Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 143808, 14 pages
doi:10.1155/2010/143808

Research Article

Non-Weakly Supercyclic Weighted Composition Operators

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Received 11 July 2010; Accepted 11 October 2010

Academic Editor: Jean Pierre Gossez

We give sufficient conditions under which a weighted composition operator on a Hilbert space of analytic functions is not weakly supercyclic. Also, we give some necessary and sufficient conditions for hypercyclicity and supercyclicity of weighted composition operators on the space of analytic functions on the open unit disc.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane. Let $\mathscr{H}$ be a Hilbert space of analytic functions defined on $\mathbb{D}$ such that $1, z \in \mathscr{H}$, and for each λ in $\mathbb{D}$, the linear functional of point evaluation at λ given by $f \mapsto f(\lambda)$ is bounded. By a Hilbert space of analytic functions $\mathscr{H}$ we mean the one satisfying the above conditions.

For $\lambda \in \mathbb{D}$, let e_λ denote the linear functional of point evaluation at λ on $\mathscr{H}$; that is, $e_\lambda(f) = f(\lambda)$ for every f in $\mathscr{H}$. Since e_λ is a bounded linear functional, the Riesz representation theorem states that

$$e_\lambda(f) = \langle f, k_\lambda \rangle \tag{1.1}$$

for some $k_\lambda \in \mathscr{H}$.

A well-known example of a Hilbert space of analytic functions is the weighted Hardy space. Let $(\beta(n))_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ is defined as the space of functions $f = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ analytic on $\mathbb{D}$ such that

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^2 [\beta(n)]^2 < \infty. \tag{1.2}$$

Since the function $\sum_{n=0}^{\infty}(n\beta(n))^{-1}z^n$ is in $H^2(\beta)$, we see that $\limsup(\beta(n)^{-1/n}) \leq 1$. These spaces are Hilbert spaces with the inner product

$$\langle f, g\rangle = \sum_{n=0}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)}\,[\beta(n)]^2, \tag{1.3}$$

for every f and g in $H^2(\beta)$.

Let $f_k(z) = z^k$ for every nonnegative integer k. Then $\{f_k\}_k$ is an orthogonal basis. As a particular consequence of this fact, the polynomials are dense in $H^2(\beta)$. It is clear that $\|f_k\| = \beta(k)$. We call the set $\{z^k/\beta(k),\ k = 0, 1, 2, \dots\}$ the standard basis for $H^2(\beta)$. The sequence of weights allows us to consider the generating function

$$k(z) = \sum_{n=0}^{\infty}\frac{z^n}{\beta(n)^2} \tag{1.4}$$

which is analytic on $\mathbb{D}$. Take $\lambda \in \mathbb{D}$. It is easy to see that for any function $f = \sum_{n=0}^{\infty}\widehat{f}(n)z^n$ in $H^2(\beta)$,

$$f(\lambda) = \langle f, k_\lambda\rangle, \tag{1.5}$$

where $k_\lambda(z) = k(\overline{\lambda}z)$; moreover,

$$\|k_\lambda\|^2 = \sum_{j=0}^{\infty}\frac{\left(|\lambda|^2\right)^j}{\beta(j)^2}. \tag{1.6}$$

The classical Hardy space, the Bergman space, and the Dirichlet space are weighted Hardy spaces with weights, respectively, given by $\beta(j) = 1$, $\beta(j) = (j+1)^{-1/2}$, and $\beta(j) = (j+1)^{1/2}$. Weighted Bergman and Dirichlet spaces are also weighted Hardy spaces.

Let φ be an automorphism of the disc. Recall that φ is elliptic if it has one fixed point in the disc and the other in the complement of the closed disc, hyperbolic if both of its fixed points are on the unit circle, and parabolic if it has one fixed point on the unit circle (of multiplicity two).

Recall that a multiplier of $\mathscr{H}$ is an analytic function w on $\mathbb{D}$ such that $w\mathscr{H} \subseteq \mathscr{H}$. The set of all multipliers of $\mathscr{H}$ is denoted by $M(\mathscr{H})$. If w is a multiplier, then the multiplication operator M_w, defined by $M_w f = wf$, is bounded on $\mathscr{H}$. Also, note that for each $\lambda \in \mathbb{D}$, $M_w^* k_\lambda = \overline{w(\lambda)}k_\lambda$. It is known that $M(\mathscr{H}) \subseteq H^\infty$. In fact, since the constant functions are in $\mathscr{H}$, every function in $M(\mathscr{H})$ is analytic on $\mathbb{D}$. Moreover, if $\lambda \in \mathbb{D}$, then

$$|w(\lambda)k_\lambda(\lambda)| = |\langle M_w k_\lambda, k_\lambda\rangle| \leq \|M_w\|\|k_\lambda\|^2, \tag{1.7}$$

which implies that $|w(\lambda)| \leq \|M_w\|$ for every $\lambda \in \mathbb{D}$, and so $w \in H^\infty$.

In what follows, suppose that $w \in M(\mathscr{H})$ and that φ is an analytic self-map of $\mathbb{D}$ such that $(f \circ \varphi)(z) = f(\varphi(z))$ is in $\mathscr{H}$ for every $f \in \mathscr{H}$. An application of the closed graph theorem shows that the weighted composition operator $C_{w,\varphi}$ defined by $C_{w,\varphi}(f)(z) = M_w C_\varphi(f)(z) = w(z)f(\varphi(z))$ is bounded. The mapping φ is called the composition map and w is called the weight. For a positive integer n, the nth iterate of φ is denoted by φ_n; also φ_0 is the identity function. We note that

$$C^n_{w,\varphi}(f) = \prod_{j=0}^{n-1} wo\varphi_j \cdot (fo\varphi_n) \tag{1.8}$$

for all $f \in \mathscr{H}$ and $n \geq 1$. Moreover,

$$C^{*n}_{w,\varphi} k_\lambda = \prod_{j=0}^{n-1} \overline{w(\varphi_j(\lambda))} k_{\varphi_n(\lambda)} \tag{1.9}$$

for every $\lambda \in \mathbb{D}$ and $n \geq 1$. The weighted composition operators come up naturally. In 1964, Forelli [1] showed that every isometry on H^p for $1 < p < \infty$ and $p \neq 2$ is a weighted composition operator. Recently, there has been a great interest in studying composition and weighted composition operators on the unit disc, polydisc, or the unit ball; see, for example, monographs [2, 3], papers [4–18].

Suppose that $x \in \mathscr{H}$ and T is an operator on $\mathscr{H}$. The set $\{x, Tx, T^2x, \ldots\}$, denoted by $\mathrm{Orb}\{T, x\}$, is called the orbit of x under T. If there is a vector $x \in \mathscr{H}$ whose orbit is (weakly) dense in $\mathscr{H}$, then T is called a (weakly) hypercyclic operator and x is called a (weakly) hypercyclic vector for T. The operator T is called (weakly) supercyclic if the set of scalar multiples of the elements of $\mathrm{Orb}\{T, x\}$ is (weakly) dense and is called cyclic if the linear span of $\mathrm{Orb}\{T, x\}$ is dense. In each case, the vector x is called, respectively, (weakly) supercyclic vector or cyclic vector for T.

Hypercyclicity of operators is studied a lot in literature. The classical hypercyclic operator is $2B$ on the space $\ell^2(\mathbb{N})$, where B is the backward shift [19]. It is proved that many famous operators are hypercyclic. For instance, certain operators in the classes of composition operators [20], the adjoint of subnormal, hyponormal, and multiplication operators [21, 22] and weighted shift operators [23] are hypercyclic. As a good source on hypercyclicity, supercyclicity, weak hypercyclicity, and weak supercyclicity of operators, one can see [24].

Recently the hypercyclicity of the adjoint of a weighted composition operator on a Hilbert space of analytic functions is discussed in [25]. Chan and Sanders [26] showed that a weakly hypercyclic operator may fail to be norm hypercyclic. Furthermore, Sanders in [27] proved that there exists a weakly supercyclic operator that fails to be norm supercyclic. In this paper, we discuss weak supercyclicity and weak hypercyclicity of a weighted composition operator.

2. Weighted Composition Operators on $\mathscr{H}$

The following proposition limits the kinds of maps that can produce weakly supercyclic weighted composition operators on $\mathscr{H}$.

Proposition 2.1. *Suppose that $C_{w,\varphi}$ is weakly supercyclic. Then*

(i) $w(\lambda) \neq 0$ *for all* $\lambda \in \mathbb{D}$;

(ii) φ *is univalent.*

Proof. Suppose that f is a weakly supercyclic vector for $C_{w,\varphi}$.

(i) Let $\lambda \in \mathbb{D}$, and let ε be a positive number. Since the set

$$V = \{h \in \mathscr{H} : |\langle h - k_\lambda, k_\lambda \rangle| < \varepsilon\} \tag{2.1}$$

is a weak neighborhood of k_λ, there exist $n \geq 1$ and a scalar α such that $\alpha C^n_{w,\varphi} f \in V$; that is,

$$\left|\left\langle \alpha C^n_{w,\varphi} f, k_\lambda \right\rangle - \|k_\lambda\|^2\right| = \left|\alpha \prod_{j=0}^{n-1} w o \varphi_j(\lambda) f o \varphi_n(\lambda) - \|k_\lambda\|^2\right| < \varepsilon. \tag{2.2}$$

Now, if $w(\lambda) = 0$, then $\|k_\lambda\| < \sqrt{\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, we get a contradiction.

(ii) Suppose that $\varphi(a) = \varphi(b)$ and put

$$g = \left(\frac{k_a}{\overline{w(a)}}\right) - \left(\frac{k_b}{\overline{w(b)}}\right). \tag{2.3}$$

For any $\varepsilon > 0$, the set

$$V = \{h \in \mathscr{H} : |\langle h - g, g \rangle| < \varepsilon\} \tag{2.4}$$

is a weak neighborhood of g, and so there exist $n \geq 1$ and a scalar α such that $\alpha C^n_{w,\varphi} f \in V$; that is,

$$\left|\left\langle \alpha C^n_{w,\varphi} f - g, g \right\rangle\right| = \left|\left\langle \alpha f, C^{*n}_{w,\varphi} g \right\rangle - \|g\|^2\right| = \|g\|^2 < \varepsilon. \tag{2.5}$$

Thus, $g = 0$, and so $\langle 1, k_a/\overline{w(a)}\rangle = \langle 1, k_b/\overline{w(b)}\rangle$; this implies that $w(a) = w(b)$. Consequently, $\langle z, k_a\rangle = \langle z, k_b\rangle$, which, in turn, implies that $a = b$. □

Hereafter, we may assume that φ is univalent and $w(z) \neq 0$ for all $z \in \mathbb{D}$. Take $a \in \mathbb{D}$, and consider an automorphism of $\mathbb{D}$ defined by $\varphi_a(z) = (a - z)/(1 - \overline{a}z), (z \in \mathbb{D})$. A Hilbert space $\mathscr{H}$ of analytic functions is called *automorphism invariant* if for every $a \in \mathbb{D}$, $f \circ \varphi_a \in \mathscr{H}$ whenever $f \in \mathscr{H}$ and $w \circ \varphi_a \in M(\mathscr{H})$ whenever $w \in M(\mathscr{H})$. Spaces such as the Hardy and the Bergman spaces are automorphism invariant.

Theorem 2.2. *Suppose that $\mathscr{H}$ is automorphism invariant. If $C_{w,\varphi}$ is weakly supercyclic and $\lambda \in \mathbb{D}$ is a fixed point of φ, then $\left(\prod_{j=0}^{n-1} w o \varphi_j(a)/(w(\lambda))^n\right)_n$ is an unbounded sequence for every $a \in \mathbb{D} - \{\lambda\}$.*

Proof. First, suppose that $\lambda = 0$. Let f be a weakly supercyclic vector for $C_{w,\varphi}$. Since the constant function 1 is in $\mathcal{H}$, the point evaluation e_λ is surjective. Therefore, the continuity of e_λ implies that the set

$$\left\{\left(\alpha C^n_{w,\varphi} f\right)(0) : n \geq 0,\ \alpha \in \mathbb{C}\right\} = \{\alpha w(0)^n f(0) : n \geq 0,\ \alpha \in \mathbb{C}\} \tag{2.6}$$

is dense in $\mathbb{C}$; hence, $f(0)$ should be nonzero.

Let $g(z) = z$, and $\varepsilon > 0$. Put

$$\begin{aligned} V &= \{h : |\langle h - g, k_0\rangle| < \varepsilon\}, \\ W &= \{h : |\langle h - g, k_a\rangle| < \varepsilon\}. \end{aligned} \tag{2.7}$$

Since $V \bigcap W$ is a weak neighborhood of g, there exist $n \geq 1$ and a scalar α such that $\alpha C^n_{w,\varphi} f \in V \bigcap W$. Therefore,

$$\begin{aligned} \left|\alpha C^n_{w,\varphi} f(0) - g(0)\right| &= |\alpha w(0)^n f(0) - g(0)| < \varepsilon, \\ \left|\alpha C^n_{w,\varphi} f(a) - g(a)\right| &= \left|\alpha \prod_{j=0}^{n-1} w(\varphi_j(a))(f o \varphi_n)(a) - g(a)\right| < \varepsilon. \end{aligned} \tag{2.8}$$

On the other hand, since $\varphi(0) = 0$, the Schwarz lemma yields that $|\varphi_k(a)| < |a|$ for every positive integer k; so there is a constant c_1 such that $|f(\varphi_n(a))| \leq c_1$ for all n. Put $a_n = \prod_{j=0}^{n-1} w o \varphi_j(a)/(w(\lambda))^n$ and assume, on the contrary, that there is a constant c_2 such that $|a_n| \leq c_2$ for all n. Thus, (2.8) imply that

$$|g(a)| \leq \varepsilon + |\ \alpha w(0)^n \| a_n \| f o \varphi_n)(a)| \leq \varepsilon + \frac{\varepsilon}{|f(0)|} c_1 c_2 \tag{2.9}$$

for every $\varepsilon > 0$, which contradicts the fact that $g(a) \neq 0$.

Now, suppose that $\lambda \neq 0$, and put

$$\psi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \qquad \phi(z) = \psi_\lambda o \varphi o \psi_\lambda, \qquad W(z) = w o \psi_\lambda. \tag{2.10}$$

The operators $C_{w,\varphi}$ and $C_{W,\phi}$ are similar; indeed, $C_{w,\varphi} = C_{\psi_\lambda} C_{W,\phi} C^{-1}_{\psi_\lambda}$. Thus, $C_{W,\phi}$ is weakly supercyclic. If $b = \psi_\lambda(a)$, then

$$W(\phi_n(b)) = w o \psi_\lambda o (\psi_\lambda o \varphi_n o \psi_\lambda)(b) = w o \varphi_n(a) \tag{2.11}$$

for every positive integer n. Moreover, $b \neq 0$ and $\phi(0) = 0$; thus, by the previous step the sequence $(W(\phi_n(b))/W(0)^n)_n$ is unbounded, and so is the sequence $(a_n)_n$. □

The following corollaries are direct consequences of Theorem 2.2.

Corollary 2.3. *Let $\mathcal{H}$ be automorphism invariant and φ have a fixed point λ in $\mathbb{D}$. If there is a point $a \in \mathbb{D} - \{\lambda\}$ and a positive integer N such that $|w(\varphi_n(a))| \leq |w(\lambda)|$ for all $n \geq N$, then $C_{w,\varphi}$ is not weakly supercyclic on $\mathcal{H}$.*

Example 2.4. Suppose that $\varphi(z) = az$, where $0 < a < 1$ and $w(z) = 1 - z$. Then $|w(\varphi_n(a))| = |1 - a^{n+1}| \leq |w(0)| = 1$ for all $n \geq 1$. Therefore, $C_{w,\varphi}$ is not weakly supercyclic on $\mathcal{H}$.

Corollary 2.5. *Suppose that φ has a fixed point $\lambda \in \mathbb{D}$. If $|w(\lambda)| \geq 1$ and $\left(\prod_{j=0}^{n-1} w o \varphi_j(a)\right)_n$ is a bounded sequence, for some $a \in \mathbb{D} - \{\lambda\}$; then $C_{w,\varphi}$ is not weakly supercyclic on $\mathcal{H}$.*

Example 2.6. For $\alpha \neq 0$ in $\mathbb{D}$, define $\varphi(z) = (\alpha - z)/(1 - \overline{\alpha} z)$ and $w(z) = \sqrt{1 - |\alpha|^2}/(1 - \overline{\alpha} z)$. Since $w \in H^\infty$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $C_{w,\varphi}$ is bounded on the Hardy space $H^2(\mathbb{D})$ and the Bergman space $L_a^2(\mathbb{D})$. An easy computation shows that φ is an elliptic automorphism with a fixed point $\lambda = (1 - \sqrt{1 - |\alpha|^2})/\overline{\alpha}$ and $w(\lambda) = 1$. Also, for the even values of j, $\varphi_j(z) = z$ and so $w(\varphi_j(z)) = w(z)$; moreover, for the odd values of j, $\varphi_j(z) = \varphi(z)$ and $w(\varphi(z)) = 1/w(z)$. Thus, $w(\varphi_j(a)) w(\varphi_{j+1}(a)) = 1$ for all $j \geq 0$ and $a \in \mathbb{D}$. Therefore, $\left(\prod_{j=0}^{n-1} w o \varphi_j(a)\right)_n$ is a bounded sequence and so $C_{w,\varphi}$ is not weakly supercyclic.

By taking $w(z) \equiv 1$ in Corollary 2.3, the following result will be obtained.

Corollary 2.7. *If $\mathcal{H}$ is automorphism invariant and φ has a fixed point in $\mathbb{D}$, then C_φ is not weakly supercyclic.*

Proposition 2.8. *Suppose that polynomials are dense in $\mathcal{H}$. If $C_{w,\varphi}$ is weakly supercyclic, then w is a cyclic vector for the operator M_z.*

Proof. Since $C_{w,\varphi}$ is weakly supercyclic, its range is weakly dense. The convexity of the range of $C_{w,\varphi}$ implies that it is norm dense, and so is the range of M_w. Suppose that $g \in \mathcal{H}$ and $\epsilon > 0$. Then there is a vector f in $\mathcal{H}$ such that $\|wf - g\| < \epsilon/2$. Moreover, there is a polynomial p such that $\|p - f\| \leq \epsilon/2\|M_w\|$. Thus,

$$\|wp - g\| \leq \|M_w(p - f)\| + \|wf - g\| \leq \|M_w\| \|p - f\| + \frac{\epsilon}{2} < \epsilon. \tag{2.12}$$

This shows that w is a cyclic vector for M_z. □

Corollary 2.9. *If $C_{w,\varphi}$ is weakly supercyclic on the Hardy space $H^2(\mathbb{D})$, then w is an outer function.*

Proof. By Beurling's theorem, the only cyclic vectors for M_z on $H^2(\mathbb{D})$ are the outer functions. □

A necessary condition for the supercyclicity of an operator, called the angle criterion, was introduced by A. Montes-Rodríguez and H. N. Salas. They used this criterion to construct the non-supercyclic vectors for certain weighted shifts. We need the Hilbert space setting version of this criterion which is presented here, for the sake of completeness.

Theorem 2.10 (the angle criterion [24]). *Let K be a Hilbert space, and suppose that T is bounded operator on K. If a vector $x \in K$ is supercyclic for T, then*

$$\limsup_{n\to\infty} \frac{|\langle y, T^n x\rangle|}{\|T^n x\| \|y\|} = 1 \tag{2.13}$$

for every nonzero vector $y \in K$.

We say that a sequence $(a_j)_j$ of complex numbers is not a Blaschke sequence if there exists j_0 such that $a_j \in \mathbb{D}$ for all $j \geq j_0$, and $\sum_{j=1}^{\infty}(1 - |a_j|)$ diverges.

Theorem 2.11. *Suppose that φ is not an elliptic automorphism and has a fixed point $\lambda \in \mathbb{D}$. If there exist $a \in \mathbb{D} - \{\lambda\}$ and a positive integer N so that $|w(\lambda)| < |wo\varphi_j(a)|$ for all $j \geq N$, then $C_{w,\varphi}$ is not supercyclic.*

Proof. If φ is the identity map, then $C_{w,\varphi} = M_w$. The point spectrum of M_w^* contains more than one point; in fact, $M_w^* k_z = \overline{w(z)} k_z$ for all $z \in \mathbb{D}$. Thus, M_w is not supercyclic [24, page 13]. So, we may assume that φ is not the identity map.

Assume that $C_{w,\varphi}$ is a supercyclic operator on $\mathscr{H}$ and $U = \{f \in \mathscr{H} : f(\lambda) \neq 0\}$. Since $U^c = \ker e_\lambda$, the set U is open in $\mathscr{H}$. We know that the set of supercyclic vectors for $C_{w,\varphi}$ is dense in $\mathscr{H}$ [24, page 9]. So, there exists a supercyclic vector f for $C_{w,\varphi}$ in U. If $(w(\lambda)/w(\varphi_j(a)))_j$ is not a Blaschke sequence, then $\prod_{j=0}^{\infty}|w(\lambda)/w(\varphi_j(a))| = 0$. On the other hand, we have

$$\begin{gathered}
\left|\prod_{j=0}^{n-1} wo\varphi_j(a)\right| |fo\varphi_n(a)| = \left|e_a\left(C_{w,\varphi}^n f\right)\right| \leq \|e_a\| \left\|C_{w,\varphi}^n f\right\|, \\
\left|\left(C_{w,\varphi}^n f\right)(\lambda)\right| = \left|\prod_{j=0}^{n-1} wo\varphi_j(\lambda)\right| |f(\varphi_n(\lambda))| = |w(\lambda)|^n |f(\lambda)|.
\end{gathered} \tag{2.14}$$

Therefore,

$$\frac{\left|\left(C_{w,\varphi}^n f\right)(\lambda)\right|}{\|C_{w,\varphi}^n f\|} \leq \frac{|w(\lambda)|^n |f(\lambda)| \|e_a\|}{\left|\prod_{j=0}^{n-1} wo\varphi_j(a)\right| |fo\varphi_n(a)|} = \prod_{j=0}^{n-1}\left|\frac{w(\lambda)}{wo\varphi_j(a)}\right| \frac{|f(\lambda)| \|e_a\|}{|fo\varphi_n(a)|}. \tag{2.15}$$

However since $\varphi_n(a) \to \lambda$ and $f(\lambda) \neq 0$, the left side of the above inequality tends to zero, but by the angle criterion,

$$\limsup_{n\to\infty} \frac{\left|\left(C_{w,\varphi}^n f\right)(\lambda)\right|}{\left\|C_{w,\varphi}^n f\right\| \|k_\lambda\|} = 1 \tag{2.16}$$

which is a contradiction.

Now, suppose that $(w(\lambda)/w(\varphi_j(a)))_j$ is a Blaschke sequence. If $g(z) = z - \lambda$, then there exists a sequence $(\alpha_k C_{w,\varphi}^{n_k})_k$ such that $\alpha_k C_{w,\varphi}^{n_k} f \to g$ as $k \to \infty$. So

$$\alpha_k \prod_{j=0}^{n_k-1} wo\varphi_j(\lambda) fo\varphi_{n_k}(\lambda) = \alpha_k (w(\lambda))^{n_k} f(\lambda) \longrightarrow g(\lambda) = 0,$$
$$\alpha_k \prod_{j=0}^{n_k-1} wo\varphi_j(a) fo\varphi_{n_k}(a) \longrightarrow g(a) \neq 0 \tag{2.17}$$

as $k \to \infty$. Thus,

$$\prod_{j=0}^{n_k-1} \frac{w(\lambda)}{wo\varphi_j(a)} \frac{f(\lambda)}{f(\varphi_{n_k}(a))} \longrightarrow 0, \tag{2.18}$$

which is a contradiction. □

The following theorem is a kind of the angle criterion for the weak supercyclicity which is called the weak angle criterion.

Theorem 2.12 (the weak angle criterion [24]). *Let K be a Hilbert space, and let T be a bounded linear operator on K. Also, let $x \in K$. Assume that one can find some nonzero $y \in K$ such that*

$$\sum_{n=0}^{\infty} \frac{|\langle y, T^n x \rangle|}{\|T^n x\|} < \infty. \tag{2.19}$$

Then, x is not a weakly supercyclic vector for T.

An operator T is bounded below if there exists $c > 0$ such that $\|Tf\| \geq c\|f\|$ for all f. It is well known that T is bounded below if and only if it has a closed range and $\ker(T) = (0)$. It can be easily seen that $\ker(C_{w,\varphi}) = (0)$; so a weighted composition operator is bounded below if and only if its range is closed.

Proposition 2.13. *Suppose that φ has a fixed point $\lambda \in \mathbb{D}$. If*

$$c = \inf\{\|C_{w,\varphi} f\| : \|f\| = 1\} > |w(\lambda)|, \tag{2.20}$$

then $C_{w,\varphi}$ is not weakly supercyclic.

Proof. If $f \in \mathscr{H}$, then $\|C_{w,\varphi}^n f\| \geq c^n \|f\|$ and

$$\frac{\left|\left\langle C_{w,\varphi}^n f, k_\lambda \right\rangle\right|}{\|C_{w,\varphi}^n f\|} \leq \frac{|w(\lambda)|^n |f(\lambda)|}{c^n \|f\|} = \left|\frac{w(\lambda)}{c}\right|^n \frac{|f(\lambda)|}{\|f\|}. \tag{2.21}$$

Consequently, $\sum_{n=1}^{\infty} (|\langle C_{w,\varphi}^n f, k_\lambda \rangle| / \|C_{w,\varphi}^n f\|) < \infty$, and theorem follows by the weak angle criterion. □

Corollary 2.14. *Let φ be an inner function of the disc with a fixed point $\lambda \in \mathbb{D}$. Suppose that w is nonconstant and there exists $m > 0$ such that $|w| \geq m$ almost everywhere on $\partial\mathbb{D}$. If $|w(\lambda)|\sqrt{1+|\varphi(0)|} < m\sqrt{1-|\varphi(0)|}$, then $C_{w,\varphi}$ is not weakly supercyclic on the Hardy space $H^2(\mathbb{D})$.*

Proof. Since φ is an inner function, $\|C_\varphi f\| \geq ((1-|\varphi(0)|)/(1+|\varphi(0)|))^{1/2}\|f\|$ for all $f \in H^2$ [2, page 123]. So

$$\|C_{w,\varphi} f\| \geq m\|f o\varphi\| \geq m\sqrt{\frac{1-|\varphi(0)|}{1+|\varphi(0)|}}\|f\|, \tag{2.22}$$

and the result follows by applying Proposition 2.13. □

In the last part of this section, we give some necessary conditions for the weak hypercyclicity of a weighted composition operator.

Proposition 2.15. *If one of the following conditions holds, then $C_{w,\varphi}$ is not weakly hypercyclic.*

(i) $\|C_\varphi\| \leq 1$, $\|w\| \leq 1$ *and* $\|k_0\| \leq 1$.

(ii) *If a is the Denjoy-Wolff point of φ, then w is continuous at a with $\|w(a)C_\varphi\| < 1$.*

(iii) *$\mathscr{H} \subseteq H^\infty$; moreover, if a is the Denjoy-Wolff point of φ, then w is continuous at a and $|w(a)| < 1$.*

(iv) *The function φ has a fixed point in $\mathbb{D}$.*

Proof. Assume, on the contrary, that $C_{w,\varphi}$ has a weakly hypercyclic vector f. If $\lambda \in \mathbb{D}$ and $n \geq 0$, then

$$\left\langle C^n_{w,\varphi} f, k_\lambda \right\rangle = \left\langle f, C^{*n}_{w,\varphi} k_\lambda \right\rangle = \prod_{j=0}^{n-1} w(\varphi_j(\lambda)) f(\varphi_n(\lambda)). \tag{2.23}$$

Suppose that (i) holds. Put $\lambda = 0$ in (2.23). Since $\|k_\lambda\| \leq 1$, we have

$$\begin{aligned}\left|C^n_{w,\varphi} f(\lambda)\right| &= \left|\left\langle C^n_{w,\varphi} f, k_\lambda \right\rangle\right| = \left| |\langle C_{\varphi_n} f, k_\lambda\rangle| \prod_{j=0}^{n-1} \left\langle C_{\varphi_j} w, k_\lambda \right\rangle \right| \\ &\leq \left\|C^n_\varphi\right\| \|f\| \prod_{j=0}^{n-1} \|C_\varphi\|^j \|w\| \leq \|f\|.\end{aligned} \tag{2.24}$$

However, the mapping e_λ is weakly continuous [28, page 167], and so the sequence $((C^n_{w,\varphi} f)(\lambda))_n$ is dense in $\mathbb{C}$, which is impossible.

If (ii) holds and $\lambda \in \mathbb{D}$, then by the Denjoy-Wolff theorem $\varphi_n(\lambda) \to a$ as $n \to \infty$. Since $\|w(a)C_\varphi\| < 1$, there is j_0 such that $|w(\varphi_j(\lambda))|\|C_\varphi\| < 1$ for all $j > j_0$. Hence, by (2.23), we have

$$\begin{aligned} \left|C_{w,\varphi}^n f(\lambda)\right| &= \left|\prod_{j=0}^{n-1} w(\varphi_j(\lambda))\right| \left|\langle C_{\varphi_n} f, k_\lambda\rangle\right| \\ &\le \|f\|\|k_\lambda\| \left|\prod_{j=0}^{n-1} w(\varphi_j(\lambda))\right| \|C_\varphi\| \\ &\le \|f\|\|k_\lambda\| \left|\prod_{j=0}^{j_0} w(\varphi_j(\lambda))\right| \|C_\varphi\|. \end{aligned} \tag{2.25}$$

Thus, the sequence $((C_{w,\varphi}^n f)(\lambda))_n$ is bounded; now, similar to part (i), we get a contradiction.

Suppose that (iii) holds. If $\lambda \in \mathbb{D}$, then $\varphi_n(\lambda) \to a$ and so $w(\varphi_n(\lambda)) \to w(a)$ as $n \to \infty$. Since $|w(a)| < 1$, $\sum_{j=0}^{\infty}(1 - |w(\varphi_j(\lambda))|\ ) = \infty$; equivalently, $\lim_{n\to\infty}\prod_{j=0}^{n-1} w(\varphi_j(\lambda)) = 0$. Now, by (2.23), the sequence $((C_{w,\varphi}^n f)(\lambda))_n$ is bounded and similar to part (i); again we get a contradiction.

At last, suppose that (iv) holds. Let $p \in \mathbb{D}$ be a fixed point of φ. Since $C_{w,\varphi}^* k_p = \overline{w(p)} k_p$, the point spectrum of $C_{w,\varphi}$ is nonempty, which is a contradiction [24, page 11]. □

We remark that if $\mathscr{H} = H^2(\beta)$, then $\|k_0\| = 1$; moreover, if $\liminf_{j\to\infty} \beta(j)^{2/j} > 1$, then $\mathscr{H} \subseteq H^\infty$. Indeed, for every $f \in \mathscr{H}$ and $\lambda \in \mathbb{D}$, we have

$$\left|f(\lambda)\right|^2 = \left|\langle f, k_\lambda\rangle\right|^2 \le \|f\|^2\|k_\lambda\|^2 \le \|f\|^2 \sum_{j=0}^{\infty} \frac{1}{\beta(j)^2} < \infty. \tag{2.26}$$

3. Weighted Composition Operators on $H(\mathbb{D})$

In this section, we give some necessary and sufficient conditions for hypercyclicity and supercyclicity of weighted composition operators on the space of analytic functions $H(\mathbb{D})$. If K is a compact subset of $\mathbb{D}$, for $f \in H(\mathbb{D})$, define $P_K(f) = \sup_{z\in K}|f(z)|$. Then $\{P_K : K \subseteq \mathbb{D}, K \text{ is compact}\}$ is a family of seminorms that makes $H(\mathbb{D})$ into a locally convex space. In fact, this topology is the topology of uniform convergence on compact subsets of the open unit disc, the usual topology for $H(\mathbb{D})$, which is an F-space.

Definition 3.1. A sequence of self-maps $(f_n)_n$ of $\mathbb{D}$ is said to be a run-away sequence if for every compact set $K \subseteq \mathbb{D}$, there exists a positive integer n such that $f_n(K) \cap K = \emptyset$.

For example, if φ is an automorphism, then $(\varphi_n)_n$ is a run-away sequence if and only if φ has no fixed point in $\mathbb{D}$; see [10].

To prove the next theorem, we apply Birkhoff's transitivity theorem.

Theorem 3.2 (Birkhoff's transitivity Theorem [24]). *Let X be a separable F-space, and let T be a bounded linear operator in X. The following statements are equivalent.*

(i) *T is hypercyclic;*

(ii) *T is topologically transitive; that is, for each pair of nonempty open subsets (U, V) of X, there exists a positive integer n such that $T^n(U) \cap V \neq \emptyset$.*

Theorem 3.3. *If $(\varphi_n)_n$ is a run-away sequence and w is in H^∞ such that $w(z) \neq 0$ for all $z \in \mathbb{D}$, then $C_{w,\varphi}$ is a hypercyclic operator on $H(\mathbb{D})$.*

Proof. Let U and V be two nonempty open subsets of $H(\mathbb{D})$. There exist $\epsilon > 0$, two compact subsets, K_1 and K_2, of $\mathbb{D}$ and two functions $f, g \in H(\mathbb{D})$ such that

$$\begin{aligned} &\left\{h \in H(\mathbb{D}) : \sup_{z\in K_1}\left|h(z) - f(z)\right| < \epsilon\right\} \subseteq U, \\ &\left\{h \in H(\mathbb{D}) : \sup_{z\in K_2}\left|h(z) - g(z)\right| < \epsilon\right\} \subseteq V. \end{aligned} \tag{3.1}$$

Put $K = K_1 \cup K_2$. Therefore,

$$\begin{aligned} &\{h \in H(\mathbb{D}) : \sup_{z\in K}\left|h(z) - f(z)\right| < \epsilon\} \subseteq U, \\ &\{h \in H(\mathbb{D}) : \sup_{z\in K}\left|h(z) - g(z)\right| < \epsilon\} \subseteq V. \end{aligned} \tag{3.2}$$

Since K is compact, we can choose two closed discs, B_1, B_2, in $\mathbb{D}$ such that $K \subseteq B_1 \subseteq B_2^o$. Since $(\varphi_n)_n$ is a run-away sequence, there exists a positive integer n such that $\varphi_n(B_2) \cap B_2 = \emptyset$. Now, consider the map R, defined by $f(z)$, if $z \in B_1$, and by $\prod_{j=1}^{n}(1/wo\varphi_{-j}(z))go\varphi_{-n}(z)$ if $z \in \varphi_n(B_1)$. The sets B_1 *and* $\varphi_n(B_1)$ are two disjoint closed subsets of $\mathbb{D}$; moreover, since by convention, φ and so φ_n are injective, the complement of $B_1 \cup \varphi_n(B_1)$ is connected. Then by Runge's theorem, there exists a polynomial p such that

$$\left|R(z) - p(z)\right| < \min\left\{\epsilon, \frac{\epsilon}{\|w\|_\infty^n}\right\} \tag{3.3}$$

for all $z \in B_1 \cup \varphi_n(B_1)$. So

$$\left|f(z) - p(z)\right| < \epsilon \quad \forall z \in B_1, \tag{3.4}$$

which implies that $p \in U$. Also,

$$\left|\prod_{j=1}^{n}\frac{1}{wo\varphi_{-j}}(z)go\varphi_{-n}(z) - p(z)\right| < \frac{\epsilon}{\|w\|_\infty^n}, \qquad \forall z \in \varphi_n(B_1), \tag{3.5}$$

which implies that

$$\left|C_{w,\varphi}^n p(z) - g(z)\right| < \epsilon, \quad \forall z \in B_1. \tag{3.6}$$

Therefore, $C_{w,\varphi}^n p \in V$. Then $p \in U \cap C_{w,\varphi}^{-n}(V)$ which, in turn, implies that $C_{w,\varphi}^n U \bigcap V$ is nonempty. Thus, Birkhoff's transitivity theorem implies that $C_{w,\varphi}$ is hypercyclic. □

Proposition 3.4. *Suppose that $C_{w,\varphi}$ is hypercyclic on $H(\mathbb{D})$ and $(\varphi_n)_n$ is not a run-away sequence. Then $\operatorname{ran} w \cap \partial\mathbb{D} \neq \emptyset$.*

Proof. Assume that for all $z \in \mathbb{D}, |w(z)| < 1$ or for all $z \in \mathbb{D}, |w(z)| > 1$. Let f be a hypercyclic vector for $C_{w,\varphi}$, and choose a subsequence $(n_k)_k$ so that $\prod_{j=0}^{n_k-1} wo\varphi_j \cdot (fo\varphi_{n_k}) \to 1$ as $k \to \infty$. Since $(\varphi_n)_n$ is not a run-away sequence there exists a compact subset K of $\mathbb{D}$ such that $\varphi_n(K) \cap K \neq \emptyset$ for every positive integer n; so one can find a sequence $(\xi_{n_k})_k \subseteq K$ satisfying $\varphi_{n_k}(\xi_{n_k}) \in K$. Since K is a compact set, there exists $\alpha \in K$ such that $\varphi_{n_k}(\xi_{n_k}) \to \alpha$ (passing, if necessary, to a subsequence of (n_k)), and so $f(\varphi_{n_k}(\xi_{n_k})) \to f(\alpha)$. Now, if $|w(z)| < 1$ for all $z \in \mathbb{D}$, then $\lim_{k\to\infty}(1 - |w(\varphi_{n_k}(\xi_{n_k}))|) \neq 0$ which implies that $\sum_{j=0}^{\infty}(1 - |w(\varphi_j(\xi_{n_k}))|) = \infty$ or, equivalently, $\prod_{j=0}^{n_k-1} wo\varphi_j(\xi_{n_k}) \to 0$ as $k \to \infty$. In the case that $|w(z)| > 1$ for all $z \in \mathbb{D}$, using a similar argument, we obtain $\prod_{j=0}^{n_k-1}((wo\varphi_j)(\xi_{n_k}))^{-1} \to 0$ as $k \to \infty$; so in both cases we have contradictions, due to the fact that $\prod_{j=0}^{n_k-1} wo\varphi_j \cdot (fo\varphi_{n_k}) \to 1$. □

It is worthy of attention to note that by a similar argument as in the proof of Theorem 2.2, one can show that it can also be applied on the space $H(\mathbb{D})$ with a further assumption that $w(\lambda) \neq 0$. Since the space $H(\mathbb{D})$ is automorphism invariant, we have the following result.

Corollary 3.5. *Suppose that φ is a univalent self-map of the disc. Then the following statements are equivalent:*

(a) *φ has no fixed point in $\mathbb{D}$.*

(b) *λC_φ is weakly hypercyclic on $H(\mathbb{D})$ for every complex number λ with $|\lambda| = 1$.*

(c) *C_φ is weakly supercyclic on $H(\mathbb{D})$.*

Proof. If (a) holds, then by Proposition 2.3 of [25], λC_φ is hypercyclic on $H(\mathbb{D})$ for every complex number λ with $|\lambda| = 1$. Clearly (b) implies (c) and by the preceding discussion (c) implies (a). □

Proposition 3.6. *Suppose that φ is a self-map of the disc with the Denjoy-Wollff point α in $\mathbb{D}$ satisfying $w(\alpha) \neq 0$. Then $C_{w,\varphi}$ is weakly supercyclic if and only if C_φ is weakly supercyclic on $H(\mathbb{D})$.*

Proof. By Proposition 2.6. of [25], $w(\alpha)$ is an eigenvalue for $C_{w,\varphi}$ with an eigenvector g which has no zero in $\mathbb{D}$; so the operator M_g is invertible. On the other hand,

$$C_{w,\varphi} M_g = M_g(w(\alpha) C_\varphi), \tag{3.7}$$

thus, $C_{w,\varphi}$ is similar to $w(\alpha)C_\varphi$. Now, the result follows from the comparison principle [24, Page 10]. □

Corollary 3.7. *Suppose that φ is a univalent self-map of the disc with the Denjoy-Wollff point α in $\mathbb{D}$ such that $|w(\alpha)| = 1$. Then $C_{w,\varphi}$ is weakly hypercyclic if and only if C_φ is weakly hypercyclic on $H(\mathbb{D})$.*

Proof. Corollary 3.5 shows that $w(\alpha)C_\varphi$ is weakly hypercyclic if and only if C_φ is weakly hypercyclic. Now, the proof is the same as the preceding proposition. □

Acknowledgment

This research was in part supported by a grant from Shiraz University Research Council.

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