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Research Article

On a Periodic Predator-Prey System with Holling III Functional Response and Stage Structure for Prey

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We propose and study the permanence of the following periodic Holling III predator-prey system with stage structure for prey and both two predators which consume immature prey. Sufficient and necessary conditions which guarantee the predator and the prey species to be permanent are obtained.

1. Introduction

The aim of this paper is to investigate the permanence of the following periodic stage-structure predator-prey system with Holling III functional response:

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t) - \frac{p_1(t)x_1^2(t)}{k_1(t)+x_1^2(t)}y_1(t) \\
&\quad - \frac{p_2(t)x_1(t)}{k_2(t)+m(t)x_1(t)+n(t)y_2(t)}y_2(t), \\
\dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t) - \frac{p_3(t)x_2^2(t)}{k_3(t)+x_2^2(t)}y_2(t), \\
\dot{y}_1(t) &= y_1(t)\left(-g_1(t) + \frac{h_1(t)x_1^2(t)}{k_1(t)+x_1^2(t)} - q_1(t)y_1(t)\right), \\
\dot{y}_2(t) &= y_2(t)\left(-g_2(t) + \frac{h_2(t)x_1(t)}{k_2(t)+m(t)x_1(t)+n(t)y_2(t)} + \frac{h_3(t)x_2^2(t)}{k_3(t)+x_2^2(t)} - q_2(t)y_2(t)\right),
\end{aligned} \tag{1.1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $p_i(t)$, $g_i(t)$, $q_i(t)$, $i = 1,2$ and $h_i(t)$, $k_i(t)$, $i = 1,2,3$, are all continuous positive ω-periodic functions. Here $x_1(t)$ and $x_2(t)$ denote the density of immature and mature prey species, respectively, and y_i is the density of the predators.

The periodic functions in (1.1) have the following biological meanings. The birth rate into the immature population is given by $a(t)x_2$; that is, it is assumed to be proportional to the existing mature population, with a proportionality coefficient $a(t)$. The death rate of the immature population is proportional to the existing immature population and to its square with coefficients $b(t)$ and $d(t)$, respectively. The death rate of the mature population is of a logistic nature, with proportionality coefficient $f(t)$. The transition rate from the immature individuals to the mature individuals is assumed to be proportional to the existing immature population, with proportionality coefficient $c(t)$. Similarly, $-g_i(t)y_i - q_i(t)y_i$ gives the density-dependent death rate of the predators. $p_i(t)$ and $h_i(t)$ give the coefficients that relate to the conversion rate of the immature prey biomass into predator biomass. More details about the biological background for (1.1) can be found in [1–10].

The function $p_i(t)x_1^2(t)/(k_i(t)+x_1^2(t))$ represents the functional response of predator to immature prey. Let $\varphi_i(t,x_1) = p_i(t)x_1^2(t)/(k_i(t)+x_1^2(t))$, $i = 1,2$, then we have

$$\frac{\partial}{\partial x_1}\varphi_i(t,x_1) \geq 0, \quad x_1(t) > 0, \quad i = 1,2. \tag{1.2}$$

The functional response of predator species $y_i(t)$ to immature prey species takes the Holling type III, that is, $p_i(i)x_1^2(t)/(k_i(t)+x_1^2(t))$. Holling type III is the third function in which Holling proposed three kinds of functional response of the predator to prey based on numerous experiments for different species. The Holling type form of functional response is intituled prey-dependent model form. It is applied to almost invertebrate that is one of the most extensive applied functional responses.

In [2], Cui and Takeuchi considered the following periodic predator-prey system with a stage structure:

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - b(t)x_2(t) - d(t)x_1^2(t) - p(t)\phi(t,x_1)x_1y(t),\\
\dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t),\\
\dot{y}(t) &= y(t)\bigl(-g(t) + h(t)\bigr)\phi(t,x_1)x_1y(t) - q(t)y(t),
\end{aligned} \tag{1.3}$$

where

$$0 < \phi(t,x_1) < L, \quad \frac{\partial}{\partial x_1}\bigl(\phi(t,x_1)x_1\bigr) \geq 0 \quad (x_1 > 0). \tag{1.4}$$

Different predators usually consume prey in different stage structures. Some predators only prey on immature prey, and some predators only prey on mature prey [5]. Based on system (1.3), we also consider another predator species which also consumes immature prey. Assuming that the predator consumes immature prey according to Holling III functional response while the other predator consumes mature prey also according to the Holling III functional response, we get model (1.1).

To the best of the authors' knowledge, for the nonautonomous case of predator-prey systems with two predators which consume immature prey and stage structure for prey, whether one could obtain the sufficient and necessary conditions which insure the permanence of the system or not is still an open problem.

The aim of this paper is, by further developing the analysis technique of Cui and Takeuchi [2], to derive a set of sufficient and necessary conditions which ensure the permanence of the system (1.1). The rest of the paper is arranged as follows. In Section 2, we introduce some lemmas and then state the main result of this paper. The result is proved in Section 3.

Throughout this paper, for a continuous ω-periodic function $f(t)$, we set

$$A_{\omega}(f) = \omega^{-1}\int_{0}^{\omega} f(t)dt. \tag{1.5}$$

2. Main Results

Definition 2.1. The system

$$\dot{x} = F(t,x), \quad x \in R^{n} \tag{2.1}$$

is said to be permanent if there exists a compact set K in the interior of $R^{n}_{+} = \{(x_1, x_2, \ldots, x_n) \in R^{n} \mid x_i \geq 0,\ i = 1, 2, \ldots, n\}$, such that all solutions starting in the interior of R^{n}_{+} ultimately enter K and remain in K.

The following lemma can be found in [4].

Lemma 2.2. *If $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $f(t)$ are all ω-periodic, then system*

$$\begin{aligned} \dot{x}_1(t) &= a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t), \\ \dot{x}_2(t) &= c(t)x_1(t) - f(t)x_2^2(t) \end{aligned} \tag{2.2}$$

has a positive ω-periodic solution $(x_1^*(t), x_2^*(t))$ which is globally asymptotically stable with respect to $R^2_{+} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$.

Lemma 2.3 (see [11]). *If $b(t)$ and $a(t)$ are all ω-periodic, and if $A_{\omega}(b) > 0$ and $A_{\omega}(a) > 0$ for all $t \in R$, then the system*

$$\dot{x} = x[b(t) - a(t)x] \tag{2.3}$$

has a positive ω-periodic solution which is globally asymptotically stable.

Theorem 2.4. *Suppose that*

$$A_{\omega}\left(-g_1(t)+\frac{h_1(t)(x_1^*(t))^2}{k_1(t)+(x_1^*(t))^2}\right)>0,$$
$$A_{\omega}\left(-g_2(t)+\frac{h_2(t)x_1^*(t)}{k_2(t)+m(t)x_1^*(t)}+\frac{h_3(t)(x_2^*(t))^2}{k_3(t)+(x_2^*(t))^2}\right)>0 \tag{2.4}$$

holds then system (1.1) *is permanent, where* $(x_1^*(t), x_2^*(t))$ *is the unique positive periodic solution of system* (2.2) *given by Lemma 2.2.*

Theorem 2.5. *System* (1.1) *is permanent if and only if* (2.4) *holds.*

3. Proof of the Main Results

We need the following propositions to prove Theorems 2.4 and 2.5. The hypothesis of the lemmas and theorems of the preceding section are assumed to hold in what follows.

Proposition 3.1. *There exist positive constants* M_x *and* M_y *such that*

$$\limsup_{t\to+\infty} x_i(t)\le M_x,\quad \limsup_{t\to+\infty} y_i(t)\le M_y,\quad i=1,2. \tag{3.1}$$

Proof. Obviously, R_+^4 is a positively invariant set of system (1.1). Given any positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.1), we have

$$\dot{x}_1(t)\le a(t)x_2(t)-b(t)x_1(t)-d(t)x_1^2(t),$$
$$\dot{x}_2(t)\le c(t)x_1(t)-f(t)x_2^2(t). \tag{3.2}$$

By Lemma 2.2, the following auxiliary equation

$$\dot{u}_1(t)=a(t)u_2(t)-b(t)u_1(t)-d(t)u_1^2(t),$$
$$\dot{u}_2(t)=c(t)u_1(t)-f(t)u_2^2(t) \tag{3.3}$$

has a globally asymptotically stable positive ω-periodic solution $(x_1^*(t), x_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.3) with $u_i(0)=x_i(0)$. By comparison, we then have

$$x_i(t)\le u_i(t),\quad i=1,2 \tag{3.4}$$

for $t \geq 0$. By (2.4), we can choose a positive $\varepsilon > 0$ small enough such that

$$A_\omega\left(-g_1(t) + \frac{h_1(t)(x_1^*(t)+\varepsilon)^2}{k_1(t)}\right) > 0,$$
$$A_\omega\left(-g_2(t) + \frac{h_2(t)\left(x_1^*(t)+\varepsilon\right)}{k_2(t)} + \frac{h_3(t)\left(x_2^*(t)+\varepsilon\right)^2}{k_3(t)}\right) > 0. \tag{3.5}$$

Thus, from the global attractivity of $(x_1^*(t), x_2^*(t))$, for the above given $\varepsilon > 0$, there exists a $T_0 > 0$ such that

$$\left|u_i(t) - x_i^*(t)\right| < \varepsilon, \quad t \geq T_0, \tag{3.6}$$

Equation (3.4) combined with (3.6) leads to

$$x_i(t) < x_i^*(t) + \varepsilon, \quad t > T_0. \tag{3.7}$$

In addition, for $t \geq T_0$, from the third and fourth equations of (1.1) and (3.7) we get

$$\begin{aligned}
\dot{y}_1(t) &\leq y_1(t)\left[-g_1(t) + \frac{h_1(t)x_1^2(t)}{k_1(t)} - q_1(t)y_1(t)\right] \\
&\leq y_1(t)\left[-g_1(t) + \frac{h_1(t)(x_1^*(t)+\varepsilon)^2}{k_1(t)} - q_1(t)y_1(t)\right], \\
\dot{y}_2(t) &\leq y_2(t)\left[-g_2(t) + \frac{h_2(t)x_1(t)}{k_2(t)} + \frac{h_3(t)x_2^2(t)}{k_3(t)} - q_2(t)y_2(t)\right] \\
&\leq y_2(t)\left[-g_2(t) + \frac{h_2(t)\left(x_1^*(t)+\varepsilon\right)}{k_2(t)} + \frac{h_3(t)(x_2^*(t)+\varepsilon)^2}{k_3(t)} - q_2(t)y_2(t)\right].
\end{aligned} \tag{3.8}$$

Consider the following auxiliary equation:

$$\begin{aligned}
\dot{v}_1(t) &= v_1(t)\left[-g_1(t) + \frac{h_1(t)(x_1^*(t)+\varepsilon)^2}{k_1(t)} - q_1(t)v_1(t)\right], \\
\dot{v}_2(t) &= v_2(t)\left[-g_2(t) + \frac{h_2(t)\left(x_1^*(t)+\varepsilon\right)}{k_2(t)} + \frac{h_3(t)(x_2^*(t)+\varepsilon)^2}{k_3(t)} - q_2(t)v_2(t)\right].
\end{aligned} \tag{3.9}$$

It follows from (3.5) and Lemma 2.3 that (3.9) has a unique positive ω-periodic solution $y_i^*(t) > 0$ which is globally asymptotically stable. Similarly to the above analysis, there exists a $T_1 > T_0$ such that for the above ε, one has

$$y_i(t) < y_i^*(t) + \varepsilon, \quad t \geq T_1. \tag{3.10}$$

Let $M_x = \max_{0\le t\le\omega}\{x_i^*(t)+\varepsilon : i = 1,2\}$, $M_y = \max_{0\le t\le\omega}\{y_i^*(t)+\varepsilon : i = 1,2\}$, then we have

$$\limsup_{t\to+\infty} x_i(t) \le M_x, \qquad \limsup_{t\to+\infty} y_i(t) \le M_y. \tag{3.11}$$

This completes the proof of Proposition 3.1. □

Proposition 3.2. *There exist positive constants $\delta_i < M_x$, $i = 1,2$, such that*

$$\liminf_{t\to+\infty} x_i(t) \ge \delta_i, \quad i = 1,2. \tag{3.12}$$

Proof. By Proposition 3.1, there exists $T_1 > 0$ such that

$$0 < x_i(t) \le M_x; \quad 0 < y_i(t) \le M_y; \quad t > T_1. \tag{3.13}$$

Hence, from the first and second equations of system (1.1), we have

$$\begin{aligned}\dot{x}_1(t) &\ge a(t)x_2(t) - \left(b(t) + \left(\frac{p_1(t)}{k_1(t)} + \frac{p_2(t)}{k_2(t)}\right)M_y\right)x_1(t) - d(t)x_1^2(t),\\ \dot{x}_2(t) &\ge c(t)x_1(t) - \left(f(t) + \frac{p_3(t)}{k_3(t)}M_y\right)x_2^2\end{aligned} \tag{3.14}$$

for $t > T_1$. By Lemma 2.2, the following auxiliary equation

$$\begin{aligned}\dot{u}_1(t) &= a(t)u_2(t) - \left(b(t) + \left(\frac{p_1(t)}{k_1(t)} + \frac{p_2(t)}{k_2(t)}\right)M_y\right)u_1(t) - d(t)u_1^2(t),\\ \dot{u}_2(t) &\ge c(t)u_1(t) - \left(f(t) + \frac{p_3(t)}{k_3(t)}M_y\right)u_2^2\end{aligned} \tag{3.15}$$

has a globally asymptotically stable positive ω-periodic solution $(\tilde{x}_1^*(t), \tilde{x}_2^*(t))$. Let $(u_1(t), u_2(t))$ be the solution of (3.15) with $(u_1(T_1), u_2(T_1)) = (x_1(T_1), x_2(T_1))$; by comparison, we have

$$x_i(t) \ge u_i(t) \quad (i = 1,2),\ t > T_1. \tag{3.16}$$

Thus, from the global attractivity of $(\tilde{x}_1^*(t), \tilde{x}_2^*(t))$, there exists a $T_2 > T_1$, such that

$$\left|u_i(t) - \tilde{x}_i^*(t)\right| < \frac{\tilde{x}_i^*(t)}{2} \quad (i = 1,2),\ t > T_2. \tag{3.17}$$

Equation (3.17) combined with (3.16) leads to

$$x_i(t) > \delta_i = \min_{0\le t\le\omega}\left\{\frac{\tilde{x}_i^*(t)}{2}\right\}, \quad i=1,2,\ t>T_2. \tag{3.18}$$

That is, we have

$$\lim_{t\to+\infty}\inf x_i(t) \ge \delta_i, \quad i=1,2. \tag{3.19}$$

This completes the proof of Proposition 3.2. □

Proposition 3.3. *There exists a positive constant δ_y such that*

$$\lim_{t\to+\infty}\sup y_i(t) \ge \delta_y, \quad i=1,2. \tag{3.20}$$

Proof. By assumption (2.4), we can choose a constant $\varepsilon_0 > 0$ and the same constant ε as in Proposition 3.1 such that

$$A_\omega\left(\varphi_{i\varepsilon_0}(t)\right) > 0, \quad i=1,2, \tag{3.21}$$

where

$$\begin{aligned}
\varphi_{1\varepsilon_0}(t) &= -g_1(t) + \frac{h_1(t)(x_1^*(t)-\varepsilon_0)^2}{k_1(t)+(x_1^*(t)-\varepsilon_0)^2} - q_1(t)\varepsilon_0, \\
\varphi_{2\varepsilon_0}(t) &= -g_2(t) + \frac{h_2(t)\left(x_1^*(t)-\varepsilon_0\right)}{k_2(t)+m(t)\left(x_1^*(t)+\varepsilon\right)+n(t)\varepsilon_0} \\
&\quad + \frac{h_3(t)(x_2^*(t)-\varepsilon_0)^2}{k_3(t)+(x_2^*(t)-\varepsilon_0)^2} - q_2(t)\varepsilon_0.
\end{aligned} \tag{3.22}$$

Consider the following equation with a parameter $\beta > 0$:

$$\begin{aligned}
\dot{x}_1(t) &= a(t)x_2(t) - \left(b(\mathrm{t}) + 2\beta\left(\frac{p_1(t)}{k_1(t)}+\frac{p_2(t)}{k_2(t)}\right)M_y\right)x_1(t) - d(t)x_1^2(t), \\
\dot{x}_2(t) &= c(t)x_1(t) - \left(f(t) + \frac{p_3(t)}{k_3(t)}2\beta\right)x_2^2.
\end{aligned} \tag{3.23}$$

By Lemma 2.2, (3.23) has a unique positive ω-periodic solution $(x_{1\beta}(t), x_{2\beta}(t))$, which is globally asymptotically stable. Let $(\overline{x}_{1\beta}(t), \overline{x}_{2\beta}(t))$ be the solution of (3.23) with initial condition $\overline{x}_{i\beta}(0) = x_i^*(0)$, $i=1,2$; then, for the above ε_0, there exists a $T_3 > T_2$, such that

$$\left|\overline{x}_{i\beta}(t) - x_{i\beta}(t)\right| < \frac{\varepsilon_0}{4}, \quad i=1,2,\ t>T_3. \tag{3.24}$$

By continuity of the solution in the parameter, we have $(\overline{x}_{1\beta}(t), \overline{x}_{2\beta}(t)) \to (x_1^*(t), x_2^*(t))$ uniformly in $[T_3, T_3+\omega]$ as $\beta \to 0$. Hence, for $\varepsilon_0 > 0$ there exists $\beta_0 = \beta_0(\varepsilon_0) > 0$ such that

$$\left|\overline{x}_{i\beta}(t) - x_i^*(t)\right| < \frac{\varepsilon_0}{4}, \quad t \in [T_3, T_3+\omega],\ 0 < \beta < \beta_0. \tag{3.25}$$

So we have

$$\left|x_{i\beta}(t) - x_i^*(t)\right| \leq \left|\overline{x}_{i\beta}(t) - x_{i\beta}(t)\right| + \left|\overline{x}_{i\beta}(t) - x_i^*(t)\right| < \frac{\varepsilon_0}{2}, \quad t \in [T_3, T_3+\omega]. \tag{3.26}$$

Since $x_{i\beta}(t)$ and $x_i^*(t)$ are all ω-periodic, we have

$$\left|x_{i\beta}(t) - x_i^*(t)\right| < \frac{\varepsilon_0}{2}, \quad t \geq 0,\ 0 < \beta < \beta_0. \tag{3.27}$$

Choose a constant β_1 $(0 < \beta_1 < \beta_0, 2\beta_1 < \varepsilon_0)$ and

$$x_{i\beta_1}(t) \geq x_i^*(t) - \frac{\varepsilon_0}{2}, \quad t \geq 0. \tag{3.28}$$

Suppose that the conclusion (3.20) is not true. Then there exists a $Z \in R_+^4$ such that, for the positive solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of (1.1) with an initial condition $(x_1(0), x_2(0), y_1(0), y_2(0)) = Z$, we have

$$\limsup_{t \to +\infty} y_i(t) < \beta_1. \tag{3.29}$$

So there exists a $T_4 > T_3$ such that

$$y_i(t) < 2\beta_1, \quad t \geq T_4. \tag{3.30}$$

By applying (3.30), from the first and second equations of system (1.1) it follows that for all $t \geq T_4$,

$$\begin{aligned} \dot{x}_1(t) &\geq a(t)x_2(t) - \left(b(t) + 2\beta_1\frac{p_1(t)}{k_1(t)} + 2\beta_1\frac{p_2(t)}{k_2(t)}\right)x_1(t) - d(t)x_1^2(t), \\ \dot{x}_2(t) &\geq c(t)x_1(t) - \left(f(t) + \frac{p_3(t)}{k_3(t)}2\beta_1\right)x_2^2(t). \end{aligned} \tag{3.31}$$

Let $(u_1(t), u_2(t))$ be the solution of (3.23) with $\beta = \beta_1$ and $u_i(T_4) = x_i(T_4)$, $i = 1, 2$; we know that $x_i(t) \geq u_i(t)$, $t \geq T_4$, $i = 1, 2$.

By the global asymptotic stability of $(x_{1\beta_1}(t), x_{2\beta_1}(t))$, for the given $\varepsilon = \varepsilon_0/2$, there exists $T_5 \geq T_4$ such that

$$\left|u_i(t) - x_{i\beta_1}(t)\right| < \frac{\varepsilon_0}{2}, \quad t \geq T_5. \tag{3.32}$$

So we have

$$x_i(t) \geq u_i(t) > x_{i\beta_1}(t) - \frac{\varepsilon_0}{2}, \quad t \geq T_5 \tag{3.33}$$

and hence

$$x_i(t) > x_i^*(t) - \varepsilon_0, \quad t \geq T_5. \tag{3.34}$$

From (3.7) and (3.34), we have

$$x_i^*(t) - \varepsilon_0 < x_i(t) < x_i^*(t) + \varepsilon, \quad t \geq T_5. \tag{3.35}$$

By (3.35) and (1.2), from the third and fourth equations of system (1.1) we have

$$\dot{y}_i(t) \geq \varphi_{i\varepsilon_0}(t) y_i(t), \quad t \geq T_5. \tag{3.36}$$

Integrating (3.36) from T_5 to t yields

$$y_i(t) \geq y_i(T_5) \exp\left\{\int_{T_5}^{t} \varphi_{i\varepsilon_0}(t) dt\right\}, \quad t \geq T_5. \tag{3.37}$$

By (3.21), we know that $y_i(t) \to +\infty$ as $t \to +\infty$, $i = 1, 2$, which is a contradiction. This completes the proof. □

Proof of Theorem 2.4. By Propositions 3.2 and 3.3, system (1.1) is uniform weak persistent [2]. From [12, Propositions 3.1 and Theorem 1.3.3], system (1.1) is permanent. This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5. The sufficiency of Theorem 2.5 now follows from Theorem 2.4. We thus only need to prove the necessity of Theorem 2.5. Suppose that

$$\begin{gathered} A_\omega\left(-g_1(t) + \frac{h_1(t)(x_1^*(t))^2}{k_1(t) + (x_1^*(t))^2}\right) \leq 0, \\ A_\omega\left(-g_2(t) + \frac{h_2(t)x_1^*(t)}{k_2(t) + m(t)x_1^*(t)} + \frac{h_3(t)(x_2^*(t))^2}{k_3(t) + (x_2^*(t))^2}\right) \leq 0. \end{gathered} \tag{3.38}$$

We will show that

$$\lim_{t\to+\infty} y_i(t) = 0, \quad i = 1,2. \tag{3.39}$$

In fact, by (3.38), we know that, for any given $0 < \varepsilon < 1$, there exist $\varepsilon_1 > 0$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned}
&A_\omega\left(-g_1(t) + \frac{h_1(t)(x_1^*(t)+\varepsilon_1)^2}{k_1(t)+(x_1^*(t)+\varepsilon_1)^2} - q_1(t)\varepsilon\right) \le -\frac{\varepsilon}{2}A_\omega(q_1(t)) \le -\varepsilon_0,\\
&A_\omega\left(-g_2(t) + \frac{h_2(t)(x_1^*(t)+\varepsilon_1)}{k_2(t)+m(t)(x_1^*(t)+\varepsilon_1)+n(t)\varepsilon} + \frac{h_3(t)(x_2^*(t)+\varepsilon_1)^2}{k_3(t)+(x_2^*(t)+\varepsilon_1)^2} - q_2(t)\varepsilon\right)\\
&\quad \le -\frac{\varepsilon}{2}A_\omega(q_2(t)) \le -\varepsilon_0.
\end{aligned} \tag{3.40}$$

Note that $q_i(t) > 0$ for $t \ge 0$. Since

$$\begin{aligned}
\dot{x}_1(t) &\le a(t)x_2(t) - b(t)x_1(t) - d(t)x_1^2(t),\\
\dot{x}_2(t) &\le c(t)x_1(t) - f(t)x_1^2(t),
\end{aligned} \tag{3.41}$$

we know that, for the given ε_1, there exists $T^{(1)} > 0$ such that

$$x_i(t) \le x_i^*(t) + \varepsilon_1, \quad t \ge T^{(1)},\ i = 1,2. \tag{3.42}$$

By (3.40), (1.2), and (3.42), we have

$$\begin{aligned}
&A_\omega\left(-g_1(t) + \frac{h_1(t)x_1^2(t)}{k_1(t)+x_1^2(t)} - q_1(t)\varepsilon\right) \le -\varepsilon_0, \quad t \ge T^{(1)},\\
&A_\omega\left(-g_2(t) + \frac{h_2(t)x_1(t)}{k_2(t)+m(t)x_1(t)+n(t)\varepsilon} + \frac{h_3(t)x_2^2(t)}{k_3(t)+x_2^2(t)} - q_2(t)\varepsilon\right) \le -\varepsilon_0, \quad t \ge T^{(1)}.
\end{aligned} \tag{3.43}$$

We now show that there must exist $T^{(2)}(> T^{(1)})$ such that $y_i(T^{(2)}) < \varepsilon$. Otherwise, by (3.43), we have

$$\begin{aligned}
&\varepsilon \le y_1(t) \le y_1\left(T^{(1)}\right)\exp\left\{\int_{T^{(1)}}^t \left[-g_1(s) + \frac{h_1(s)(x_1(s))^2}{k_1(s)+(x_1(s))^2} - q_1(s)\varepsilon\right]ds\right\} \longrightarrow 0, \quad t \longrightarrow +\infty,\\
&\varepsilon \le y_2(t) \le y_2\left(T^{(1)}\right)\exp\left\{\int_{T^{(1)}}^t \left[-g_2(s) + \frac{h_2(s)x_1(s)}{k_2(s)+m(s)x_1(s)+n(s)\varepsilon} + \frac{h_3(s)x_2^2(s)}{k_3(s)+x_2^2(s)} - q_2(s)\varepsilon\right]ds\right\}\\
&\quad \longrightarrow 0, \quad t \longrightarrow +\infty.
\end{aligned} \tag{3.44}$$

This implies $\varepsilon \leq 0$, which is a contradiction.

Let

$$M_1(\varepsilon) = \max_{0\leq s\leq\omega}\left\{-g_1(s) + \frac{h_1(s)(x_1(s))^2}{k_1(s) + (x_1(s))^2} - q_1(s)\varepsilon\right\},$$
$$M_2(\varepsilon) = \max_{0\leq s\leq\omega}\left\{-g_2(s) + \frac{h_2(s)x_1(s)}{k_2(s) + m(s)x_1(s) + n(s)\varepsilon} + \frac{h_3(s)x_2^2(s)}{k_3(s) + x_2^2(s)} - q_2(s)\varepsilon\right\}. \tag{3.45}$$

We know that $M_i(\varepsilon)$ is bounded (Proposition 3.1 given). We now show that

$$y_i(t) \leq \varepsilon \exp\{M_i(\varepsilon)\omega\}, \quad t \geq T^{(2)},\ i = 1, 2. \tag{3.46}$$

Otherwise, there exists $T^{(3)} > T^{(2)}$ such that

$$y_i\left(T^{(3)}\right) > \varepsilon \exp\{M_i(\varepsilon)\omega\}. \tag{3.47}$$

By the continuity of $y_i(t)$, there must exist $T^{(4)} \in (T^{(2)}, T^{(3)})$ such that $y_i(T^{(4)}) = \varepsilon$ and $y_i(t) > \varepsilon$ for $t \in (T^{(4)}, T^{(3)}]$. Let P_1 be the nonnegative integer such that $T^{(3)} \in (T^{(4)}+P_1\omega, T^{(4)}+(P_1+1)\omega]$; by (3.43) we have

$$\begin{aligned}
\varepsilon \exp\{M_1(\varepsilon)\omega\} &< y_1\left(T^{(3)}\right) \\
&< y_1\left(T^{(4)}\right) \exp\left\{\int_{T^{(4)}}^{T^{(3)}}\left[-g_1(t) + \frac{h_1(t)(x_1(t))^2}{k_1(t) + (x_1(t))^2} - q_1(t)\varepsilon\right]dt\right\} \\
&= \varepsilon \exp\left\{\int_{T^{(4)}}^{T^{(4)}+P_1\omega} + \int_{T^{(4)}+P_1\omega}^{T^{(3)}}\right\}\left[-g_1(t) + \frac{h_1(t)(x_1(t))^2}{k_1(t) + (x_1(t))^2} - q_1(t)\varepsilon\right]dt \\
&\leq \varepsilon \exp\left\{\int_{T^{(4)}+P_1\omega}^{T^{(3)}}\left[-g_1(t) + \frac{h_1(t)(x_1(t))^2}{k_1(t) + (x_1(t))^2} - q_1(t)\varepsilon\right]dt\right\} \\
&< \varepsilon \exp\{M_1(\varepsilon)\omega\},
\end{aligned}$$

$$\begin{aligned}
\varepsilon \exp\{M_2(\varepsilon)\omega\} &< y_2\left(T^{(3)}\right) \\
&< y_2\left(T^{(4)}\right) \exp\left\{\int_{T^{(4)}}^{T^{(3)}} \left[-g_2(t) + \frac{h_2(t)x_1(t)}{k_2(t)+m(t)x_1(t)+n(t)\varepsilon}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{h_3(t)x_2^2(t)}{k_3(t)+x_2^2(t)} - q_2(t)\varepsilon\right] dt\right\} \\
&= \varepsilon \exp\left\{\int_{T^{(4)}}^{T^{(4)}+P_1\omega} + \int_{T^{(4)}+P_1\omega}^{T^{(3)}}\right\} \left[-g_2(t) + \frac{h_2(t)x_1(t)}{k_2(t)+m(t)x_1(t)+n(t)\varepsilon}\right. \\
&\qquad\qquad \left. + \frac{h_3(t)x_2^2(t)}{k_3(t)+x_2^2(t)} - q_2(t)\varepsilon\right] dt \\
&\le \varepsilon \exp\left\{\int_{T^{(4)}+P_1\omega}^{T^{(3)}} \left[-g_2(t) + \frac{h_2(t)x_1(t)}{k_2(t)+m(t)x_1(t)+n(t)\varepsilon} + \frac{h_3(t)x_2^2(t)}{k_3(t)+x_2^2(t)} - q_2(t)\varepsilon\right] dt\right\} \\
&< \varepsilon \exp\{M_2(\varepsilon)\omega\},
\end{aligned} \tag{3.48}$$

which is a contradiction. This implies that (3.46) holds. We then conclude, by the arbitrariness of ε, that $y_i(t) \to 0$ as $t \to +\infty$, $i = 1, 2$. This completes the proof of Theorem 2.5. □

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