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Research Article

Some Normal Criteria about Shared Values with Their Multiplicity Zeros

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Let $\mathcal{F}$ be a family of meromorphic functions in the domain D, all of whose zeros are multiple. Let n ($n \geq 2$) be an integer and let a, b be two nonzero finite complex numbers. If $f + a(f')^n$ and $g + a(g')^n$ share b in D for every pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.

1. Introduction and Main Results

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}(= \mathbb{C} \cup \{\infty\})$ to denote the extended complex plane, and D to denote a domain in $\mathbb{C}$. With renewed interest in normal families of analytic and meromorphic functions in plane domains, mainly because of their role in complex dynamics, it has become quite interesting to talk about normal families in their own right.

We will be concerned with the analytic maps (i.e., meromorphic functions)

$$f : (D, |\cdot|_{R^2}) \longrightarrow \left(\widehat{\mathbb{C}}, X\right) \tag{1.1}$$

from D (endowed with the Euclidean metric) to the extended complex plane $\widehat{\mathbb{C}}$ endowed with the spherically metric X given by

$$X(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}} \quad z, z' \in \widehat{\mathbb{C}}. \tag{1.2}$$

A family $\mathcal{F}$ of meromorphic functions defined in D is said to be normal, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally and uniformly in D to a meromorphic function or ∞. Clearly, $\mathcal{F}$ is normal in D if and only if it is normal at every point of D (see [1, 2]).

Let f and g be two nonconstant meromorphic functions in D, and $a \in \widehat{\mathbb{C}}$. We say that f and g share the value a in D, if $f - a$ and $g - a$ have the same zeros (ignoring multiplicities). When $a = \infty$ the zeros of $f - a$ means the poles of f (see [3]).

Influenced from Bloch's principle [4], every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to a constant, makes a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [5]), many authors proved normality criterion for families of meromorphic functions corresponding to Liouville-Picard type theorem (see [1, 2, 6]).

It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schiff [1] first proved an interesting result that a family of meromorphic functions in a domian is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, Sun [7] proved that a family of meromorphic functions in a domian is normal if in which each pair of functions share three fixed distinct valus, which is an improvement of the famous Montel's Normal Criterion [8] by the ideas of shared values. More results about normality criteria concerning shared values can be found, for instance, (see [9–12]) and so on.

In 1989, Schwick [13] proved that let $\mathcal{F}$ be a family of meromorphic functions in a domain D, if $(f^n)^{(k)} \neq 1$ for every $f \in \mathcal{F}$, where n, k are two positive integers and $n \geq k+3$, then $\mathcal{F}$ is normal in D.

Recently, by the ideas of shared values, Li and Gu [14] proved the following.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in D. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D for each pair of functions f, g in $\mathcal{F}$, where n, k are two positive integers such that $n \geq k+2$ and a is a finite nonzero complex number, then $\mathcal{F}$ is normal in D.*

In 1998, Wang and Fang [15] obtained the following result.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in D. Let k be a positive integer and b be a nonzero finite complex number. If, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+2$, and $f^{(k)}(z) \neq b$ on D, then $\mathcal{F}$ is normal in D.*

Remark 1.1. By a counter-example, Wang and Fang [15] show Theorem B is not valid if all zeros of f have multiplicity less than $k+2$.

It is natural to ask whether Theorem B can be improved by the idea of shared values. In this paper we investigate this problem and obtain the following result.

Theorem 1.2. *Let k $(k \geq 2)$ be an integer and b be a nonzero finite complex number, and let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least $k+2$. If, for every pair $f, g \in \mathcal{F}$, all zeros of $f^{(k)}(z), g^{(k)}(z)$ are multiple, $f^{(k)}(z)$ and $g^{(k)}(z)$ share b in D, then $\mathcal{F}$ is normal in D.*

Remark 1.3. Comparing with Theorem A, Theorem 1.2 releases the condition that the poles of $f(z)$ have multiplicity at least $k+2$, which improves Theorem A in some sense.

Example 1.4. Take $D = \{z : |z| < 1\}$ and

$$\mathcal{F} = \left\{ f_m = mz^{k+2} \mid m = 1, 2, \ldots \right\} \tag{1.3}$$

or

$$\mathcal{F} = \{ f_m = e^{mz} \mid m = 1, 2, \ldots \}. \tag{1.4}$$

Obviously, any f_m has zeros of multiplicity at least $k+2$ and $f_m^{(k)}(z)$ has zeros of multiplicity at least 2. For any f_m and g_m in $\mathcal{F}$, we have $f_m^{(k)}(z)$ and $g_m^{(k)}(z)$ share 0. However the families $\mathcal{F}$ is not normal at $z = 0$.

Remark 1.5. Example 1.4 shows that the condition $b \neq 0$ in Theorem 1.2 is inevitable.

In 1959, Hayman [16] proved that let f be a meromorphic function in $\mathbb{C}$, if $f' - af^n \neq b$, where n is a positive integer and a, b are two finite complex numbers such that $n \geq 5$ and $a \neq 0$, then f is a constant. On the other hand, Mues [17] showed that for $n = 3, 4$ the conclusion is not valid.

The following theorem which confirmed a Hayman's well-known conjecture about normal families in [18].

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer and a, b be two finite complex numbers such that $a \neq 0$. If $n \geq 3$ and for each function $f \in \mathcal{F}$, $f' - af^n \neq b$, then $\mathcal{F}$ is normal in D.*

In 2008, by the ideas of shared values, Zhang [12] proved the following.

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer and a, b be two finite complex numbers such that $a \neq 0$, If $n \geq 4$ and for every pair of functions f and g in $\mathcal{F}$, If $f' - af^n$ and $g' - ag^n$ share the value b, then $\mathcal{F}$ is normal in D.*

In 1994, Ye [19] considered a similar problem and obtained that if f is a transcendental meromorphic function and a is a nonzero finite complex number, then $f + a(f')^n$ assumes every finite complex value infinitely often for $n \geq 3$. Ye [19] also asked whether the conclusion remains valid for $n = 2$.

In 2008, Fang and Zalcman [20] solved this problem and obtained the following result.

Theorem E. *Let f be a transcendental meromorphic function and a be a nonzero complex number. Then $f + a(f')^n$ assumes every complex value infinitely often for each positive integer $n \geq 2$.*

Remark 1.6. By a special example, Fang and Zalcman [20] show Theorem E is not valid for $n = 1$.

On the basis of the above results, Fang and Zalcman [20] obtained the normality criterion corresponding to Theorem E.

Theorem F. *Let $\mathcal{F}$ be a family of meromorphic functions in D. Let n $(n \geq 2)$ be an integer, and let a $(a \neq 0)$, b be two finite complex numbers. If, for each $f \in \mathcal{F}$, all zeros of f are multiple and $f + a(f')^n \neq b$ in D, then $\mathcal{F}$ is normal in D.*

Likewise Theorem D, it is natural to ask whether Theorem F can be improved by the ideas of shared values. In this paper we investigate this problem and obtain the following result.

Theorem 1.7. *Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros are multiple and let n ($n \geq 2$) be an integer and a, b be two nonzero finite complex numbers. If $f + a(f')^n$ and $g + a(g')^n$ share b in D for every pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Here we will generalize above results by allowing $f(z) + a(f'(z))^n - b$ to have zeros. For the sake of convenience, we give the following notations:

$$E(\mathcal{F}, n) := \{z \in D \mid f(z) + a(f'(z))^n - b = 0, f(z) \in \mathcal{F}\};$$
$$\overline{U(z_0, r)} := \{z \in D \mid |z - z_0| \leq r\}; \tag{1.5}$$
$$E(z_0, r, \mathcal{F}, n) := \bigcup_{f \in \mathcal{F}} \{z \in U(z_0, r) \mid f(z) + a(f'(z))^n - b = 0, f(z) \in \mathcal{F}\};$$

$\sharp E$ denotes the number of the elements in the set E.

Corollary 1.8. *Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros are multiple and let n ($n \geq 2$) be an integer, and let a, b be two nonzero finite complex numbers. If, for each $f \in \mathcal{F}$, there exists a positive constant M such that $|f(z)| \geq M$ whenever $z \in E(\mathcal{F}, n)$, then $\mathcal{F}$ is normal in D.*

Corollary 1.9. *Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros are multiple and let n ($n \geq 2$) be an integer, and let a ($a \neq 0$), b be two finite complex numbers. Suppose that*

(i) *there are two distinct complex numbers $c, d \in \mathbb{C} \cup \{\infty\}$ such that $f(z) \neq c, d$ for every $f(z) \in \mathcal{F}$;*

(ii) *for arbitrary $z_0 \in D$, there is $r > 0$ such that $U(r, z_0) \subset D$ and $\sharp E(z_0, r, \mathcal{F}, n) < \infty$. Then $\mathcal{F}$ is normal in D.*

Example 1.10 (see [20]). Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m\}$ where $f_m := -(1/4)z^2 + mz + 1/2$ and m is a positive integer. Then for every pair of functions $f, g \in \mathcal{F}$, $f_m + (f'_m)^2 \equiv m^2 + 1/2$ and $g_m + (g'_m)^2 \equiv m^2 + 1/2$ share any point in D. However $\mathcal{F}$ is not normal at $z = 0$.

Remark 1.11. Example 1.10 shows the condition that all zeros of $f \in \mathcal{F}$ are multiple is necessary in Theorem 1.2.

Example 1.12. Take $D = \{z : |z| < 1\}$,

$$\mathcal{F} = \left\{f_m = mz^2 \mid m = 1, 2, \ldots\right\}. \tag{1.6}$$

Remark 1.13. For $n = 2$, we obtain $f_m + a(f'_m)^2 = mz^2 + (2m)^2 az^2$. So $f + a(f')^2$ and $g + a(g')^2$ share 0 in D for every pair functions $f, g \in \mathcal{F}$. But $\mathcal{F}$ is not normal in D.

For $n = 2$, from $mz^2 + a(2mz)^2 = b$, we deduce $(m + 4am^2)z^2 = b$. Obviously, If m sufficiently large, then $m + 4am^2 \neq 0$, $|f_m(z)| = |mz^2| = |mb/(m + 4am^2)| = |b/(1 + 4am)|$ and then $|f_m(z)| \to 0$. Example 1.12 shows that the condition $b \neq 0$ in Theorem 1.7 is inevitable for

$n = 2$. Example 1.12 also shows the hypothesis there exists a positive constant M such that $|f(z)| \geq M$ for all $f \in \mathcal{F}$ whenever $z \in E(\mathcal{F}, n)$ cannot be omitted in Corollary 1.8.

Remark 1.14. Some ideas of this paper are based on [9, 12, 21–23].

2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas.

First, we need the following well-known Pang-Zalcman lemma, which is the local version of [10, 24].

Lemma 2.1. *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ with the property that for each $f(z) \in \mathcal{F}$, all zeros of multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) \in \mathcal{F}$ and $f(z) = 0$. If $\mathcal{F}$ is not normal at a point $z_0 \in \Delta$, then for $0 \leq \alpha \leq k$, there are*

(1) *a sequence of complex numbers $z_n \in \Delta$, $z_n \to z_0$;*

(2) *a sequence of functions $f_n \in \mathcal{F}$;*

(3) *a sequence of positive numbers $\rho_n \to 0^+$;*

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ converge locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k, $g^{\sharp}(\xi) \leq g^{\sharp}(0) = kA + 1$. In particular, g has order at most 2.

In Lemma 2.1, the order of g is defined by using the Nevanlinna's characteristic function $T(r, g)$:

$$\rho(g) = \lim_{r \to \infty} \sup \frac{\log T(r, g)}{\log r}. \tag{2.1}$$

Here $g^{\sharp}(\xi)$ denotes the spherical derivative

$$g^{\sharp}(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|^2}. \tag{2.2}$$

Lemma 2.2 (see [25]). *Let $f(z)$ be a transcendental meromorphic function of finite order in $\mathbb{C}$. If $f(z)$ has no simple zero, then $f'(z)$ assumes every nonzero finite value infinitely often.*

Lemma 2.3 (see [15]). *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$. If all zeros of $f(z)$ have multiplicity at least 3, for any positive integer k, then $f^{(k)}(z)$ assumes every nonzero finite value infinitely often.*

Lemma 2.4. *Let $k \geq 2$ be an integer and b be a nonzero finite complex number and let $f(z)$ be a nonconstant rational meromorphic function in $\mathbb{C}$, all zeros of $f(z)$ have multiplicity at least $k + 2$. If all zeros of $f^{(k)}(z)$ are multiple, then $f^{(k)}(z) - b$ has at least two distinct zeros.*

Proof. In the following, we consider two cases.

Case 1. Assume, to the contrary, that $f^{(k)}(z) - b$ has at most one zero z_0.

Subcase 1.1 ($f(z)$ is a nonconstant polynomial). Set $f^{(k)}(z) - b = K(z - z_0)^l$, where K is a nonzero constant, l is a positive integer. Because all zeros of $f^{(k)}(z)$ are multiple, we obtain $l \geq 2$ and $f^{(k+1)}(z) = Kl(z - z_0)^{l-1}$ $(l - 1 \geq 1)$. Thus, $f^{(k+1)}(z)$ has exactly one zero. Since all zeros of $f(z)$ have multiplicity at least $k + 2$, we derive $f^{(k)}(z)$ has the same zero z_0. Hence $f^{(k)}(z_0) = 0$, which contradicts with $f^{(k)}(z_0) = b \neq 0$.

Subcase 1.2 ($f(z)$ is rational but not a polynomial). By the assumption of Lemma 2.4, we may set

$$f^{(k)}(z) = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}} = \frac{P(z)}{Q(z)}, \tag{2.3}$$

where A is a nonzero constant. Since all zeros of $f^{(k)}(z)$ are multiple, we find $m_i \geq 2$ $(i = 1, 2, \ldots, s)$, $n_j \geq (k+1)$ $(j = 1, 2, \ldots, t)$ $(k \geq 2)$.

For simplicity, we denote

$$\begin{aligned} m_1 + m_2 + \cdots + m_s &= M \geq 2s, \\ n_1 + n_2 + \cdots + n_t &= N \geq (k+1)t, \end{aligned} \tag{2.4}$$

where $P(z)$ and $Q(z)$ are coprime polynomials of degree M, N, respectively, in (2.3).

Since $f^{(k)}(z) - b$ has just a zero z_0, from (2.3) we obtain

$$f^{(k)}(z) = b + \frac{B(z-z_0)^l}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}} = \frac{P(z)}{Q(z)}. \tag{2.5}$$

By $b \neq 0$, we deduce $z_0 \neq \alpha_i$ $(i = 1, \ldots, s)$, where B is a nonzero constant.

From (2.3), we get

$$f^{(k+1)}(z) = \frac{(z-\alpha_1)^{m_1-1}(z-\alpha_2)^{m_2-1}\cdots(z-\alpha_s)^{m_s-1}g_1(z)}{(z-\beta_1)^{n_1+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.6}$$

where $g_1(z)$ is polynomial of degree at most $s + t - 1$.

Differentiating (2.5) yields

$$f^{(k+1)}(z) = \frac{(z-z_0)^{l-1}g_2(z)}{(z-\beta_1)^{n_1+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.7}$$

where $g_2(z) = B(l - N)z^t + b_{t-1}z^{t-1} + \cdots + b_0$, ($b_{t-1} \cdots b_0$ are constants).

Next we distinguish two subcases.

Subcase 1.2.1 ($l \neq N$). By using (2.5), we deduce $\deg P(z) \geq \deg Q(z)$, that is, $M \geq N$. Since $z_0 \neq \alpha_i$, that (2.6) and (2.7) imply $M - s \leq \deg g_2 = t$. By using (2.4), we get

$$M \leq s + t \leq \frac{M}{2} + \frac{N}{k+1} \leq \frac{M}{2} + \frac{M}{k+1} < M \quad (k \geq 2). \tag{2.8}$$

Which is a contradiction since $k \geq 2$.

Subcase 1.2.2 ($l = N$). We further distinguish two subcases:

Subcase 1.2.2.1 ($M \geq N$). By using (2.6) and (2.7), we obtain $M - s \leq \deg g_2 \leq t - 1 < t$. Similar to Subcase 1.2.1, we obtain a contradiction $M < M$.

Subcase 1.2.2.2 ($M < N$). By using (2.6) and (2.7) again, we deduce $l - 1 \leq \deg g_1 \leq s + t - 1$, and hence

$$N = l \leq s + t - 1 + 1 = s + t \leq \frac{M}{2} + \frac{N}{k+1} < N \tag{2.9}$$

this is impossible for $k \geq 2$.

Case 2 ($f^{(k)}(z) - b \neq 0$ ($k \geq 2$)). By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T\left(r, f^{(k)}\right) &\leq \overline{N}\left(r, f^{(k)}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S\left(r, f^{(k)}\right) \\ &\leq \frac{1}{k+1} N\left(r, f^{(k)}\right) + \frac{1}{2} N\left(r, \frac{1}{f^{(k)}}\right) + S\left(r, f^{(k)}\right) \\ &\leq \left(\frac{1}{2} + \frac{1}{k+1}\right) T\left(r, f^{(k)}\right) + S\left(r, f^{(k)}\right). \end{aligned} \tag{2.10}$$

It follows that $T(r, f^{(k)}) = S(r, f^{(k)})$, a contradiction.

The proof of Lemma 2.4 is complete. □

Example 2.5. Take

$$f(z) = \frac{(z+1)^4}{z^2}. \tag{2.11}$$

Remark 2.6. By a simple computation, for $k = b = 2$, we deduce $f''(z) - 2$ has only one zero at $z = -3/4$. Example 2.5 shows that the condition all of zeros of $f^{(k)}(z)$ are the multiple seems not to be omitted.

Lemma 2.7. *Let $g(z)$ be a nonconstant rational meromorphic function in $\mathbb{C}$ and let n ($n \geq 2$) be an integer. If all zeros of $g(z)$ are multiple, then $(g'(z))^n - c = 0$ has at least two distinct zeros (where c is a nonzero constant).*

Proof. Suppose that $(g'(z))^n - c \neq 0$, we set $c_1, c_2, \ldots, c_n$ the distinct solutions of $w^n = c$ $(n \geq 2)$. By Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r,g') &\leq \overline{N}(r,g') + \overline{N}\left(r, \frac{1}{g'-c_1}\right) + \cdots + \overline{N}\left(r, \frac{1}{g'-c_n}\right) + S(r,g') \\ &\leq \overline{N}(r,g') + S(r,g') \leq \frac{1}{2}N(r,g') + S(r,g') \\ &\leq \frac{1}{2}T(r,g') + S(r,g'). \end{aligned} \tag{2.12}$$

It follows that $T(r,g') = S(r,g')$, a contradiction.

Assume, to the contrary, that $(g'(z))^n - c$ has exactly one zero z_0.

If $g(z)$ is a nonconstant polynomial, then we set $(g'(z))^n - c = K(z-z_0)^l$, where K is nonzero constant and l is a positive integer. Because all zeros of $g(z)$ are multiple and $n \geq 2$, we have $l \geq 2$ and $((g'(z))^n)' = Kl(z-z_0)^{l-1}$ $(l-1 \geq 1)$. Thus, $((g'(z))^n)'$ has exactly one zero. Noting that all zeros of $(g'(z))^n$ are multiple, we deduce $(g'(z))^n$ has the same zero z_0. Hence $(g'(z_0))^n = 0$, which contradicts with $(g'(z_0))^n = c \neq 0$.

If g is a rational function but not a polynomial, by all zeros of g are multiple, then we know g' is not a constant. For $n \geq 2$, we obtain $(g'-c_1)(g'-c_2)\cdots(g'-c_n) = 0$ (where $c_1, \ldots, c_n$ are distinct roots of $w^n = c$). So there exists a z_0 such that $g'(z_0) = c_j$ and $g'(z_0) \neq c_k$ (where $1 \leq j < k \leq n$). We have

$$g'(z) = c_j + \frac{A(z-z_0)^n}{P(z)} \equiv c_k + \frac{B}{P(z)} \quad (n \geq 2, A, B \in \mathbb{C} \setminus \{0\}). \tag{2.13}$$

So we obtain

$$(c_k - c_j)P(z) + B \equiv A(z-z_0)^n, \tag{2.14}$$

for some nonconstant polynomial $P(z)$ with $P(z_0) \neq 0$. Then $c_0 P(z) + B \equiv A(z-z_0)^n$, where $c_0 = c_j - c_k$ is a nonzero constant. Furthermore,

$$c_0 P'(z) \equiv An(z-z_0)^{n-1}. \tag{2.15}$$

Observing that g' has only multiple poles, and by (2.13), we obtain $P(z)$ has multiple zeros and $P'(z_0) \neq 0$. Putting z_0 into (2.15), we get that $P'(z_0) = 0$, which is a contradiction.

The proof of Lemma 2.7 is complete. □

3. Proof of Theorems

Proof of Theorem 1.2. We may assume that $D = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there are a sequence of complex numbers z_j, $z_j \to 0$ $(j \to \infty)$; a sequence of functions $f_j \in \mathcal{F}$; and a sequence of positive numbers $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi)$ converges uniformly with respect to the spherical metric to a nonconstant mermorphic function $g(\xi)$ in $\mathbb{C}$ and all zeros of $g(\xi)$ have the multiplicity at least $k+2$. Moreover, $g(\xi)$ is at most of order 2.

From the above, we get $g_j^{(k)}(\xi) = f_j^{(k)}(z_j + \rho_j\xi)$. Noting that all zeros of $f_j^{(k)}(z)$ are multiple, by Hurwitz's theorem, then all zeros of $g_j^{(k)}(\xi)$ have the multiplicity at least 2.

On every compact subsets of $\mathbb{C}$ which contains no poles of g, we have

$$f_j^{(k)}(z_j + \rho_j\xi) - b = g_j^{(k)}(\xi) - b \tag{3.1}$$

converges uniformly with respect to the spherical metric to $g^{(k)}(\xi) - b$ $(k \geq 2)$.

If $g^{(k)}(\xi) - b \equiv 0$, then g is a polynomial of degree k. This contradicts with that all zeros of f have multiplicity at least $k+2$.

Since g is a nonconstant meromorphic function, by Lemmas 2.3 and 2.4, we deduce that $g^{(k)}(\xi) - b$ has at least two distinct zeros.

Next we will prove that $g^{(k)}(\xi) - b$ has just a unique zero. On the contrary, let ξ_0 and ξ_0^* be two distinct solutions of $g^{(k)}(\xi) - b$, and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \emptyset$ where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (3.1) and Hurwitz's theorem, there are points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$\begin{aligned} &f_j^{(k)}(z_j + \rho_j\xi_j) - b = 0, \\ &f_j^{(k)}\left(z_j + \rho_j\xi_j^*\right) - b = 0. \end{aligned} \tag{3.2}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $f^{(k)}(\xi)$ and $g^{(k)}(\xi)$ share b in D, we know that for any positive integer m

$$\begin{aligned} &f_m^{(k)}(z_j + \rho_j\xi_j) - b = 0, \\ &f_m^{(k)}\left(z_j + \rho_j\xi_j^*\right) - b = 0. \end{aligned} \tag{3.3}$$

Fix m, take $j \to \infty$, and in view of that $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, we have

$$f_m^{(k)}(0) - b = 0. \tag{3.4}$$

Since the zeros of $f_m^{(k)} - b$ have no accumulation point, we have that $z_j + \rho_j\xi_j = 0$ and $z_j + \rho_j\xi_j^* = 0$.

Hence

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}. \tag{3.5}$$

This contradicts with $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \emptyset$ So $g^{(k)}(\xi) - b$ has just a unique zero, which contradicts the fact that $g^{(k)}(\xi) - b$ has at least two distinct zeros.

The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.7. Likewise the proof of Theorem 1.2, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there are a sequence of complex numbers z_j, $z_j \to 0$ $(j \to \infty)$, a sequence of functions $f_j \in \mathcal{F}$, and a sequence of positive numbers $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-1} f_j(z_j + \rho_j \xi)$ converges uniformly with respect to the spherical metric to a nonconstant mermorphic functions $g(\xi)$, all zeros of $g(\xi)$ are multiple. Moreover, $g(\xi)$ is of order at most 2.

On every compact subsets of $\mathbb{C}$ which contains no poles of g, we have

$$f_j(z_j + \rho_j \xi) + a\Big(f_j'(z_j + \rho_j \xi)\Big)^n - b = \rho_j g_j(\xi) + a\Big(g_j'(\xi)\Big)^n - b \tag{3.6}$$

converges uniformly with respect to the spherical metric to $a(g'(\xi))^n - b$ $(n \geq 2)$.

If $a(g'(\xi))^n - b \equiv 0$, then g is a polynomial of degree 1, which contradicts with the zeros of $g(\xi)$ are multiple.

By Lemmas 2.2 and 2.7, we derive that $a(g'(\xi))^n - b$ has at least two distinct zeros.

Next we will prove that $a(g'(\xi))^n - b$ has just a unique zero. On the contrary, let ξ_0 and ξ_0^* be two distinct solutions of $(g'(\xi))^n - c = 0$ (where $c = b/a \neq 0$), and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \emptyset$ where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (3.6), by Hurwitz's theorem, there are points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$\begin{gathered} f_j(z_j + \rho_j \xi_j) + a\Big(f_j'(z_j + \rho_j \xi_j)\Big)^n - b = 0, \\ f_j\Big(z_j + \rho_j \xi_j^*\Big) + a\Big(f_j'\Big(z_j + \rho_j \xi_j^*\Big)\Big)^n - b = 0. \end{gathered} \tag{3.7}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $f + a(f')^n$ and $g + a(g')^n$ share b in D, we know that for any positive integer m

$$\begin{gathered} f_m(z_j + \rho_j \xi_j) + a(f_m'(z_j + \rho_j \xi_j))^n - b = 0, \\ f_m\Big(z_j + \rho_j \xi_j^*\Big) + a\Big(f_m'\Big(z_j + \rho_j \xi_j^*\Big)\Big)^n - b = 0. \end{gathered} \tag{3.8}$$

Fix m, take $j \to \infty$, and in view of that $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$, then

$$f_m(0) + a\big(f_m'(0)\big)^n - b = 0. \tag{3.9}$$

Since the zeros of $f_m + a(f_m')^n - b$ have no accumulation point, we have that $z_j + \rho_j \xi_j = 0$, $z_j + \rho_j \xi_j^* = 0$.

Hence

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}. \tag{3.10}$$

This contradicts with $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \emptyset$. So $(g'(\xi))^n - c$ has just a unique zero which contradicts with the fact $(g'(\xi))^n - c$ has at least two distinct zeros.

The proof of Theorem 1.7 is complete. $\square$

Proof of Corollary 1.8. Similar to the proof of Theorem 1.2, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then by Lemma 2.1, there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$, such that

$$g_j(\xi) = \rho_j^{-1} f_j(z_j + \rho_j \xi) \longrightarrow g(\xi), \tag{3.11}$$

locally uniformly with respect to the spherical metric, where $g(\xi)$ is nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least 2.

On every compact subsets of $\mathbb{C}$ which contains no poles of g, we have

$$f_j(z_j + \rho_j \xi_j) + a\left(f_j'(z_j + \rho_j \xi_j)\right)^n - b = \rho_j g_j(\xi) + a\left(g_j'(\xi)\right)^n - b \longrightarrow a(g'(\xi))^n - b. \tag{3.12}$$

If $a(g')^n - b \equiv 0$, that is $(g')^n \equiv b/a$. Then g is a polynomial with degree 1. It contradicts the fact that all zeros of g have multiplicity at least 2.

By Lemmas 2.2 and 2.7, we know there exists some $\xi_0 \in \mathbb{C}$ such that

$$a(g'(\xi_0))^n - b = 0. \tag{3.13}$$

From the above discussion, we get $g(\xi_0) \neq \infty$. Since $g_j(\xi) \to g(\xi)$ uniformly in a closed disk $\overline{U(\xi_0;\delta)}$, then by Hurwitz's theorem, there is $\xi_j \to \xi_0$ $(j \to \infty)$ such that

$$\begin{aligned} &f_j(z_j + \rho_j \xi_j) + a\left(f_j'(z_j + \rho_j \xi_j)\right)^n - b \\ &\quad = \rho_j g_j(\xi_j) + a\left(g_j'(\xi_j)\right)^n - b \longrightarrow a(g'(\xi_0))^n - b = 0. \end{aligned} \tag{3.14}$$

For j sufficiently large, we obtain $f_j(z_j + \rho_j \xi_j) + a(f_j'(z_j + \rho_j \xi_j))^n - b = 0$. By the assumption, we get $|f_j(z_j + \rho_j \xi_j)| \geq M$. This implies

$$\left|g_j(\xi_j)\right| \geq \rho_j^{-1} M. \tag{3.15}$$

Noting that g is holomorphic at ξ_0, then $|g(\xi)| \leq C$ for some positive constant C, and for all $\xi \in \overline{U(\xi_0;\eta)}$. Again by $g_j \to g$ and for all $\varepsilon > 0$, there are some j_0 such that for all $\xi \in \overline{U(\xi_0;\eta)}$, we have

$$\left|g_j(\xi) - g(\xi)\right| < \varepsilon. \tag{3.16}$$

For all $j \geq j_0$, by (3.15), we find that

$$C \geq \left|g(\xi_j)\right| \geq \left|g_j(\xi_j)\right| - \left|g_j(\xi_j) - g(\xi_j)\right| \geq \rho_j^{-1} M - \varepsilon. \tag{3.17}$$

That is

$$C \geq \rho_j^{-1} M - \varepsilon \tag{3.18}$$

for all $j \geq j_0$, which is impossible.

Thus, the proof of Corollary 1.8 is proved. □

Proof of Corollary 1.9 (For any $z_0 \in D$, we will prove that $\mathcal{F}$ is normal at z_0). From the assumptions in the theorem, we can choose a sufficiently small $r > 0$ such that $E(z_0, r, \mathcal{F}, n) = \emptyset$ or $\{z_0\}$. Thus we see that $f + a(f')^n \neq b$ in $U(z_0, r) \setminus \{z_0\}$ for any $f(z) \in \mathcal{F}$. Theorem F gives that $\mathcal{F}$ is normal in $U(z_0, r) \setminus \{z_0\}$. Next we will prove that $\mathcal{F}$ is normal at z_0.

Since the Möbius transformation for $\mathcal{F}$ does not change the normality, without loss of generality, we may assume that $(c, d) = (0, \infty)$.

Set $C_{r/2} := \{z \in U(z_0, r) : |z - z_0| = r/2\}$. By the given condition, we see that $\mathcal{F}$ is normal on $C_{r/2}$. Hence for any sequence of functions $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, there is a subsequence, say $\{f_{n_k}(z)\} \subset \{f_n(z)\}$, such that as $k \to \infty$,

$$f_{n_k}(z) \longrightarrow g(z) \tag{3.19}$$

uniformly convergence on $C_{r/2}$.

If $g(z) \not\equiv \infty$, by $f_{n_k}(z) \neq 0, \infty$, we have $g(z)$ is analytic on $C_{r/2}$. Hence for all $\varepsilon > 0$, there is an integer K such that

$$\left| f_{n_k}(z) - f_{n_m}(z) \right| < \varepsilon \tag{3.20}$$

for all $z \in C_{r/2}$. For arbitrary numbers $k, m > K$, by the maximum modular theorem, we have

$$\left| f_{n_k}(z) - f_{n_m}(z) \right| < \varepsilon \tag{3.21}$$

for all $|z| \leq C_{r/2}$. Thus $\mathcal{F}$ is normal at $z = z_0$.

If $g(z) \equiv \infty$, then there is an integer K and a positive number M such that

$$\left| f_{n_k}(z) \right| \geq M \tag{3.22}$$

for all $k \geq K$, $z \in C_{r/2}$. Noting that $f_{n_k} \neq 0$ in $U(z_0, r)$, we obtain

$$\left| f_{n_k}(z) \right| \geq M \tag{3.23}$$

for all $k \geq K$, $|z| \leq C_{r/2}$ by the minimum modular theorem. Hence, we have

$$f_{n_k} \longrightarrow \infty \quad (k \longrightarrow \infty) \tag{3.24}$$

uniformly on $|z| \leq r/2$. Therefore $\mathcal{F}$ is normal at $z = z_0$. □

Appendix

Using exactly the same argument as in the proof of Lemma 2.4, we can show that the following result.

Theorem A.1. *Let k be a positive integer and b be a nonzero finite complex number and let $f(z)$ be a nonconstant rational meromorphic function in $\mathbb{C}$. If all zeros of $f(z)$ have multiplicity at least $k+2$ and all zeros of $f^{(k)}(z)$ have multiplicity at least 3, then $f^{(k)}(z)-b$ has at least two distinct zeros.*

Also with the same method as in the proof of Theorem 1.2, we obtain the next conclusion.

Theorem A.2. *Let k be a positive integer and b be a nonzero finite complex number, and let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros are of multiplicity at least $k+2$. If, for every pair $f, g \in \mathcal{F}$, all zeros of $f^{(k)}(z), g^{(k)}(z)$ have multiplicity at least 3, $f^{(k)}(z)$ and $g^{(k)}(z)$ share b on D, then $\mathcal{F}$ is normal in D.*

For further study, we pose three questions.

Question 1. Whether the condition all zeros of $f^{(k)}(z)$ have multiplicity at least 2 in Theorem 1.2 can be weakened?

Question 2. Whether the conclusion of Theorem 1.2 still holds for $k=1$?

Question 3. Whether the condition $b \neq 0$ in Theorem 1.7 is necessary for $n>2$?

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