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ON CONTINUOUS EXTENSIONS

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Abstract. We consider various possibilities concerning the continuous extension of continuous functions taking values in an ultrametric space. In Section 1 we consider Tietze-type extension theorems concerning continuous extendibility of continuous functions from compact and closed subsets to the whole space. In Sections 2 and 3 we consider extending "separated" continuous functions in such a way that certain continuous extensions remain separated. Functions taking values in a complete ultravalued field are dealt with in Section 2, and the real and complex cases in Section 3.

1. A Tietze-Type Extension Theorem

An *ultranormal* topological space T is a Hausdorff space in which disjoint closed subsets may be separated by clopen sets. As shown by R. Ellis ([1], cf. [2], Th. 8.27, p. 258) the following version of the Tietze extension theorem obtains in the nonarchimedean setting:

> (Bounded) continuous functions x mapping a closed subset K of the ultranormal space T into a complete separable metric space have (bounded) continuous extensions to all of T.

We prove a similar result below—we eliminate the separability requirement on the domain T but consider an *ultra*metric space Y as the codomain. Thus, the result below is not more general, just different. For the sake of the proof we define *a clopen n-partition* of a subset M of T *with respect to* $x : M \to Y$ to be a finite pairwise disjoint collection of clopen subsets V_i of T such that $M \subset \bigcup_{i=1}^{n} V_i$ and for all $s, t \in V_i \cap M,\ d\left(x\left(s\right), x\left(t\right)\right) \leq 1/n$. As in many of the classical constructions of functions with certain properties (continuous but nowhere differentiable, for example) we create a sequence of functions each of which almost has the desired property, whose uniform limit does.

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Theorem 1.1. *Let y be a point of the complete ultrametric space (Y, d). In each part below y could belong to K.*

(a) *A continuous map x of a compact subset K of the ultraregular space T into Y may be continuously extended to $\bar{x}$ defined on all of T with $\bar{x}(T) = x(K) \cup \{y\}$.*

(b) *A continuous map x of a closed subset K of the ultranormal space T into Y with relatively compact range may be continuously extended to $\bar{x}$ defined on all of T with $\bar{x}(T) \subset [\operatorname{cl} x(K)] \cup \{y\}$ where "cl" denotes topological closure.*

Proof. (a) Let $C(p, r)$ denote the closed ball of radius $r > 0$ about $p \in Y$. For each $t \in K$, let $B(t, 1/n) = \{s \in K : d(x(s), x(t)) \le 1/n\} = x^{-1}(C(x(t), 1/n))$. Since K is compact, there exist $t_1, \ldots, t_{n_1} \in K$ such that $K = \bigcup_{i=1}^{n_1} B(t_i, 1)$. In the standard way, we may rewrite the relatively clopen subsets of K, $B(t_i, 1)$, $1 \le i \le n_1$, to get a pairwise disjoint relatively clopen cover $\{A_i\}$ of K with the property that, for any $1 \le i \le n_1$ and $s, t \in A_i$, $d(x(s), x(t)) < 1$. We continue to assume that the A_i are nonempty and that $t_i \in A_i$ for $1 \le i \le n_1$; otherwise, we would reindex the sets and choose some new points t_i as needed. Since T is ultraregular and K is compact, we may choose pairwise disjoint clopen subsets U_i of T such that $U_i \cap K = A_i$ for each $i = 1, 2, \ldots, n_1$. Evidently the family $\{U_i\}$ is a clopen 1-partition of K with respect to x. We define the continuous "step" function

$$x_1(t) = \begin{cases} x(t_i), & t \in U_i,\ 1 \le i \le n_1, \\ y, & t \notin \bigcup_{i=1}^{n_1} U_i. \end{cases}$$

For any $t \in K$, for some $1 \le i \le n_1$, $t \in U_i$ and $x_1(t) = x(t_i)$; hence

$$\begin{aligned} d(x_1(t), x(t)) &\le \max[d(x_1(t), x(t_i)), d(x(t_i), x(t))] \\ &= \max[d(x(t_i), x(t_i)), d(x(t_i), x(t))] \\ &= d(x(t_i), x(t)) \\ &\le 1. \end{aligned}$$

It follows that $d_K(x_1, x) = \sup\{d(x_1(t), x(t)) : t \in K\} \le 1$. For $n = 2$, we decompose each A_i into a clopen 2-partition with respect to x. On A_1, for example, we get nonempty disjoint clopen subsets $B_1, \ldots, B_{r_1}$ of T such that each is a subset of U_1 and $(\bigcup_{i=1}^{r_1} B_i) \cap K = A_1$. Choose $s_i \in B_i \cap A_1$ for $1 \le i \le r_1$ and define

$$x_2(t) = \begin{cases} x(s_i), & t \in B_i,\ 1 \le i \le r_1, \\ x_1(t) = x(t_1), & t \notin U_1 - \bigcup_{i=1}^{r_1} B_i, \\ y, & t \notin \bigcup_{i=1}^{n_1} U_i \end{cases}$$

and define x_2 in a similar way on the rest of the A_i. In so doing, we create a continuous step function $x_2 : T \to Y$ such that $d_K(x_2, x) \le 1/2$ and $d_T(x_1, x_2) \le 1$. We continue in this fashion to construct a sequence (x_n)

of functions continuous on T such that, for every n, $d_K(x_n, x) \leq 1/n$ and $d_T(x_n, x_{n-1}) \leq 1/(n-1)$. We denote the pointwise limit of (x_n) as $\bar{x}$. As $x_n \to \bar{x}$ uniformly, it follows that $\bar{x}$ is continuous; clearly $\bar{x} = x$ on K.

(b) As $\mathrm{cl}\, x(K)$ is compact, there exist $y_1, y_2, \ldots, y_{n_1} \in \mathrm{cl}\, x(K)$ such that the $C(y_i, 1)$ cover $\mathrm{cl}\, x(K)$. Consequently $K = \bigcup_{i=1}^{n_1} x^{-1}[C(y_i, 1)]$. Since T is ultranormal there are pairwise disjoint clopen sets U_i such that $x^{-1}[C(y_i, 1)] \subset U(i, 1)$ for $1 \leq i \leq n_1$. Choose $t_i \in x^{-1}[C(y_i, 1)]$ for $1 \leq i \leq n_1$ and define x_1 as was done in (a) and continue the process to obtain the sequence (x_n) and the limit $\bar{x}$. $\square$

Corollary 1.2. *Let T, y and Y be as in the theorem.* (a) *Let x and w be continuous functions defined on the disjoint compact subsets K and L of T taking values in Y. For any disjoint clopen supersets U and V of K and L, respectively, x and w can be continuously extended to $\bar{x}$ and $\bar{w}$, respectively, defined on T, with $\bar{x}(T) \subset x(K) \cup \{y\}$, $\bar{w}(T) \subset w(L) \cup \{y\}$ and $x(t) = w(t) = y$ on $\mathbf{C}(U \cup V)$.*

(b) *If x and w are continuous functions with relatively compact range defined on the disjoint closed subsets K and L of the ultranormal space T taking values in Y, then x and w can be continuously extended to $\bar{x}$ and $\bar{w}$, respectively, defined on T, with $\bar{x}(T) \subset \mathrm{cl}\, x(K) \cup \{y\}$, $\bar{w}(T) \subset \mathrm{cl}\, w(L) \cup \{y\}$ and $x(t) = w(t) = y$ on $\mathbf{C}(U \cup V)$.*

Proof. (a) By the theorem, x and w may be continuously extended to U and V, respectively. It only remains to define those extensions to assume the value y on $\mathbf{C}(U \cup V)$ to obtain the desired extensions to all of T. The result of (b) follows directly from part (b) of the theorem. $\square$

For any topological space T and any semigroup G we say that two functions $x, y : T \to G$ are *strongly separated* if $\mathrm{cl}\,\mathrm{coz}\, x \cap \mathrm{cl}\,\mathrm{coz}\, y = \varnothing$, and *separated* if $\mathrm{coz}\, x \cap \mathrm{coz}\, y = \varnothing$, where $\mathrm{coz}\, x$ denotes the cozero set of x. Note that x and y are separated if and only if $xy = 0$. As has been developed in a number of papers ([3]–[8]) there is ample reason for interest in additive maps $H : C(T) \to C(S)$ between spaces of scalar-valued continuous functions $C(T)$ and $C(S)$ which are [*weakly*] *separating* in the sense that [strongly] separated functions x and y have separated images Hx and Hy. For that reason it is of importance to know when continuous functions which are separated or weakly separated on a subset have continuous extensions which remain separated, the subject of the corollary below; the point y of the theorem is taken to be the zero element of the semigroup.

Corollary 1.3. *Let T be ultraregular and let Y be a semigroup on which a complete ultrametric is defined* (*not necessarily a topological semigroup*).

(a) *If let x and w are strongly separated continuous functions defined on a compact subset K of T then there are separated continuous extensions $\bar{x}$ and $\bar{w}$ defined on T such that $\bar{x}\bar{w} = 0$.*

(b) *Two strongly separated continuous functions x and w with relatively compact range defined on a closed subset K of the ultranormal space T have continuous extensions $\bar{x}$ and $\bar{w}$ defined on T such that $\bar{x}\bar{w} = 0$.*

2. The Case of Complete Valued Fields

Let T be ultraregular and let $C(T,F)$ denote the space of continuous functions taking values in the complete, nonarchimedean nontrivially valued field F. We show in Theorem 2.2 that pairwise strongly separated functions $x_1, x_2, \ldots, x_n \in C(K,F)$ have pairwise separated extensions $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \in C(T,F)$ for two types of subset K.

Lemma 2.1. *Let K be a closed subset of the compact ultraregular space T and let $x_1, x_2, \ldots, x_n$ be elements of $C(T,F)$ which are pairwise separated on K. Then there exist pairwise separated elements $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ in $C(T,F)$ such that $\bar{x}_i \mid K = x_i \mid K$ for $i = 1, \ldots, n$. (If the x_i were only defined on K then they could be continuously extended to T by Corollary* 1.3.)

Proof. Assume that $x, y \in C(T,F)$ are separated on K. Let M bound x and y. Let $G(x,i) = x^{-1}\{a \in F : |a| \leq M/i\}$ for each positive integer i, with $G(y,i)$ defined analogously. Let $U(x,i) = G(x,i) - G(x,i+1)$ and let $U(y,i)$ be defined similarly. Let $C(x,i)$ and $C(y,i)$ denote the intersections of $U(x,i)$ and $U(y,i)$, respectively, with K. Note that the unions of $C(x,i)$ and $C(y,i)$, $i \in N$, yield $\operatorname{coz} x$ and $\operatorname{coz} y$, respectively. Since x and y are separated on K, $C(x,i)$ and $C(y,j)$ are disjoint for all positive integers i and j. Let $G(i)$ denote the clopen set $G(x,i) \cap G(y,i)$ for each i. Since x and y are separated on K, $C(x,m)$ and $C(y,m)$ are disjoint subsets of $G(i)$ for all $m \geq i$; $C(x,m)$ and $C(y,m)$ are subsets of $CG(i)$, the complement of $G(i)$, for all $m < i$. Thus there exist pairwise disjoint clopen sets $V(x,i)$ and $V(y,i)$ such that $C(x,i) \subset V(x,i) \subset U(x,i)$ and $C(y,i) \subset V(y,i) \subset U(y,i)$. Let $k(x,i)$ and $k(y,i)$ denote the characteristic functions of $V(x,i)$ and $V(y,i)$, respectively. The functions $x_m = \sum_{i=1}^m k(x,i)x$ and $y_m = \sum_{i=1}^m k(y,i)y$ are uniform Cauchy sequences on T. Call their limits $\bar{x}$ and $\bar{y}$ in $C(T,F)$, respectively. Since x_m and y_m are separated for each m, $\bar{x}$ and $\bar{y}$ are separated as well; the restrictions of $\bar{x}$ and $\bar{y}$ to K are just x and y, as is easy to verify. The method for extending this to a finite number of functions—rather than just two—is a simple modification of this idea. □

In the next result we drop the assumption of compactness and get an extension theorem.

Theorem 2.2. *For compact K and ultraregular T [or closed K and ultranormal T] and pairwise strongly separated functions $x_1, x_2, \ldots, x_n \in C(K,F)$ [with relatively compact range], there exist pairwise separated extensions $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \in C(T,F)$ of $x_1, x_2, \ldots, x_n$.*

Proof. It suffices to consider two separated functions x and y in $C(K,F)$. Let $\beta_0 T$ denote the Banaschewski compactification of T, and continuously extend x and y to βx and βy in $C(\beta_0 T, F)$ by Corollary 1.3; then replace βx and βy by functions $\bar{x}$ and $\bar{y}$ as described in the preceding lemma. Finally, restrict $\bar{x}$ and $\bar{y}$ to T. $\square$

3. The Real and Complex Case

Let K be a compact subset of the Tihonov space T, and let $C(K,F)$ and $C(T,F)$ denote the spaces of F-valued functions on K and T, where $F = R$ or $\mathbf{C}$. Let H be a finite subset of $C(K,F)$ which is pairwise separated in the sense that $xy = 0$ for each x and y in H. We show in Theorem 3.3 that each x in H has an extension $\bar{x}$ in $C(T,F)$ such that the set $\bar{H}$ of extensions is pairwise separated. We begin with two lemmas about nonnegative functions.

Lemma 3.1. *If the functions in H are nonnegative then each x in H has an extension $\bar{x} \in C(T,R)$ such that the set $\bar{H}$ of such extensions is pairwise separated.*

Proof. We proceed by induction. We begin by considering the case in which H has two elements x and y. Let βv denote the continuous extension of the difference $v = x - y$ to the Stone-Cech compactification βT of T. Let βv^+ and βv^- denote the positive and negative parts of βv, respectively, so that $\beta v = \beta v^+ - \beta v^-$. Since x and y are nonnegative and $xy = 0$, βv^+ and βv^- are continuous extensions of x and y, respectively; clearly $(\beta v^+)(\beta v^-) = 0$. It only remains to restrict βv^+ and βv^- to T to complete the argument in this case. Now suppose that H has $n+1 \geq 3$ elements and choose $x_1, x_2, \ldots, x_n$ from H. By the induction hypothesis, there exist continuous, separated extensions $w_1, w_2, \ldots, w_n$ of $x_1, x_2, \ldots, x_n$ to T. We now repeatedly pair x_{n+1} with each of the w_i, $i = 1, \ldots, n$, to get separated continuous extensions (by the first part of the argument) z_i of x_i and v_i of x_{n+1} for $i = 1, \ldots, n$. Now let $\bar{x}_i = \inf(w_i, z_i)$ for $i = 1, \ldots, n$ and let $\bar{x}_{n+1} = \inf\{v_1, \ldots, v_n\}$. These are the desired extensions of $x_1, x_2, \ldots, x_n$. $\square$

In the next result we assume that each x is separated from each y but do not suppose that the x's are a pairwise separated set, nor are the y's.

Lemma 3.2. *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ denote families of nonnegative functions in $C(K)$ such that x_i is separated from y_j for all i and j. Then there exist continuous extensions $\bar{x}_i$ and $\bar{y}_i$, $i = 1, \ldots, n$, of these functions to T with the same property.*

Proof. By Lemma 3.1 there exist separated extensions $x(i,j)$ and $y(i,j)$ of x_i and y_j, respectively, for each i each j. The functions $\bar{x}_i = \inf\{x(i,j) : 1 \leq j \leq n\}$ and $\bar{y}_j = \inf\{y(i,j) : 1 \leq i \leq n\}$ are separated on all of T for all i and j. $\square$

Theorem 3.3. *With $F = \mathbb{R}$ or $\mathbb{C}$, and H as before Lemma* 3.1*, then each x in H has an extension $\bar{x}$ in $C(T, F)$ such that the set $\bar{H}$ of extensions is pairwise separated.*

Proof. Let $F = R$ and let x and y be separated members of $C(K, \mathbb{R})$. The positive and negative parts x^+, x^-, y^+ and y^- of x and y, respectively, are pairwise separated. Since they are nonnegative, they can be extended to a pairwise separated collection $\overline{x^+}$, $\overline{x^-}$, $\overline{y^+}$ and $\overline{y^-}$ on T as in Lemma 3.1. The desired extensions are now given by $\bar{x} = \overline{x^+} - \overline{x^-}$ and $\overline{y} = \overline{y^+} - \overline{y^-}$. It is straightforward to extend this to the case of more than two functions. To handle the case where the field is $\mathbb{C}$, let x and y be separated continuous functions on K. Each of the functions $(\operatorname{Re} x)^+$, $(\operatorname{Re} x)^-$, $(\operatorname{Im} x)^+$, $(\operatorname{Im} x)^-$ is separated from each of the functions $(\operatorname{Re} y)^+$, $(\operatorname{Re} y)^-$, $(\operatorname{Im} y)^+$, $(\operatorname{Im} y)^-$ on K and Lemma 3.2 can be applied to these functions. The separated extensions to T of those functions can be used to construct separated extensions of x and y. $\square$

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