t)) \end{pmatrix}, \\ \varkappa(t) = \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, \quad \Delta z(t) = \begin{pmatrix} \Delta x(t) \\ \Delta y(t) \end{pmatrix}, \quad L_k(z) = \begin{pmatrix} I_k(z) \\ J_k(z) \end{pmatrix}, \quad k \in \mathbb{N}, \end{gathered} \tag{12}$$

$x(t)$ and $y(t)$ are ℓ- and m-dimensional vector-functions, and $\ell + m = n$.

For problem (1), (12) various schemes for partial averaging are possible, which lead to averaged systems of differential equations not containing supremum.

First scheme for partial averaging. Let the following limits exist:

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T X(t, z, z, 0)\, dt = \overline{X}(z) \tag{13}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0<\tau_k<T} I_k(z) = \overline{I}(z). \tag{14}$$

Then with the initial problem (1), (12) we associate the partially averaged impulsive system of differential equations (4) with

$$\overline{Z}(t, z) = \begin{pmatrix} \overline{X}(z) \\ Y(t, z, z, 0) \end{pmatrix}, \tag{15}$$

$$\overline{L}_k(z) = \begin{pmatrix} \overline{I}(z) \\ J_k(z) \end{pmatrix}, \quad k \in \mathbb{N}. \tag{16}$$

Theorem 2. *Let the following conditions hold:*

1. *The functions $X(t, z, u, v)$, $Y(t, z, u, v)$, $I_k(z)$, $J_k(z)$, $k \in \mathbb{N}$, are continuous in the respective projections of the domain $\{t \ge 0,\ z, u \in G,\ v \in H\}$, where G and H are open domains in $\mathbb{R}^n$. The functions $\varphi(t)$, $\dot\varphi(t)$, $\psi(t)$, $\dot\psi(t)$ are continuous in the interval $[-h, 0]$, $h = const > 0$, $\varphi_i(t)$, $\dot\varphi_i(t)$ $(i = 1, \dots, \ell)$, $\psi_j(t)$, $\dot\psi_j(t)$ $(j = 1, \dots, m)$ have a finite number of extremums in the interval $[-h, 0]$, and $\varkappa(t) = (\varphi(t), \psi(t)) \in G$, $\dot\varkappa(t) = (\dot\varphi(t), \dot\psi(t)) \in H$ for $t \in [-h, 0]$.*

2. *There exist positive constants* λ *and* M *such that for all* $t \geq 0$, $z, u, z', u' \in G$, $v, v' \in H$, $k \in \mathbb{N}$, *the following inequalities are valid:*

$$\begin{aligned}
&\|X(t,z,u,v)\| + \|Y(t,z,u,v)\| + \|I_k(z)\| + \|J_k(z)\| \leq M,\\
&\|X(t,z,u,v) - X(t,z',u',v')\| \leq \lambda\big(\|z-z'\| + \|u-u'\| + \|v-v'\|\big),\\
&\|Y(t,z,u,v) - Y(t,z',u',v')\| \leq \lambda\big(\|z-z'\| + \|u-u'\| + \|v-v'\|\big),\\
&\|I_k(z) - I_k(z')\| + \|J_k(z) - J_k(z')\| \leq \lambda\|z-z'\|
\end{aligned}$$

and for $t \in [-h, 0]$ *the estimate* $\|\varphi(t)\| + \|\dot{\varphi}(t)\| + \|\psi(t)\| + \|\dot{\psi}(t)\| \leq M$ *is valid.*

3. *For each* $z \in G$ *there exist the limits* (13) *and* (14).

4. *There exists a positive constant* θ *such that for* $k \in \mathbb{N}$ *the inequality* $\tau_k - \tau_{k-1} \geq \theta$ *holds, where* $\tau_0 = 0$.

5. *For any* $\varepsilon \in (0, \varepsilon^*]$, $\varepsilon^* = const > 0$ *the initial-value problem* (1), (12) *has a unique solution* $z(t)$ *which is defined for* $t \geq 0$; $z_i(t)$ *and* $\dot{z}_i(t)$ $(i = 1, \dots, n)$ *have a finite number of extremums in each interval of length* h; $z(t)$ *and* $\dot{z}(t)$ *satisfy respectively the conditions* $z(0+0) = \varkappa(0)$ *and* $\dot{z}(0+0) = \dot{\varkappa}(0)$.

6. *For any* $\varepsilon \in (0, \varepsilon^*]$ *the initial-value problem* (4), (15), (16) *has a unique solution* $\zeta(t)$ *which is defined for* $t \geq 0$ *and for* $t \geq 0$ *belongs to the compact* $Q \subset G$ *together with some* ρ-*neighborhood of it* ($\rho = const > 0$).

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon_0 \in (0, \varepsilon^*]$ ($\varepsilon_0 = \varepsilon_0(\eta, L)$) *such that for* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the inequality* $\|z(t) - \zeta(t)\| < \eta$ *holds.*

Proof. From (13), (14) and the conditions of Theorem 2 it follows that in the domain G the functions $\overline{X}(z)$ and $\overline{I}(z)$ are bounded, continuous, and satisfy the Lipschitz condition. Hence for the functions $Z(t,z,u,v)$, $\overline{Z}(t,z)$, $L_k(z)$, and $\overline{L}_k(z)$, $k \in \mathbb{N}$, of problem (1), (12), conditions 1 and 2 of Theorem 1 are met. Further on, the proof of Theorem 2 is analogous to the proof of Theorem 1. □

Second scheme for partial averaging. Let there exist the limit (13) Then with the initial-value problem (1), (12) we associate the partially averaged impulsive system of differential equations (4) with $\overline{Z}(t,z)$ determined by (15) and

$$\overline{L}_k(z) = L_k(z), \quad k \in \mathbb{N}. \tag{17}$$

Theorem 3. *Let the following conditions hold:*

1. *Conditions* 1, 2, 4, *and* 5 *of Theorem* 2 *are met.*
2. *For each* $z \in G$ *there exists the limit* (13).

3. *For any* $\varepsilon \in (0, \varepsilon^*]$ *the initial-value problem* (4), (15), (17) *has a unique solution* $\zeta(t)$ *which is defined for* $t \geq 0$ *and for* $t \geq 0$ *belongs to the compact* $Q \subset G$ *together with some* ρ*-neighborhood of it* ($\rho = const > 0$).

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon_0 \in (0, \varepsilon^*]$ ($\varepsilon_0 = \varepsilon_0(\eta, L)$) *such that for* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the inequality* $\|z(t) - \zeta(t)\| < \eta$ *holds.*

The proof of Theorem 3 is analogous to the proof of Theorem 1.

Third scheme for partial averaging. Let there exist (13), (14) and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0<\tau_k<T} J_k(z) = \overline{J}(z). \tag{18}$$

Then with the initial-value problem (1), (12) we associate the averaged system of ordinary differential equations

$$\dot{\chi}(t) = \varepsilon\big[\overline{Z}(t, \chi(t)) + \overline{L}(\chi(t))\big] \tag{19}$$

with the initial condition

$$\chi(0) = \varkappa(0), \tag{20}$$

where $\overline{Z}(t, z)$ is determined by (15) and

$$\overline{L}(z) = \begin{pmatrix} \overline{I}(z) \\ \overline{J}(z) \end{pmatrix}. \tag{21}$$

Theorem 4. *Let the following conditions hold:*

1. *Conditions* 1, 2, 4, *and* 5 *of Theorem* 2 *are met.*
2. *For each* $z \in G$ *there exist the limits* (13), (14), (16).
3. *For any* $\varepsilon \in (0, \varepsilon^*]$ *the initial-value problem* (4), (15), (17) *and the initial-value problem* (19), (20), (15), (21) *have unique solutions* $\zeta(t)$ *and* $\chi(t)$, *respectively, which are defined for* $t \geq 0$ *and lie in the compact* $Q \subset G$, *together with some of its* ρ*-neighborhood* ($\rho = const > 0$).

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon_0 \in (0, \varepsilon^*]$ ($\varepsilon_0 = \varepsilon_0(\eta, L)$) *such that for* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the inequality* $\|z(t) - \chi(t)\| < \eta$ *holds.*

Proof. From (13), (14), (16) and the conditions of Theorem 4 it follows that in the domain G the functions $\overline{X}(z)$, $\overline{I}(z)$, and $\overline{J}(z)$ are bounded, continuous, and satisfy the Lipschitz condition, namely for all $z, z' \in G$ the inequalities

$$\begin{aligned}
&\|\overline{X}(z)\| \leq M, \quad \|\overline{I}(z)\| \leq \frac{M}{\theta}, \quad \|\overline{J}(z)\| \leq \frac{M}{\theta}, \\
&\|\overline{X}(z) - \overline{X}(z')\| \leq 2\lambda\|z - z'\|,
\end{aligned}$$

$$\|\overline{I}(z) - \overline{I}(z')\| \le \frac{\lambda}{\theta}\,\|z - z'\|, \quad \|\overline{J}(z) - \overline{J}(z')\| \le \frac{\lambda}{\theta}\,\|z - z'\|$$

are valid.

By Theorem 3 for each $\eta > 0$ and $L > 0$ there exists a number $\widehat{\varepsilon}_0 \in (0, \varepsilon^*]$ $(\widehat{\varepsilon} = \widehat{\varepsilon}_0(\eta, L))$ such that for $\varepsilon \in (0, \widehat{\varepsilon}_0]$ and $t \in [0, L\varepsilon^{-1}]$ the inequality

$$\|z(t) - \zeta(t)\| < \frac{1}{2}\min(\eta, \rho) \tag{22}$$

holds, where $z(t)$ is a solution of the initial-value problem (1), (12), and $\zeta(t)$ is a solution of the initial-value problem (4), (15), (17).

From the conditions of Theorem 4 it follows that for the compact $Q \subset G$ there exists a continuous function $\alpha(T)$ which monotonically tends to zero as $T \to \infty$ and is such that for $z \in Q$ the inequality

$$\Big\| \sum_{0<\tau_k<T} L_k(z) - \overline{L}(z) \cdot T \Big\| \le T\alpha(T) \tag{23}$$

holds.

For $t \in T_\varepsilon = [0, L\varepsilon^{-1}]$, from (4) and (19), (20) we have

$$\zeta(t) = \varkappa(0) + \varepsilon \int_0^t \overline{Z}(\tau, \zeta(\tau))d\tau + \varepsilon \sum_{0<\tau_k<t} L_k(\zeta(\tau_k)), \tag{24}$$

$$\chi(t) = \varkappa(0) + \varepsilon \int_0^t \big[\overline{Z}(\tau, \chi(\tau)) + \overline{L}(\chi(\tau))\big]d\tau. \tag{25}$$

Subtracting (25) from (24) for $t \ge 0$ we obtain

$$\begin{aligned} \|\zeta(t) - \chi(t)\| \le \varepsilon \int_0^t \|\overline{Z}(\tau, \zeta(\tau)) - \overline{Z}(\tau, \chi(\tau))\|d\tau + \\ + \varepsilon \sum_{0<\tau_k<t} \|L_k(\zeta(\tau_k)) - L_k(\chi(\tau_k))\| + \\ + \varepsilon \Big\| \sum_{0<\tau_k<t} L_k(\chi(\tau_k)) - \int_0^t \overline{L}(\chi(\tau))d\tau \Big\|. \end{aligned} \tag{26}$$

Denote by $\omega(t)$ the third summand in the right-hand side of (26).

In order to estimate $\omega(t)$ for $t \in T_\varepsilon = [0, L\varepsilon^{-1}]$, we partition the interval T_ε into q equal parts by means of the points $t_i = \frac{iL}{\varepsilon q}$, $i = 0, 1, \dots, q$.

Let t be an arbitrary chosen and fixed number of the interval T_ε and let $t \in (t_s, t_{s+1}]$, where $s \in \mathbb{N}$ and $0 \le s \le q-1$. Then, using the conditions of Theorem 4 and inequality (23), we obtain

$$\begin{aligned}
\omega(t) \le{}& \varepsilon \sum_{0<\tau_k<t_1} \|L_k(\chi(\tau_k)) - L_k(\chi(0))\| + \varepsilon \int_0^{t_1} \|\overline{L}(\chi(\tau)) - \overline{L}(\chi(0))\| d\tau + \\
&+ \varepsilon \sum_{i=1}^{s-1} \Big(\sum_{t_i \le \tau_k < t_{i+1}} \|L_k(\chi(\tau_k)) - L_k(\chi(t_i))\| + \\
&+ \int_{t_i}^{t_{i+1}} \|\overline{L}(\chi(\tau)) - \overline{L}(\chi(t_i))\| d\tau \Big) + \\
&+ \varepsilon \Big\| \sum_{0<\tau_k<t_1} L_k(\chi(0)) - \int_0^{t_1} \overline{L}(\chi(0)) d\tau \Big\| + \\
&+ \varepsilon \sum_{i=1}^{s-1} \Big\| \sum_{t_i \le \tau_k < t_{i+1}} L_k(\chi(t_i)) - \int_{t_i}^{t_{i+1}} \overline{L}(\chi(t_i))\| d\tau \Big\| + \\
&+ \varepsilon \sum_{t_s \le \tau_k < t} \|L_k(\chi(t_k))\| + \varepsilon \int_{t_s}^{t} \|\overline{L}(\chi(\tau))\| d\tau \le \\
&\le \varepsilon\lambda \sum_{0<\tau_k<t_1} \|\chi(\tau_k) - \chi(0)\| + \varepsilon \frac{2\lambda}{\theta} \int_0^{t_1} \|\chi(\tau) - \chi(0)\| d\tau + \\
&+ \varepsilon\lambda \sum_{i=1}^{s-1} \Big(\sum_{t_i \le \tau_k < t_{i+1}} \|\chi(\tau_k) - \chi(t_i)\| + \frac{2}{\theta} \int_{t_i}^{t_{i+1}} \|\chi(\tau) - \chi(t_i)\| d\tau \Big) + \\
&+ \varepsilon t_1 \alpha(t_1) + \varepsilon \sum_{i=1}^{s-1} \Big\| \sum_{0<\tau_k<t_{i+1}} L_k(\chi(t_i)) - t_{i+1}\overline{L}(\chi(t_i)) \Big\| + \\
&+ \varepsilon \sum_{i=1}^{s-1} \Big\| \sum_{0<\tau_k<t_i} \big[L_k(\chi(t_i)) - t_i \overline{L}(\chi(t_i))\big] \Big\| + \\
&+ \varepsilon \sum_{i=1}^{s-1} \sum_{\tau_k = t_i} \|L_k(\chi(t_i))\| + \frac{3LM}{\theta q} \le \\
&\le \frac{2\varepsilon^2 \lambda M(1+\theta)}{\theta} \sum_{i=1}^{s-1} \Big(\sum_{t_i \le \tau_k < t_{i+1}} (\tau_k - t_i) + \frac{2}{\theta} \int_{t_i}^{t_{i+1}} (\tau - t_i) d\tau \Big) +
\end{aligned}$$

$$+\varepsilon t_1\alpha(t_1)+\varepsilon\sum_{i=1}^{s-1} t_{i+1}\alpha(t_{i+1})+\varepsilon\sum_{i=1}^{s-1} t_i\alpha(t_i)+2\varepsilon(s-1)M+\frac{3LM}{\theta q}\le$$
$$\le\frac{2\varepsilon^2\lambda M(1+\theta)}{\theta}\sum_{i=0}^{s-1}\Big(\sum_{t_i\le\tau_k<t_{i+1}}(\tau_k-t_i)\Big)+\frac{2s\lambda L^2M(1+\theta)}{\theta^2q^2}+$$
$$+\frac{2s^2L}{q}\,\alpha\Big(\frac{L}{\varepsilon q}\Big)+2\varepsilon(s-1)M+\frac{3LM}{\theta q}.$$

From (26), the conditions of Theorem 4, and the estimate obtained for $\omega(t)$ it follows that for $t\in T_\varepsilon$ the inequality

$$\|\zeta(t)-\chi(t)\|\le b(q,\varepsilon)+c(q,\varepsilon)+4\varepsilon\lambda\int_0^t\|\zeta(\tau)-\chi(\tau)\|d\tau+$$
$$+\varepsilon\lambda\sum_{0<\tau_k<t}\|\zeta(\tau_k)-\chi(\tau_k)\| \tag{27}$$

holds, where

$$b(q,\varepsilon)=\Big(3+\frac{2\lambda L(1+\theta)}{\theta}\Big)\frac{LM}{\theta q}+$$
$$+\frac{2\varepsilon^2\lambda M(1+\theta)}{\theta}\sum_{i=0}^{m-1}\Big(\sum_{t_i\le\tau_k<t_{i+1}}(\tau_k-t_i)\Big),$$
$$c(q,\varepsilon)=2qL\alpha\Big(\frac{L}{\varepsilon q}\Big)+2\varepsilon qM.$$

Choose successively a sufficiently large $q_0\in\mathbb{N}$ and a sufficiently small $\varepsilon_1\in(0,\widehat{\varepsilon}_0]$ so that the inequalities

$$\Big(3+\frac{5\lambda L(1+\theta)}{\theta}\Big)\frac{LM}{\theta q_0}\le\frac{1}{4}\exp\Big\{-\lambda L\big(4+\frac{1}{\theta}\big)\Big\}\min(\eta,\rho),$$
$$\varepsilon_1 q_0\le\frac{L}{\theta} \tag{28}$$

holds.

For $q=q_0$ and $\varepsilon\in(0,\varepsilon_1]$ we apply Lemma 2 to the summand on the right-hand side of $b(q,\varepsilon)$ and obtain

$$\frac{2\varepsilon^2\lambda M(1+\theta)}{\theta}\sum_{i=0}^{q_0-1}\Big(\sum_{t_i\le\tau_k<t_{i+1}}(\tau_k-t_i)\Big)\le\frac{3\lambda^2L^2M(1+\theta)}{\theta^2q_0}. \tag{29}$$

From (27), (28), and (29) for $q = q_0$, $\varepsilon \in (0, \varepsilon_1]$, and $t \in T_\varepsilon$ there follows the inequality

$$\|\zeta(t) - \chi(t)\| \le b(q_0, \varepsilon) + c(q_0, \varepsilon) + 4\varepsilon\lambda \int\limits_0^t \|\zeta(\tau) - \chi(\tau)\| d\tau + \\ + \varepsilon\lambda \sum_{0<\tau_k<t} \|\zeta(\tau_k) - \chi(\tau_k)\|, \tag{30}$$

where $b(q_0, \varepsilon) \le \frac{1}{4} \exp\{-\lambda L(4 + \frac{1}{\theta})\} \min(\eta, \rho)$.

Applying Lemma 1 to (30), we get

$$\begin{aligned}\|\zeta(t) - \chi(t)\| &\le \big(b(q_0, \varepsilon) + c(q_0, \varepsilon)\big)(1 + \varepsilon\lambda)^{i(0,t)} \exp\{4\varepsilon\lambda t\} \le \\ &\le \big(b(q_0, \varepsilon) + c(q_0, \varepsilon)\big) \exp\Big\{4\varepsilon\lambda t + \frac{t}{\theta} \ln(1 + \varepsilon\lambda)\Big\} \le \\ &\le \big(b(q_0, \varepsilon) + c(q_0, \varepsilon)\big) \exp\Big\{\lambda L\big(4 + \frac{1}{\theta}\big)\Big\}.\end{aligned}$$

Choose a sufficiently small $\check{\varepsilon}_0 \in (0, \varepsilon_1]$ so that for $\varepsilon \in (0, \check{\varepsilon}_0]$ we have

$$c(q_0, \varepsilon) < \frac{1}{4} \exp\Big\{ - \lambda L\big(4 + \frac{1}{\theta}\big)\Big\} \min(\eta, \rho).$$

Then for $\varepsilon \in (0, \check{\varepsilon}_0]$ $(\check{\varepsilon}_0 = \check{\varepsilon}_0(\eta, L))$ and $t \in T_\varepsilon$ the inequality

$$\|\zeta(t) - \chi(t)\| < \frac{1}{2} \min(\eta, \rho) \tag{31}$$

holds.

From (22) and (31) it follows that for $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 = \min(\widehat{\varepsilon}_0, \check{\varepsilon}_0)$, and $t \in T_\varepsilon$ the inequality

$$\|z(t) - \chi(t)\| \le \|z(t) - \zeta(t)\| + \|\zeta(t) - \chi(t)\| \le \min(\eta, \rho)$$

holds.

Hence for $\varepsilon \in (0, \varepsilon_0]$ and $t \in T_\varepsilon$, $z(t)$ lies in the domain G, and the estimate $\|z(t) - \chi(t)\| < \eta$ is valid. □

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Authors' addresses:

D. Bainov
Academy of Medicine
Sofia, Bulgaria

Yu. Domshlak
Ben Gurion University of the Negev
Israel

S. Milusheva
Technical University
Sofia, Bulgaria