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Research Article

On the Fredholm Alternative for the Fučík Spectrum

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We consider resonance problems for the one-dimensional p-Laplacian assuming Dirichlet boundary conditions. In particular, we consider resonance problems associated with the first three curves of the Fučík Spectrum. Using variational arguments based on linking theorems, we prove sufficient conditions for the existence of at least one solution. Our results are related to the classical Fredholm Alternative for linear operators. We also provide a new variational characterization for points on the third Fučík curve.

1. Introduction

In this paper, we study the solvability of the problem

$$-\left(|u'|^{p-2}u'\right)' - \alpha(u^+)^{p-1} + \beta(u^-)^{p-1} = f \quad \text{in } (0,T), \qquad u(0) = u(T) = 0, \tag{1.1}$$

where $p > 1$, $T > 0$, $(\alpha, \beta) \in \mathbb{R}^2$, and $f \in L^1(0,T)$. A solution of (1.1) is defined as a function $u \in C_0^1[0,T]$, such that $|u'|^{p-2}u'$ is absolutely continuous and satisfies (1.1) a.e. in $(0,T)$.

It is helpful to restate the given problem as an operator equation. Let $X := W^{1,p}(0,T)$ and let $\langle\cdot,\cdot\rangle$ represent the duality pairing between X and X^*. Define the operators $J, S : X \to X^*$ by

$$\langle J(u), v\rangle = \int_0^T |u'|^{p-2}u'v', \qquad \langle S(u), v\rangle = \int_0^T |u|^{p-2}uv, \quad \text{for } u, v \in X. \tag{1.2}$$

In [1, page 306], it is proved that J is an isomorphism (in particular, J^{-1} exists and is continuous) and that S is continuous and compact. Using an integration by parts argument, it is easy to verify [2, page 120] that solutions of (1.1) are in a one-to-one correspondence with solutions of the operator equation

$$J(u) - \alpha S(u^+) + \beta S(u^-) = F, \tag{1.3}$$

where $F \in X^*$ is defined by

$$\langle F, v\rangle = \int_0^T fv, \quad v \in X. \tag{1.4}$$

We are interested in the case where $(\alpha,\beta) \in \Sigma_p$ which represents the Fučík Spectrum associated with (1.1), that is, the set of all $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$J(u) - \alpha S(u^+) + \beta S(u^-) = 0 \tag{1.5}$$

has a nontrivial solution. An explicit form of Σ_p is given in [2, page 132]. For convenience, we recall the first parts of it. First we note that for $\alpha = \beta = \lambda$, we have $(\lambda,\lambda) \in \Sigma_p$ if and only if λ is an eigenvalue of

$$J(u) - \lambda S(u) = 0. \tag{1.6}$$

These eigenvalues can be expressed explicitly as

$$\lambda = \lambda_k = (p-1)\left(\frac{k\pi_p}{T}\right)^p, \quad k = 1, 2, \ldots, \tag{1.7}$$

where $\pi_p := 2\pi/p\sin(\pi/p)$. The associated eigenfunctions are scalar multiples of

$$u_k(t) = \sin_p\left(\frac{k\pi_p t}{T}\right), \quad k = 1, 2, \ldots, \tag{1.8}$$

where $\sin_p$ is defined by the implicit formula

$$\tau = \int_0^{\sin_p(\tau)} \frac{ds}{(1-s^p)^{1/p}}, \quad \tau \in \left[0, \frac{\pi_p}{2}\right], \tag{1.9}$$

which is extended to $[0,\pi_p]$ and then $[0,2\pi_p]$ by symmetry, and then to all of $\mathbb{R}$ as a $2\pi_p$ periodic function. See, for example, [3, 4]. Note that, we have $u_1 > 0$ in $(0,T)$, and u_1 is a nontrivial solution of (1.5) for $(\alpha,\beta) = (\lambda_1,\beta)$, with arbitrary $\beta \in \mathbb{R}$. Obviously, this implies that $\lambda_1 \times \mathbb{R} \subset \Sigma_p$. Similarly, $\mathbb{R} \times \lambda_1 \subset \Sigma_p$ with a corresponding nontrivial solution, $-u_1 < 0$ in $(0,T)$. It is helpful to separate this so-called *trivial* part of the Fučík Spectrum into

$$\begin{aligned} \mathcal{C}_1^- &:= \{(\lambda_1,\beta) \in \mathbb{R}^2 : \beta \le \lambda_1\} \cup \{(\alpha,\lambda_1) \in \mathbb{R}^2 : \alpha \ge \lambda_1\}, \\ \mathcal{C}_1^+ &:= \{(\lambda_1,\beta) \in \mathbb{R}^2 : \beta \ge \lambda_1\} \cup \{(\alpha,\lambda_1) \in \mathbb{R}^2 : \alpha \le \lambda_1\}. \end{aligned} \tag{1.10}$$

We set $\mathcal{C}_1 = \mathcal{C}_1^- \cup \mathcal{C}_1^+$. The set $\mathcal{C}_1$ is the component, that is, maximal connected subset, of Σ_p which contains (λ_1,λ_1). The other components of Σ_p lie in the first quadrant. Hence, from now on we assume $\alpha > 0$, $\beta > 0$.

The next component of Σ_p, which contains (λ_2,λ_2), is called $\mathcal{C}_2$. It is a curve (p-hyperbola) which passes through (λ_2,λ_2) and has the asymptotes $\alpha = \lambda_1$ and $\beta = \lambda_1$. More precisely,

$$\mathcal{C}_2 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{\alpha^{1/p}} + \frac{1}{\beta^{1/p}} = (p-1)^{-1/p}\frac{T}{\pi_p}\right\}. \tag{1.11}$$

For $(\alpha,\beta) \in \mathcal{C}_2$, a corresponding nontrivial solution, $u_{\alpha\beta}$, of (1.5) is a *two-bump* function in $(0,T)$ which can be constructed in a piecewise fashion using appropriately shifted and scaled $\sin_p$ functions as the pieces. In particular, for $\alpha > \beta$, $u_{\alpha\beta}$ is a positive multiple of either ϕ_{21} or ϕ_{22}, both L^p normalized, as shown in Figure 1. We note that $\phi_{22}(t) = \phi_{21}(T-t)$, $t \in [0,T]$. For $\alpha < \beta$, the situation is similar but the positive bump is now bigger than the negative one. For further reference, we denote

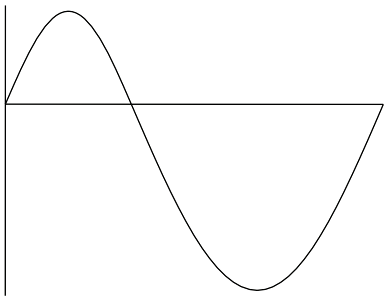
$$\begin{aligned} \mathcal{C}_2^- &:= \{(\alpha,\beta) \in \mathcal{C}_2 : \alpha \ge \beta\}, \\ \mathcal{C}_2^+ &:= \{(\alpha,\beta) \in \mathcal{C}_2 : \alpha \le \beta\}. \end{aligned} \tag{1.12}$$

In the special case where $\alpha = \beta = \lambda_2$, we have $\phi_{21} = cu_2$ and $\phi_{22} = -cu_2$, where $c > 0$ is an L^p normalizing constant. $\mathcal{C}_2$ is called the first nontrivial part of the Fučík Spectrum.

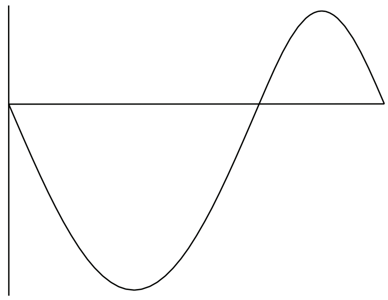
The component of Σ_p containing (λ_3,λ_3) is denoted by $\mathcal{C}_3$ and consists of two p-hyperbolas which intersect at (λ_3,λ_3).

$$\begin{aligned} \mathcal{C}_3^1 &:= \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{2}{\alpha^{1/p}} + \frac{1}{\beta^{1/p}} = (p-1)^{-1/p}\frac{T}{\pi_p}\right\}, \\ \mathcal{C}_3^2 &:= \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{\alpha^{1/p}} + \frac{2}{\beta^{1/p}} = (p-1)^{-1/p}\frac{T}{\pi_p}\right\}. \end{aligned} \tag{1.13}$$

Note that the asymptote for $\mathcal{C}_3^1$ as $\alpha \to \infty$ is $\beta = \lambda_1$, while as $\beta \to \infty$ the asymptote is $\alpha = \lambda_2$. The set $\mathcal{C}_3^2$ is just the reflection of $\mathcal{C}_3^1$ with respect to the diagonal line $\alpha = \beta$. In Theorem 1.4 below we will consider $(\alpha,\beta) \in \mathcal{C}_3^1$ with $\alpha > \beta$, with a corresponding normalized eigenfunction, ϕ_{31}, as depicted in Figure 2.



(a) ϕ_{21}



(b) ϕ_{22}

Figure 1: Eigenfunctions for $(\alpha, \beta) \in \mathcal{C}^2$ with $\alpha > \beta$.

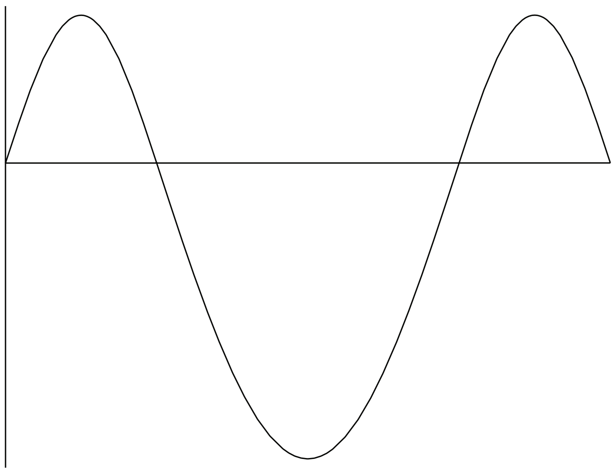


Figure 2: Eigenfunction for $(\alpha, \beta) \in C_3^1$ with $\alpha > \beta$, that is, ϕ_{31}.

Components $\mathcal{C}_n$ of Σ_p containing the points (λ_n, λ_n), $n > 3$, are obtained similarly (see [2]). Note that a nontrivial solution of (1.5) associated with $(\alpha, \beta) \in \mathcal{C}_n$ consists of n bumps in the interval $(0, T)$, and can be expressed explicitly in terms of the $\sin_p$ function.

We adopt a convention used in [5] to identify the Fučík eigenvalues and eigenfunctions referred to in our theorems and proofs. Fix a positive constant s and consider the intersection of the line $(\lambda + s, \lambda)$ with Σ_p. This produces a sequence $0 < \lambda_{11} < \lambda_{12} < \lambda_{21} = \lambda_{22} < \lambda_{31} < \lambda_{32} < \cdots$ with associated normalized eigenfunctions $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}, \phi_{31}, \phi_{32}, \ldots$. See Figure 3.

There is an interesting literature developing for problems such as (1.1). There are two different cases which have to be distinguished: $(\alpha, \beta) \notin \Sigma_p$, the so-called *nonresonance* case; $(\alpha, \beta) \in \Sigma_p$, the so-called *resonance* case.

The nonresonance case is well understood. If (α, β) belongs to a component of $\mathbb{R}^2 \setminus \Sigma_p$ containing a point (λ, λ), then (1.1) has a solution for arbitrary $f \in L^1(0, T)$. (The proof is based on the homotopy invariance property of Leray-Schauder degree.) On the other hand, if the component of $\mathbb{R}^2 \setminus \Sigma_p$ does not contain a point (λ, λ), then there exists $f \in L^1(0, T)$ for which (1.1) does not have a solution (see, e.g., [2]).

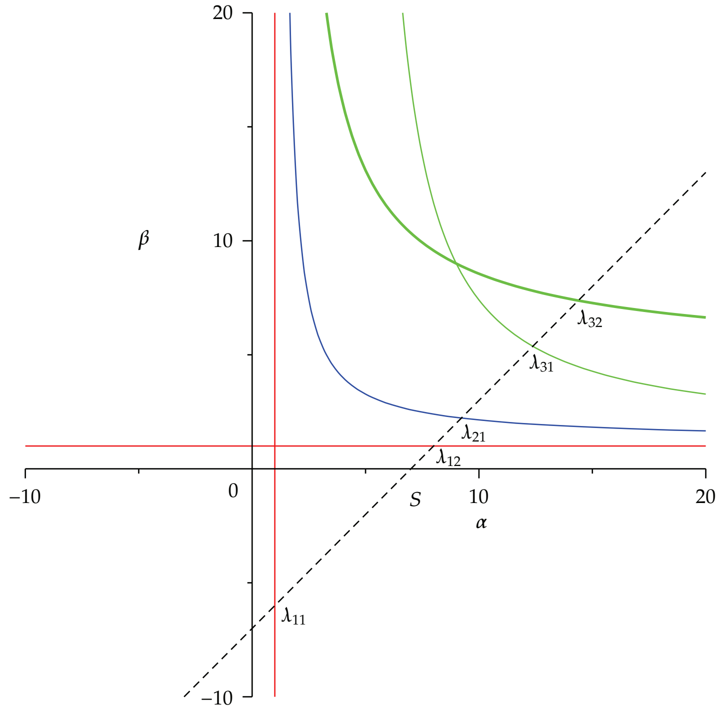


Figure 3: First few components in the Fučík Spectrum.

The resonance case is much more involved. The case where $p \neq 2$ and $(\alpha, \beta) = (\lambda_1, \lambda_1)$ was studied in detail in [4] and generalized to the PDE case by [6–10]. The case where $(\alpha, \beta) = (\lambda_2, \lambda_2)$ was studied in [11] and there are no generalizations of their results to the PDE case available so far. It is particularly interesting to note that these papers show that in some respects the semilinear ($p = 2$) and quasilinear ($p \neq 2$) cases can be quite different.

Our work is also related to results in [12–15], where solvability conditions in the spirit of Landesman and Lazer, [16], or Ahmad et al. [17], are derived for nonlinear perturbations of (1.1). Theorem 6.1 on page 334 of [12] is identical to our Theorem 1.2. In this case the novelty of our theorem is that it is proved using variational techniques rather than the method of upper and lower solutions. Problem (1.1) represents a subcase of the problems considered in [13–15]. We note that [13, 15] deal with the semilinear case, $p = 2$, and [14] deals with the general quasilinear case, $1 < p < \infty$. Moreover, the solvability conditions in these papers reduce to the solvability conditions in our theorems. However, the given papers make significant use of the additional assumption that $(\alpha, \beta) \in (\lambda_{k-1}, \lambda_{k+1})^2$ for some $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$. Our arguments do not rely on this local restriction and thus allow us to consider any point on the first three curves of Σ_p. More specifically, we can consider $\lambda = \lambda_{11}, \lambda_{12}, \lambda_{21}$, and λ_{31} with no further restriction.

We proceed with the statements of our main theorems, assuming throughout that $(\alpha, \beta) = (\lambda + s, \lambda) \in \Sigma_p$, where $s > 0$ is fixed. The theorems do not apply if $s = 0$. Our assumption that $s > 0$ restricts attention to those points of Σ_p which are below the diagonal $\alpha = \beta$. If we consider $s < 0$, then the part of Σ_p above the diagonal is considered and (1.14), (1.15), and (1.18) are reversed by the symmetry of the problem. The assumption (1.17) remains unchanged for both cases.

Theorem 1.1. *Suppose that* $\lambda = \lambda_{11}$ (λ_{12}) *and assume*

$$\int_0^T f\phi_{11} < 0 \cdot \left(\int_0^T f\phi_{12} > 0\right). \tag{1.14}$$

Then (1.1) *has at least one solution.*

In the special case $p = 2$, this theorem can be significantly strengthened.

Theorem 1.2. *Suppose that* $p = 2$ *and that* $\lambda = \lambda_{11}$ (λ_{12})*. Then* (1.1) *has a solution if and only if*

$$\int_0^T f\phi_{11} \leq 0 \cdot \left(\int_0^T f\phi_{12} \geq 0\right). \tag{1.15}$$

The statements above are written so as to keep notation consistent. Notice that $\phi_{21} = -\phi_{11}$, so the given inequalities in (1.14) are actually the same and can be restated as

$$\int_0^T f u_1 < 0. \tag{1.16}$$

The inequalities in (1.15) are also identical.

Theorem 1.3. *Suppose that* $\lambda = \lambda_{21}$ *and assume*

$$\left(\int_0^T f\phi_{21}\right) \cdot \left(\int_0^T f\phi_{22}\right) > 0. \tag{1.17}$$

Then (1.1) *has at least one solution.*

Notice that this theorem contains two cases, one where both integrals in (1.17) are positive and one where both integrals are negative. These cases correspond in some sense to resonance above and below the Fučík Spectrum.

Our last theorem is the following.

Theorem 1.4. *Suppose that* $\lambda = \lambda_{31}$ *and assume*

$$\int_0^T f\phi_{31} < 0. \tag{1.18}$$

Then (1.1) *has at least one solution.*

The proofs of our theorems will be variational. Note that (1.1) is the Euler-Lagrange equation associated with the functional

$$\begin{aligned} E(u) :&= \frac{1}{p}\int_0^T |u'|^p - \frac{\alpha}{p}\int_0^T (u^+)^p - \frac{\beta}{p}\int_0^T (u^-)^p - \int_0^T fu \\ &= \frac{1}{p}\int_0^T |u'|^p - \frac{\lambda}{p}\int_0^T |u|^p - \frac{s}{p}\int_0^T (u^+)^p - \int_0^T fu, \quad u \in X. \end{aligned} \tag{1.19}$$

Solutions of (1.1) are critical points of E. One part of the proof of Theorem 1.1 will locate a critical point via minimization. All other arguments will characterize critical points as saddle points. This requires us to establish the appropriate compactness and geometric properties of the functional. In particular, saddle point arguments require the appropriate geometry on *linked* sets.

In order to understand the functional E it is important to first understand the functional

$$I(u) := \int_0^T |u'|^p - s\int_0^T |u^+|^p, \quad u \in X. \tag{1.20}$$

It is well known that $\lambda_{11}, \lambda_{12}, \lambda_{21}, \ldots$ are critical values of I restricted to the L^p unit sphere $\mathcal{S} := \{u \in X : \int_0^T |u|^p = 1\}$. The corresponding critical points are $\phi_{11}, \phi_{12}, \phi_{21}, \ldots$. In particular, ϕ_{11} is a global minimum, ϕ_{12} is a local minimum, and ϕ_{21} is a saddle point with λ_{21} characterized as the minimax over continuous curves on $\mathcal{S}$ connecting ϕ_{11} and ϕ_{12}. See [5] for details. Throughout the paper we will make use of the notations $I^c := \{u \in X : I(u) \geq c\int_0^T |u|^p\}$ and $I_c := \{u \in X : I(u) \leq c\int_0^T |u|^p\}$, to describe super- and sub-level sets.

The linking property that is crucial for the proofs of Theorems 1.3 and 1.4 also leads to a new variational characterization of λ_{31}, which we discuss in the last section. A similar characterization was proved in [18], but we note that this was assuming $(\lambda_{31} + s, \lambda_{31}) \in (\lambda_2, \lambda_4)^2$, that is, $\lambda_{31} + s < \lambda_4$. Let us emphasize that our characterization does not restrict the value of s.

Finally, since our arguments rely on variational structures that are also available in the related PDE case, it should be clear that the given theorems generalize under appropriate circumstances. For example, Theorems 1.1 and 1.2 generalize to the PDE case with no further assumptions. Generalizing Theorem 1.3 is more complicated. We would need for λ_{21} to be isolated, and to have an associated set of solutions, that is, an *eigenspace*, composed of positive multiples of a pair of two-bump solutions such as ϕ_{21}, ϕ_{22}. We note that it is known that eigenfunctions associated with λ_{21} have only two nodes, see [5] or [19], but it is not true in general that the solution set can be spanned by just the two given solutions. It is not known if λ_{21} is necessarily isolated in the PDE case.

2. A Discussion of the Semilinear Case

We provide a brief discussion of the special properties of the semilinear case, that is, $p = 2$, with $(\alpha, \beta) \in \mathcal{C}_1$. In this case (1.1) becomes

$$\begin{gathered} -u'' - \alpha u^+ + \beta u^- = f \quad \text{in } (0, T), \\ u(0) = u(T) = 0. \end{gathered} \tag{2.1}$$

For the first case we assume that $\lambda = \lambda_{11}$, so $\alpha = \lambda_1$ and $\beta = \lambda_1 - s$ for some $s > 0$. If we multiply equation (2.1) by the positive eigenfunction ϕ_{11} and integrate by parts, we obtain

$$-s \int_0^T u^- \phi_{11} = \int_0^T f \phi_{11}, \tag{2.2}$$

and the necessity of (1.15) follows. A similar argument applies if $\lambda = \lambda_{12}$. In this case it is important to recall that $\phi_{12} = -\phi_{11} < 0$.

On the other hand let us assume that

$$\int_0^T f \phi_{11} = \int_0^T f u_1 = 0. \tag{2.3}$$

Then the linear problem

$$\begin{gathered} -u'' - \lambda_1 u = f \quad \text{in } (0, T), \\ u(0) = u(T) = 0, \end{gathered} \tag{2.4}$$

has a solution u. If we choose $c > 0$ large enough then $\overline{u} = u + c\phi_{11}$ is a positive solution of this linear problem. It follows that $\overline{u}$ is a solution of

$$\begin{gathered} -u'' - \lambda_1 u^+ + \beta u^- = f \quad \text{in } (0, T), \\ u(0) = u(T) = 0, \end{gathered} \tag{2.5}$$

for arbitrary $\beta \in \mathbb{R}$. Similarly, we can choose $c < 0$ large enough so that $\underline{u} = u + c\phi_{11} < 0$ in $(0, T)$, and $\underline{u}$ is then a solution of

$$\begin{gathered} -u'' - \alpha u^+ + \lambda_1 u^- = f \quad \text{in } (0, T), \\ u(0) = u(T) = 0 \end{gathered} \tag{2.6}$$

for arbitrary $\alpha \in \mathbb{R}$. Hence (2.3) is a sufficient condition. The sufficiency of the strict inequalities is postponed until a later section where it will be proved for the more general case, $p > 1$. We note that our proof of Theorem 1.1, in a later section, will also complete the proof of Theorem 1.2.

3. The Palais-Smale Condition on Σ_p

In this section, we prove that under conditions (1.14), (1.17), and (1.18) the energy functional E satisfies the Palais-Smale condition (PS), that is, if $\{u_n\} \subset X$ such that $\{E(u_n)\}$ is bounded and $E'(u_n) \to 0$ in X^*, then $\{u_n\}$ contains a converging subsequence. In fact, we prove the more general result stated below.

Proposition 3.1. *Let $(\alpha, \beta) \in \Sigma_p$ and let $f \in L^1(0,T)$ such that*

$$\int_0^T u_{\alpha\beta} f \neq 0, \tag{3.1}$$

for any $u_{\alpha\beta}$ which is a nontrivial solution of (1.5). *Then E satisfies (PS).*

Proof. Let $\{u_n\} \subset X$ be a sequence such that $E(u_n)$ is bounded and $E'(u_n) \to 0$ in X^*. (We will refer to such a sequence as a PS-sequence.) We proceed in two steps. First we will show that $\{u_n\}$ is bounded in X. Then we will select a converging subsequence.

We argue by contradiction, so assume $\|u_n\|_X \to \infty$. Let $w_n := u_n/\|u_n\|_X$, and, without loss of generality, assume that $w_n \rightharpoonup w$ in X. Our assumptions imply

$$0 = \lim_{n\to\infty} \frac{E'(u_n)}{\|u_n\|_X^{p-1}} = \lim_{n\to\infty} \left(J(w_n) - \alpha S(w_n^+) + \beta S(w_n^-) - \frac{F}{\|u_n\|_X^{p-1}} \right). \tag{3.2}$$

The compactness of S and (3.2) yield

$$\lim_{n\to\infty} J(w_n) = \alpha S(w^+) - \beta S(w^-), \tag{3.3}$$

and the continuity of J^{-1} subsequently leads to

$$w = \lim_{n\to\infty} w_n = J^{-1}(\alpha S(w^+) - \beta S(w^-)), \tag{3.4}$$

that is,

$$J(w) - \alpha S(w^+) + \beta S(w^-) = 0. \tag{3.5}$$

In order to rule out the case $w = 0$, we observe in (3.2) that

$$\lim_{n\to\infty} (J(w_n) - \alpha S(w_n^+) + \beta S(w_n^-)) = 0. \tag{3.6}$$

This together with $w = 0$ would imply

$$0 = \lim_{n\to\infty} \langle J(w_n), w_n \rangle = \lim_{n\to\infty} \int_0^T |w_n'|^p = 1, \tag{3.7}$$

a contradiction. Thus w is a nontrivial solution of (3.5), that is, $w = u_{\alpha\beta}$. Our assumptions also imply that

$$\int_0^T f u_n = pE(u_n) - \langle E'(u_n), u_n \rangle = o(\|u_n\|_X) \tag{3.8}$$

as $n \to \infty$. Dividing through by $\|u_n\|_X$ we get that

$$0 = \lim_{n\to\infty} \int_0^T f w_n = \int_0^T f u_{\alpha\beta} \neq 0, \tag{3.9}$$

a contradiction. We have proved that $\{u_n\}$ is bounded.

Without loss of generality we assume that there is a $u \in X$ such that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(0,T)$. Our assumptions imply that

$$0 = \lim_{n\to\infty} E'(u_n) = \lim_{n\to\infty} \left(J(u_n) - \alpha S(u_n^+) + \beta S(u_n^-) - F \right). \tag{3.10}$$

Using the continuity of J^{-1} and the compactness of S we now have

$$u = \lim_{n\to\infty} u_n = J^{-1}\left(\alpha S(u^+) - \beta S(u^-) + F \right). \tag{3.11}$$

Hence every PS-sequence has a converging subsequence and the proof is finished. □

4. Proof of Theorems 1.1 and 1.2

We will treat two cases.

Case 1 ($\lambda = \lambda_{11} = \lambda_1 - s$). If $\lambda = \lambda_{11}$, then

$$E(u) = \frac{1}{p} \int_0^T |u'|^p - \frac{\lambda_1}{p} \int_0^T |u|^p + \frac{s}{p} \int_0^T (u^-)^p - \int_0^T f u. \tag{4.1}$$

Lemma 4.1. *E is bounded below.*

Proof. Assume the contrary, that is, there exists a sequence $\{u_n\} \subset X$ such that

$$\lim_{n\to\infty} E(u_n) = -\infty. \tag{4.2}$$

As a consequence, $\|u_n\|_X \to \infty$. Let $w_n = u_n/\|u_n\|_X$. Then

$$\begin{aligned}
0 &\geq \liminf_{n\to\infty}\frac{E(u_n)}{\|u_n\|_X^p}\\
&= \liminf_{n\to\infty}\left(\frac{1}{p}\int_0^T |w_n'|^p - \frac{\lambda_1}{p}\int_0^T |w_n|^p + \frac{s}{p}\int_0^T (w_n^-)^p - \frac{1}{\|u_n\|_X^p}\int_0^T fu_n\right)\\
&= \liminf_{n\to\infty}\left(\frac{1}{p}\int_0^T |w_n'|^p - \frac{\lambda_1}{p}\int_0^T |w_n|^p + \frac{s}{p}\int_0^T (w_n^-)^p\right).
\end{aligned} \tag{4.3}$$

Without loss of generality we assume that there exists $w \in X$ such that $w_n \rightharpoonup w$ in X and $w_n \to w$ in $L^p(0,T)$. Thus

$$0 \geq \liminf_{n\to\infty}\left(\frac{1}{p}\int_0^T |w_n'|^p\right) - \frac{\lambda_1}{p}\int_0^T |w|^p + \frac{s}{p}\int_0^T (w^-)^p. \tag{4.4}$$

If $w = 0$, then (4.4) yields $\lim_{n\to\infty}\|w_n\|_X = 0$, a contradiction. Hence $w \neq 0$. By the weak lower semicontinuity of the norm we deduce from (4.4) that

$$0 \geq \frac{1}{p}\int_0^T |w'|^p - \frac{\lambda_1}{p}\int_0^T |w|^p + \frac{s}{p}\int_0^T (w^-)^p. \tag{4.5}$$

The variational characterization and simplicity of λ_1 yield that w must be a positive multiple of u_1, and thus it follows that $w = \phi_{11}$. Since

$$\lim_{n\to\infty}\int_0^T fw_n = \int_0^T f\phi_{11} < 0, \tag{4.6}$$

there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $\int_0^T fu_n < 0$. Using this and the variational characterization of λ_1, we see that

$$\frac{1}{p}\int_0^T |u_n'|^p - \frac{\lambda_1}{p}\int_0^T |u_n|^p + \frac{s}{p}\int_0^T (u_n^-)^p - \int_0^T fu_n \geq 0, \tag{4.7}$$

for $n \geq n_0$, and thus we arrive at a contradiction with (4.2).

The boundedness of E from below together with the (PS) condition yields the existence of a critical point, that is, a minimizer, of E, and thus of a solution to (1.1). The proof of the first case is finished. □

Case 2 ($\lambda = \lambda_{12} = \lambda_1$). If $\lambda = \lambda_{12}$, then we can write (1.19) as

$$E(u) = \frac{1}{p}\int_0^T |u'|^p - \frac{\lambda_1}{p}\int_0^T |u|^p - \frac{s}{p}\int_0^T (u^+)^p - \int_0^T fu. \tag{4.8}$$

We will show that E has a saddle point.

For $t > 0$, we have

$$E(tu_1) = -\frac{st^p}{p}\int_0^T (u_1)^p - t\int_0^T fu_1, \tag{4.9}$$

and for $t < 0$, we have

$$E(tu_1) = -t\int_0^T fu_1 = -tc\int_0^T f\phi_{11}, \tag{4.10}$$

for some $c > 0$, so it is clear that

$$\lim_{|t|\to\infty} E(tu_1) = -\infty. \tag{4.11}$$

Observe that for $u \in I^{\lambda_{21}}$, we have

$$E(u) \geq \frac{\lambda_{21} - \lambda_1}{p}\int_0^T |u|^p - \int_0^T fu. \tag{4.12}$$

Since $\lambda_{21} > \lambda_1$, the functional E is bounded below by some K on $I^{\lambda_{21}}$. Hence, we obtain from (4.11) that, for $R > 0$ large enough,

$$\max\{E(Ru_1), E(-Ru_1)\} < K \leq \inf_{u\in I^{\lambda_{21}}} E(u). \tag{4.13}$$

Let $\Gamma := \{h : [-1,1] \to X : h \text{ is continuous, and } h(\pm 1) = \pm Ru_1\}$. We claim that if $h \in \Gamma$, then $h([-1,1]) \cap I^{\lambda_{21}}$ is nonempty. If $h(t) = 0$ for some t, then the conclusion is immediate. If $h(t) \neq 0$ for all t, then it follows from the variational characterization of λ_{21} in [5] that

$$\max_{t\in[-1,1]} I\left(\frac{h(t)}{\|h(t)\|_{L^p}}\right) \geq \lambda_{21}, \tag{4.14}$$

and so our claim is established. It follows that

$$\max_{t\in[-1,1]} E(h(t)) \geq K \quad \forall h \in \Gamma, \tag{4.15}$$

and so

$$c = \inf_{h\in\Gamma}\max_{t\in[-1,1]} E(h(t)) \geq K. \tag{4.16}$$

Since the (PS) condition has already been established, we can conclude that c is a critical value and thus (1.1) has a solution.

The proof of the second case is finished, and thus both Theorems 1.1 and 1.2 have been proved.

5. Proof of Theorem 1.3

Consider two cases.

Case 1 ($\int_0^T f\phi_{21} > 0$ and $\int_0^T f\phi_{22} > 0$). This is the most interesting case. The plan is to construct a copy of S^1, the unit circle in $\mathbb{R}^2$, in X that links with $I^{\lambda_{31}}$. We will then argue that these sets enjoy the appropriate saddle-point geometry. Since the (PS) condition has already been established for our functional, we will have proved the existence of a saddle point.

The construction is accomplished via the union of six curves. Let

$$\gamma_1 := \left\{ \alpha\phi_{21}^+ - \beta\phi_{21}^- : \alpha \geq 0, \beta \geq 0, \alpha^p \int_0^T |\phi_{21}^+|^p + \beta^p \int_0^T |\phi_{21}^-|^p = 1 \right\}, \tag{5.1}$$

a curve in $\mathcal{S}$ connecting $\phi_{21}^+/\|\phi_{21}^+\|_{L_p}$ and $-\phi_{21}^-/\|\phi_{21}^-\|_{L_p}$. A straight-forward calculation shows that $I \equiv \lambda_{21}$ on γ_1. (See [19].)

Let

$$\gamma_2 := \left\{ \left((1-t)\left(\frac{\phi_{21}^+}{\|\phi_{21}^+\|_{L^p}} \right)^p + t\phi_{11}^p \right)^{1/p} : 0 \leq t \leq 1 \right\}, \tag{5.2}$$

a curve in $\mathcal{S}$ connecting $\phi_{21}^+/\|\phi_{21}^+\|_{L_p}$ to ϕ_{11}. Another straight-forward calculation shows that $I|_{\gamma_2}$ decreases from λ_{21} to λ_{11} as t goes from 0 to 1. Similarly, we construct γ_3 connecting $-\phi_{21}^-/\|\phi_{21}^-\|_{L_p}$ to $\phi_{12} = -\phi_{11}$.

The curves γ_4, γ_5 and γ_6 are constructed similarly to γ_1, γ_2, and γ_3 using ϕ_{22} in place of ϕ_{21}. Let $\partial Q := \gamma_1 \cup \cdots \cup \gamma_6$. It is clear that ∂Q is homeomorphic to S^1, and for convenience we name a homeomorphism $h_0 : S^1 \to \partial Q$ such that $h_0(0,-1) = \phi_{11}$, $h_0(0,1) = \phi_{12}$, $h_0(-1,0) = \phi_{21}$, and $h_0(1,0) = \phi_{22}$.

Before proceeding it is useful to modify ∂Q by lowering its I-energy in a way that will help with later estimates. The construction so far shows that $I \leq \lambda_{21}$ on ∂Q, but it will be convenient to have a strict inequality on most of the curve ∂Q. We begin with the following lemma whose proof is clear.

Lemma 5.1. *There exists an $\epsilon > 0$ and a $\delta > 0$ such that*

$$\int_0^T fu > \epsilon, \tag{5.3}$$

for any $u \in B_\delta(\phi_{21}) \cup B_\delta(\phi_{22})$.

Now let $C := \partial Q \setminus (B_{\delta/2}(\phi_{21}) \cup B_{\delta/2}(\phi_{22}))$, let $B := \emptyset$, and apply the basic deformation lemma for C^1-manifolds from [20, Lemma 3.7, page 55], where I is the C^1 functional in question and $\mathcal{S}$ is the Finsler manifold. Let $\alpha(t,x)$ be the deformation guaranteed by the lemma and let $\partial Q' := \alpha(t, \partial Q)$ for any small fixed $t > 0$. We have $I(\alpha(t,u)) \leq I(u)$ for all u, so $I \leq \lambda_{21}$ on all of $\partial Q'$. It also follows that there is a $\kappa > 0$ such that $I(u) \leq \lambda_{21} - \kappa$ on $\partial Q' \setminus (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$. For convenience, we rename $\partial Q'$ as ∂Q again.

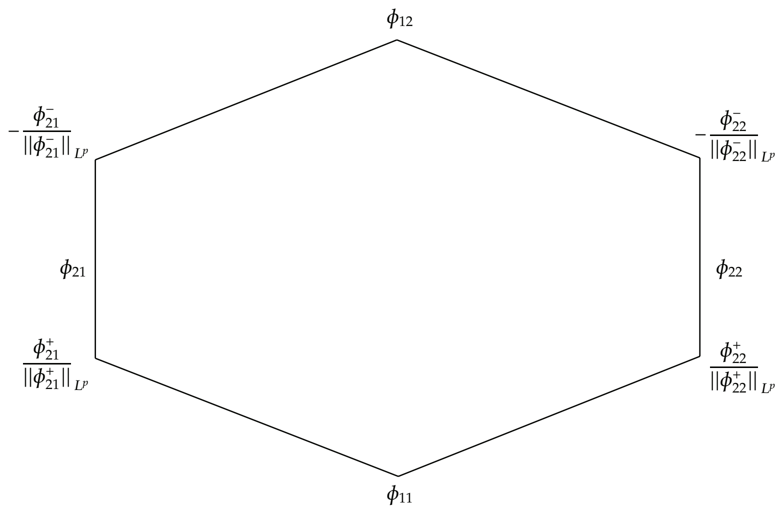


Figure 4: ∂Q.

Recall that

$$E(u) = \frac{1}{p}\int_0^T |u'|^p - \frac{s}{p}\int_0^T (u^+)^p - \frac{\lambda_{21}}{p}\int_0^T |u|^p - \int_0^T fu, \quad u \in X. \tag{5.4}$$

The following lemmas establish the desired saddle geometry.

Lemma 5.2. *E is bounded below on $I^{\lambda_{31}}$ by some constant K.*

Proof. Observe that for $u \in I^{\lambda_{31}}$, we have

$$E(u) \geq \frac{\lambda_{31} - \lambda_{21}}{p}\int_0^T |u|^p - \int_0^T fu. \tag{5.5}$$

The lemma follows. □

Lemma 5.3. *There is an $R > 0$ such that $\max_{R\partial Q} E(u) < K$, where $R\partial Q := \{Ru : u \in \partial Q\}$.*

Proof. For $u \in \partial Q \cap (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$ and $t > 0$, we have

$$E(tu) \leq -t\int_0^T fu < -t\epsilon. \tag{5.6}$$

For $u \in \partial Q \setminus (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$ and $t > 0$, we have

$$E(tu) \leq -t^p\frac{\kappa}{p} - t\int_0^T fu. \tag{5.7}$$

The lemma follows. □

Now we prove a *linking* lemma. It is clear that $\partial Q \cap I^{\lambda_{31}} = \emptyset$, so what remains to be proved is the following. For convenience we set $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Lemma 5.4. *Given any continuous $h : D \to X$ such that $h|_{S^1} \equiv h_0$, we have $h(D) \cap I^{\lambda_{31}} \neq \emptyset$.*

Proof. Suppose not. If $0 \in h(D)$, then we are done. If $0 \notin h(D)$, then without loss of generality we may replace h by its projection onto $\mathcal{S}$ and restrict the remainder of the argument to that surface. If $h(D) \cap I^{\lambda_{31}} = \emptyset$ then there is a $\delta > 0$ such that $\max_D I(h(x)) \leq \lambda_{31} - \delta$. Using the results of Corvellec, [21], there exists a strong deformation retract $\eta : I_{\lambda_{31}-\delta/2} \times [0,1] \to I_{\lambda_{21}}$. (See the proof of Theorem 4 in [21] where the existence of a strong deformation retract is proved as one step towards proving the existence of a weak deformation retract.) Replacing h with $\eta(h(\cdot),1)$, and calling it h again, we may now assume that $\max_D I(h(x)) \leq \lambda_{21}$. Observe that η has not changed the values of h on S^1, and so $h \equiv h_0$ on S^1.

We will now establish the existence of a path, γ, in D connecting $(0,\pm 1)$. The image, via h, of this path will connect ϕ_{11} to ϕ_{12} in $(I_{\lambda_{21}} \cap \mathcal{S}) \setminus \{\phi_{21}, \phi_{22}\}$. This will be a contradiction of the variational characterization of λ_{21} established in [5], and so the lemma will be proved.

Our tool for constructing γ is the Hex Theorem as stated in [22]. Let $E := \{(a,b) \in S^1 : a > 0, -\sqrt{2}/2 \leq b \leq \sqrt{2}/2\}$, and $W := \{(a,b) \in S^1 : a < 0, -\sqrt{2}/2 \leq b \leq \sqrt{2}/2\}$. The other two components of S^1 are N, the component containing $(0,1)$, and S, the component containing $(0,-1)$. It is clear that given any n there is a homeomorphism from D to the standard $n \times n$ Hex board with edges corresponding to E, W, N, and S in a natural way. We will refer to such a homeomorphism as an $n \times n$ hexagonal tiling of D. It is also clear that given any $d > 0$ there is an $n \times n$ hexagonal tiling of D such that the maximum diameter of the hexagons is less than d.

Since $h(E) \cap h(W) = \emptyset$, we have $r := \operatorname{dist}(h(E), h(W)) > 0$. Choose $\delta > 0$ such that $|x - y| < \delta$ implies $\|h(x) - h(y)\|_X < r$. Now tile D with an $n \times n$ collection of hexagons with maximum diameter $\delta/3$. If a hexagon contains a point in $h^{-1}(\phi_{21}) \cup h^{-1}(\phi_{22})$, then color it red. Otherwise color the hexagon blue. By the Hex Theorem, we either have a path of red hexagons connecting E and W, or a path of blue hexagons connecting N and S. Suppose that there is a path of red hexagons connecting E and W. Consider a hexagon at the beginning of the path so that it is touching E. Since all of the points of this hexagon are within distance $\delta/3$ of E, the image of this hexagon via h cannot intersect $h(W)$, and in particular cannot contain ϕ_{21}. Hence this first red hexagon contains a point in $h^{-1}(\phi_{22})$. Similarly, the last hexagon in the path contains a point of $h^{-1}(\phi_{21})$. Somewhere along this path there must be a pair of hexagons that share an edge and where one hexagon contains a point $x \in h^{-1}(\phi_{22})$ and the other a point $y \in h^{-1}(\phi_{21})$. However, this implies that $|x - y| < \delta$ with $\|h(x) - h(y)\|_X = \|\phi_{22} - \phi_{21}\|_X > r$, a contradiction. Hence there must be a path of blue hexagons connecting N and S. This clearly implies that there is a continuous path in D connecting $(0,1)$ to $(0,-1)$ where no point on the path is in $h^{-1}(\phi_{22}) \cup h^{-1}(\phi_{21})$. The image of this path via h is the γ that we were looking for. The lemma is proved, and hence the theorem is proved. □

It is important to observe that a similar argument establishes the linking of $R\partial Q$ and $I^{\lambda_{31}}$ for any $R \geq 1$.

Case 2 ($\int_0^T f\phi_{21} < 0$ and $\int_0^T f\phi_{22} < 0$).

Lemma 5.5. *E is bounded below by some K on $I^{\lambda_{21}}$.*

Proof. Let $\epsilon > 0$ and $\delta > 0$ such that for $u \in (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$, we have $\int_0^T fu < -\epsilon$. There is a $\kappa > 0$ such that $I(u) \geq \lambda_{21} + \kappa$ for $u \in (\mathcal{S} \cap I^{\lambda_{21}}) \setminus (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$. Therefore $E(tu) \geq \epsilon t$ for

$u \in (\mathcal{S} \cap I^{\lambda_{21}}) \cap (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$ and $E(tu) \geq (\kappa/p)t^p - t\int_0^T fu$ for $u \in (\mathcal{S} \cap I^{\lambda_{21}}) \setminus (B_\delta(\phi_{21}) \cup B_\delta(\phi_{22}))$. The lemma follows. □

Lemma 5.6. *There is an* $R > 0$ *such that* $\max\{E(R\phi_{11}), E(R\phi_{12})\} < K$.

Proof. For $t > 0$, we have $E(t\phi_{12}) = ((\lambda_{12} - \lambda_{21})/p)t^p - t\int_0^T \phi_{12} f$ and $E(t\phi_{11}) = ((\lambda_{11} - \lambda_{21})/p)t^p - t\int_0^T \phi_{11} f$, so the lemma follows easily. □

Lemma 5.7. *Every continuous path connecting* $R\phi_{11}$ *to* $R\phi_{21}$ *must intersect* $I^{\lambda_{21}}$.

Proof. This is a direct consequence of the variational characterization of λ_{21} proved in [5]. Hence the theorem is proved. □

6. Proof of Theorem 1.4

In this section we assume that $\lambda = \lambda_{31}$. Once again we must establish the appropriate saddle geometry and linking. Most of the arguments follow from work done in previous sections. The one exception is the following lemma.

Lemma 6.1. *E is bounded below by some K on* $I^{\lambda_{31}}$.

Proof. There is an $\epsilon > 0$ and a $\delta > 0$ such that if $u \in B_\delta(\phi_{31})$, then $\int_0^T fu < -\epsilon$. For $u \in (\mathcal{S} \cap I^{\lambda_{31}}) \setminus B_\delta(\phi_{31})$ there is a $\kappa > 0$ such that $I(u) \geq \lambda_{31} + \kappa$. It follows that $E(tu) \geq \epsilon t$ for $u \in B_\delta(\phi_{31}) \cap \mathcal{S} \cap I^{\lambda_{31}}$ and that $E(tu) \geq (\kappa/p)t^p - t\int_0^T uf$ for $u \in (\mathcal{S} \cap I^{\lambda_{31}}) \setminus B_\delta(\phi_{31})$. The lemma follows. □

Once again we consider ∂Q as constructed in the previous section. We can employ the ∂Q with or without the modifying flow.

Lemma 6.2. *There is an* $R > 0$ *such that* $\max_{R\partial Q} E(u) < K$, *where* $R\partial Q := \{Ru : u \in \partial Q\}$.

Proof. This follows easily from the fact that for $u \in \partial Q$, we have $I(u) \leq \lambda_{21}$ and so $E(tu) \leq ((\lambda_{21} - \lambda_{31})/p)t^p - t\int_0^T fu$.

The linking of $I^{\lambda_{31}}$ and $R\partial Q$ has already been established in Lemma 5.4. Hence the theorem is proved. □

7. Variational Characterization of λ_{31}

Let $\mathcal{Q} := \{Q \subset \mathcal{S} :$ there exists a continuous $h : D \to \mathcal{S}$ such that $h(D) = Q, h : S^1 \to h(S^1)$ is a homeomorphism into $I_{\lambda_{21}}$, and $h(S^1)$ contains $\phi_{11}, \phi_{21}, \phi_{12}, \phi_{22}$ in that order$\}$. For convenience we refer to $h(S^1)$ as ∂Q.

Lemma 5.4 shows that

$$c := \inf_{Q \in \mathcal{Q}} \max_{u \in Q} I(u) \geq \lambda_{31}. \tag{7.1}$$

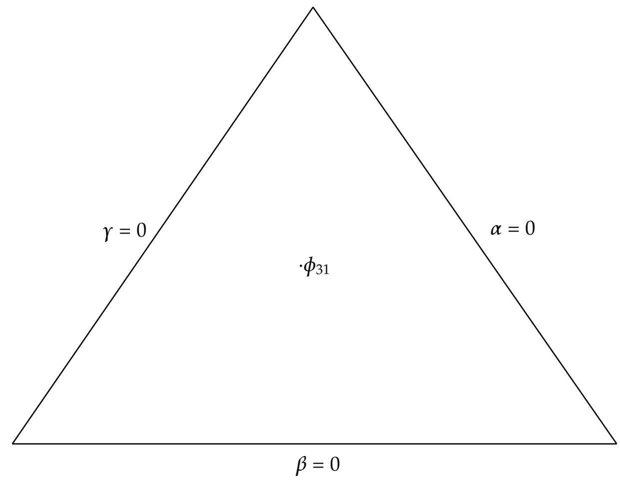


Figure 5: P.

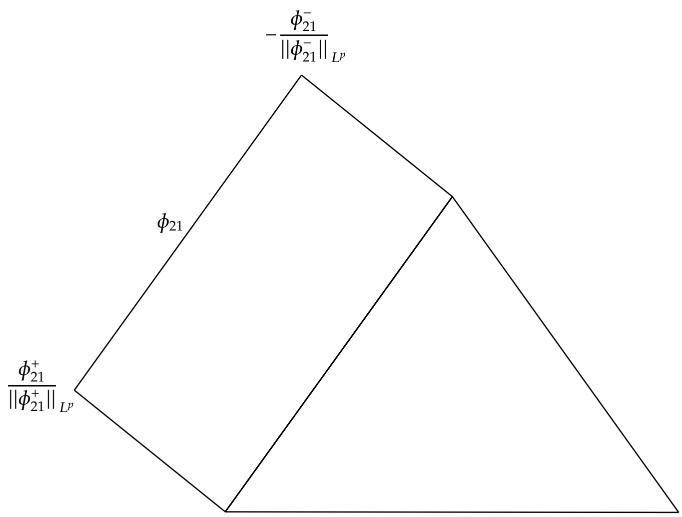


Figure 6: P'.

The set $\mathcal{Q}$ is clearly preserved by pseudogradient flows on $\mathcal{S}$ associated with I, and I satisfies the (PS) condition. Hence c is a critical value of I. To show that $c = \lambda_{31}$, and thus that λ_{31} has the given variational characterization, it suffices to construct an element $Q \in \mathcal{Q}$ such that $\max_Q I(u) = \lambda_{31}$.

We begin by constructing a *triangle* using the positive and negative parts of ϕ_{31}. For convenience we write

$$\phi_{31} := \begin{cases} u(x), & x \in [0,a], \\ v(x), & x \in [a,b], \\ w(x), & x \in [b,T], \end{cases} \tag{7.2}$$

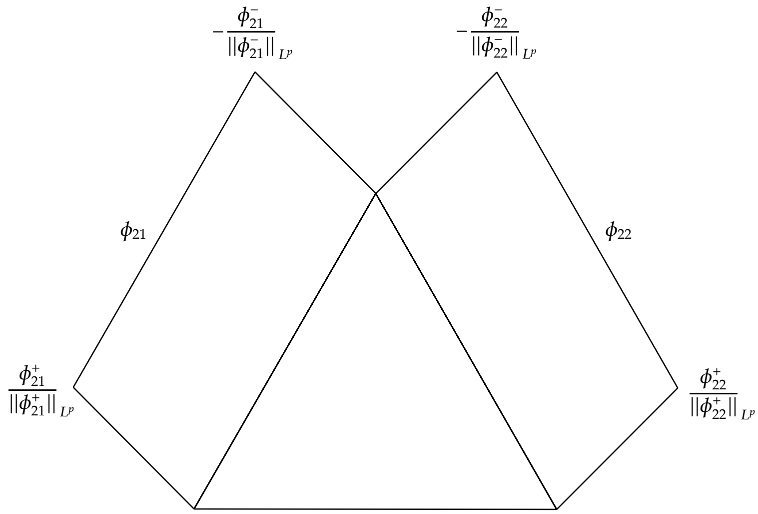


Figure 7: P''.

where u, w are the positive parts of ϕ_{31} and v is the negative part. Each of u, v, w is an appropriately scaled multiple of $\sin_p$ on the appropriate interval, and is 0 everywhere else. Now let $P = \{\alpha u + \beta v + \gamma w : \alpha, \beta, \gamma \geq 0,\ \alpha^p \int_0^T |u|^p + \beta^p \int_0^T |v|^p + \gamma^p \int_0^T |w|^p = 1\}$. See Figure 5. A straight forward computation shows that $I \equiv \lambda_{31}$ on P.

Consider a point $z = \alpha u + \beta v \in \partial P$, that is, a point on the boundary of P such that $\gamma = 0$. We construct the curve $\{t^{1/p} z(tx) : b/T \leq t \leq 1\}$. For convenience we denote $z_t(x) = t^{1/p} z(tx)$ and $z_t(x) = \alpha u_t(x) + \beta v_t(x)$. Elementary computations show that $\int_0^T |z_t|^p = 1$ and $\int_0^T |(z_t)'|^p = t^p \int_0^T |(z_1)'|^p$. Hence $I(z_t)$ decreases as t goes from 1 to b/T. Observe that if $\alpha = \beta$ then z is a positive multiple of ϕ_{31} on the interval $[0, b]$, and is thus a solution of

$$\begin{gathered} -\left(\left|y'\right|^{p-2} y'\right)' - (\lambda_{31} + s)\left(y^+\right)^{p-1} + \lambda_{31}\left(y^-\right)^{p-1} = 0 \quad \text{in } (0, b), \\ y(0) = y(b) = 0. \end{gathered} \tag{7.3}$$

It follows that z_t is a normalized two-node Fučík eigenfunction on $[0, b/t]$ with positive first node. In particular we must have $z_{b/T} = \phi_{21}$. Moreover, if we now consider $\{\alpha u_{b/T} + \beta v_{b/T} : \alpha, \beta \geq 0$ and $\alpha^p \int_0^T |u_{b/T}|^p + \beta^p \int_0^T |v_{b/T}|^p = 1\}$ we see that this must be identical to $\{\alpha \phi_{21}^+ - \beta \phi_{21}^- : \alpha, \beta \geq 0$ and $\alpha^p \int_0^T |\phi_{21}^+|^p + \beta^p \int_0^T |\phi_{21}^-|^p = 1\}$, which is precisely the segment that we used to construct ∂Q in a previous section. Note that, we have a new figure $P' = P \cup \{\alpha u_t + \beta v_t : \alpha v + \beta u \in P,\ b/T \leq t \leq 1\}$, which is still homeomorphic to D, but now includes ϕ_{21} on its boundary. See Figure 6. A similar construction is possible on the edge of P where $\alpha = 0$, so that we arrive at P'' which is homeomorphic to D and contains both ϕ_{21} and ϕ_{22}. See Figure 7.

Observe that the component of the boundary of P'' between $\phi_{21}^+/\|\phi_{21}^+\|_{L^p}$ and $\phi_{22}^+/\|\phi_{22}^+\|_{L^p}$ contains only nontrival nonnegative elements. For convenience call this part of the boundary $(\partial P'')^+$. For each $z \in (\partial P'')^+$ we form $\{(tz^p + (1-t)\phi_{11}^p)^{1/p} : 0 \leq t \leq 1\}$. It is easy to see that this is a curve on $\mathcal{S}$ connecting z to ϕ_{11} such that I decreases from z to ϕ_{11}. A similar construction is possible along the boundary of P'' between $-\phi_{21}^-/\|\phi_{21}^-\|_{L^p}$ and $-\phi_{22}^-/\|\phi_{22}^-\|_{L^p}$, which we call $(\partial P'')^-$.

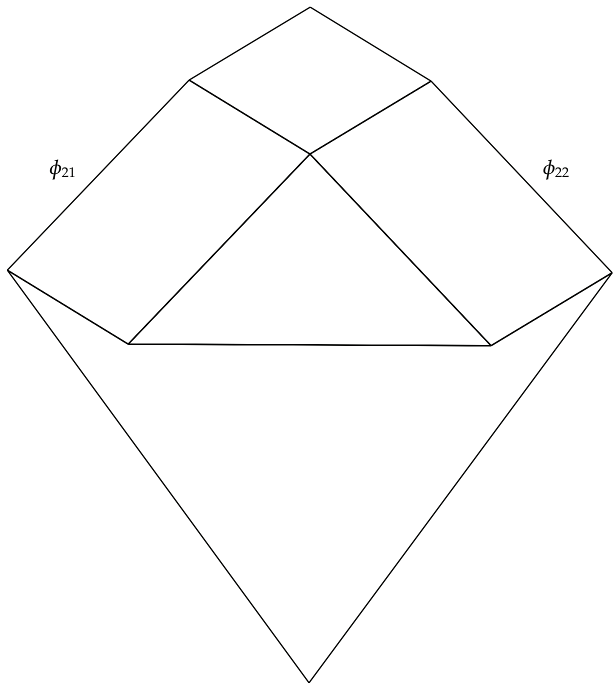


Figure 8: P'''.

Hence we can form $P''' = P'' \cup \{(tz^p + (1-t)\phi_{11}^p)^{1/p} : z \in (\partial P'')^+,\ 0 \le t \le 1\} \cup \{(-t|z|^p - (1-t)|\phi_{12}|^p)^{1/p} : z \in (\partial P'')^-, 0 \le t \le 1\}$. See Figure 8. Once again this set is homeomorphic to D. Moreover, its boundary contains the points ϕ_{11}, ϕ_{21}, ϕ_{12}, and ϕ_{22} in that order. Hence $P''' \in \mathcal{Q}$. Since with each addition to P, we have decreased energy we see that $\max_{P'''} I(u) = \max_P I(u) = I(\phi_{31}) = \lambda_{31}$. Hence

$$\inf_{Q \in \mathcal{Q}} \max_{u \in Q} I(u) = \lambda_{31}, \tag{7.4}$$

and, we have the desired variational characterization.

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