Research Article

Path Convergence and Approximation of Common Zeroes of a Finite Family of m-Accretive Mappings in Banach Spaces

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Let E be a real Banach space which is uniformly smooth and uniformly convex. Let K be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q is the sunny nonexpansive retraction. If E admits weakly sequentially continuous duality mapping j, path convergence is proved for a nonexpansive mapping $T : K \rightarrow K$. As an application, we prove strong convergence theorem for common zeroes of a finite family of m-accretive mappings of K to E. As a consequence, an iterative scheme is constructed to converge to a common fixed point (assuming existence) of a finite family of pseudocontractive mappings from K to E under certain mild conditions.

1. Introduction

Let E be a real Banach space with dual E^* and K a nonempty, closed and convex subset of E. A mapping $T : K \rightarrow K$ is said to be *nonexpansive* if for all $x, y \in K$, we have

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1.1}$$

A point $x \in K$ is called *a fixed point* of T if $Tx = x$. The fixed points set of T is the set $F(T) := \{x \in K : Tx = x\}$.

Construction of fixed points of nonexpansive mappings is an important subject in nonlinear mapping theory and its applications; in particular, in image recovery and signal processing (see, e.g., [1–3]). Many authors have worked extensively on the approximation

of fixed points of nonexpansive mappings. For example, the reader can consult the recent monographs of Berinde [4] and Chidume [5].

We denote by J the *normalized duality mapping* from E to 2^{E^*} defined by

$$Jx := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between members of E and E^*. It is well known that if E^* is strictly convex then J is single valued (see, e.g., [5, 6]). In the sequel, we will denote the single-valued normalized duality mapping by j.

A mapping $A : D(A) \subseteq E \to E$ is called *accretive* if, for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \tag{1.3}$$

By the results of Kato [7], (1.3) is equivalent to

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\|, \quad \forall s > 0. \tag{1.4}$$

If E is a Hilbert space, accretive mappings are also called *monotone*. A mapping A is called *m-accretive* if it is accretive and $R(I+rA)$, range of $(I+rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $\mathrm{cl}(D(A)) \subseteq R(I + rA)$, for all $r > 0$, where $\mathrm{cl}(D(A))$ denotes the closure of the domain of A. A is said to be *maximal accretive* if it is accretive and the inclusion $G(A) \subset G(B)$, where $G(A)$ is a graph of A, with B accretive, implies $G(A) = G(B)$. It is known (see e.g., [8]) that every maximal accretive mapping is m-accretive and the converse holds if E is a Hilbert space. Interest in accretive mappings stems mainly from their firm connection with equations of evolution. It is known (see, e.g., [9]) that many physically significant problems can be modelled by initial-value problems of the following form:

$$u'(t) + Au(t) = 0, \qquad u(0) = u_0, \tag{1.5}$$

where A is an accretive mapping in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrödinger equations. One of the fundamental results in the theory of accretive mappings, due to Browder [10], states that if A is locally Lipschitzian and accretive, then A is m-accretive. This result was subsequently generalized by Martin [11] to the continuous accretive mappings. If in (1.5), $u(t)$ is independent of t, then (1.5) reduces to

$$Au = 0, \tag{1.6}$$

whose solutions correspond to the equilibrium points of the system (1.5). Consequently, considerable research effects have been devoted, especially within the past 30 years or so, to iterative methods for approximating these equilibrium points.

Closely related to the class of accretive mappings is the class of pseudocontractive mappings. A mapping T with domain $D(T)$ in E and range $R(T)$ in E is called *pseudocontractive* if $A := I - T$ is accretive. It is then clear that any zero of A is a fixed point

of T. Consequently, the study of approximating fixed points of pseudocontractive mappings, which correspond to equilibrium points of the system (1.5), became a flourishing area of research for numerous mathematicians (see, e.g., [12–14] and the references therein).

It is not difficult to deduce from (1.4) that the mapping A is accretive if and only if $(I + rA)^{-1}$, for all $r > 0$, is nonexpansive on the range of $(I + rA)$. Thus, in particular, $J_A := (I + A)^{-1}$ is nonexpansive and single valued on the range of $(I + A)$. Furthermore, $F(J_A) := N(A) := \{x \in D(A) : Ax = 0\}$. It is well known that every nonexpansive mapping is pseudocontractive and the converse does not, however, hold.

Very recently, Yao et al. [15] proved path convergence for a nonexpansive mapping in a real Hilbert space. In particular, they proved the following theorem.

Theorem 1.1 (Yao et al. [15]). *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : K \rightarrow K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t \in (0,1)$, let the net $\{x_t\}$ be generated by $x_t = TP_K[(1 - t)x_t]$, then as $t \rightarrow 0$, the net $\{x_t\}$ converges strongly to a fixed point of T.*

Furthermore, they applied Theorem 1.1 to prove the following theorem.

Theorem 1.2 (Yao et al. [15]). *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : K \rightarrow K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For an arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$y_n = P_K[(1 - \alpha_n)x_n],$$
$$x_{n+1} = (1 - \beta_n)x_n + \beta_n Ty_n, \quad n \geq 1. \tag{1.7}$$

Suppose that the following conditions are satisfied:

(a) $\lim \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.7) converges strongly to a fixed point of T.

Motivated by the results of Yao et al. [15], we proved path convergence for a nonexpansive mapping in a uniformly smooth real Banach space which is also uniformly convex and E admits weakly sequentially continuous duality mapping j. As an application, a strong convergence is proved for common zeroes of a finite family of m-accretive mappings of K to E. As a consequence, an iterative scheme is constructed to converge to a common fixed point (assuming existence) of a finite family of pseudocontractive mappings from K to E under certain mild conditions.

2. Preliminaries

Let E be a real Banach space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup\left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 \; : \; \|x\| = 1, \|y\| = \tau \right\}. \tag{2.2}$$

E is equivalently said to be *smooth* if $\rho_E(\tau) > 0$, for any $\tau > 0$.

Let $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1; \; \epsilon = \|x - y\| \right\}. \tag{2.3}$$

E is *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|(1/2)(x + y)\| \leq 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. E is called *strictly convex* if for all $x, y \in E, x \neq y, \|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, for all $\lambda \in (0, 1)$. It is known that every uniformly convex Banach space is reflexive.

Let $K \subset E$ be closed and convex and Q be a mapping of E onto K. Then Q is said to be *sunny* if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = (z)$ for every $z \in R(Q)$, where $R(Q)$ is the *range of Q*. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K. If $E = H$, the *metric projection* P_K is a sunny nonexpansive retraction from H to any closed and convex subset of H. But this is not true in a general Banach spaces. We note that if E is smooth and Q is retraction of K onto $F(T)$, then Q is sunny and nonexpansive if and only if for each $x \in K$ and $z \in F(T)$ we have $\langle Qx - x, J(Qx - z) \rangle \leq 0$, (see [16–18] for more details).

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \rightarrow p$ then $Tx = p$.

Suppose that J is single valued. Then, J is said to be *weakly sequentially continuous* if for each $\{x_n\}_{n=1}^{\infty} \subset E$ which converges weakly to x implies $J(x_n)$ converges in weak* to $J(x)$.

We need the following lemmas in the sequel.

Lemma 2.1 (Browder [19], Goebel and Kirk [20]). *Let E be a real uniformly convex Banach space and let K be a nonempty, closed, and convex subset of E and $T : K \rightarrow K$ is a nonexpansive mapping such that $F(T) \neq \emptyset$, then, $I - T$ is demiclosed at zero.*

Lemma 2.2 (Suzuki [21]). *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in a Banach space X and let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\rightarrow\infty} \beta_n \leq \limsup_{n\rightarrow\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\rightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\rightarrow\infty}\|y_n - x_n\| = 0$.*

Lemma 2.3 (Chidume [5], Reich [22]). *Let E be a uniformly real smooth Banach space, then there exists a nondecreasing continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0^+} \beta(s) = 0$ and $\beta(cs) \leq c\beta(s)$ for $c \geq 1$ such that for all $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|). \tag{2.4}$$

Lemma 2.4 (Xu [23]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers which satisfies the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n, \quad n \geq 1, \tag{2.5}$$

where $\{a_n\}_{n=1}^{\infty} \subset [0, 1]$ and $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ satisfying the following:

(i) $\sum \alpha_n = \infty$,

(ii) $\lim \sup \sigma_n \leq 0$,

then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.5 (Cioranescu [8]). *Let A be a continuous accretive mapping defined on a real Banach space E with $D(A) = E$, then A is m-accretive.*

Lemma 2.6 (Zegeye and Shahzad [24]). *Let K be a nonempty, closed, and convex subset of a real strictly convex Banach space E. For each $r = 1, 2, \ldots, N$ let $A_r : K \to E$ be an m-accretive mapping such that $\bigcap_{r=1}^{N} N(A_i) \neq \emptyset$. Let $a_0, a_1, a_2, \ldots, a_N$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^{N} a_i = 1$, and let $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_N J_{A_N}$, with $J_{A_r} := (I + A_r)^{-1}$, then S_N is nonexpansive and $F(S_N) = \bigcap_{r=1}^{N} N(A_r)$.*

3. Path Convergence Theorem

Let K be a nonempty, closed, and convex sunny nonexpansive retract of a uniformly smooth Banach space E which is also uniformly convex where Q_K is the sunny nonexpansive retraction of E onto K. Let $T : K \to K$ be nonexpansive. For each $t \in (0, 1)$, we define the mapping $T_t : K \to K$ by

$$T_t x := TQ_K[(1 - t)x]. \tag{3.1}$$

We will show that T_t is a contraction.

From (3.1), we have

$$\begin{aligned} \|T_t x - T_t y\| &\leq \|Q_K(1 - t)x - Q_K(1 - t)y\| \\ &\leq (1 - t)\|x - y\|, \end{aligned} \tag{3.2}$$

which implies that T_t is a contraction. Therefore, by the Banach contraction mapping principle, there exists a unique fixed point z_t of T_t in K. That is,

$$z_t = TQ_K[(1 - t)z_t]. \tag{3.3}$$

Next, we prove that $\{z_t\}$ is bounded. Let $x^* \in F(T)$, then using (3.3), we have

$$
\begin{aligned}
\|z_t - x^*\| &= \|TQ_K(1-t)z_t - TQ_Kx^*\| \\
&\leq \|Q_K(1-t)z_t - Q_Kx^*\| \\
&\leq \|(1-t)z_t - tx^* + tx^* - x^*\| = \|(1-t)(z_t - x^*) - tx^*\| \\
&\leq (1-t)\|z_t - x^*\| + t\|x^*\|.
\end{aligned}
\tag{3.4}
$$

Thus, $\|z_t - x^*\| \leq \|x^*\|$. This implies that $\{z_t\}$ is bounded.

We next show that $\|z_t - Tz_t\| \to 0$ as $t \to 0$, as follows:

$$
\begin{aligned}
\|z_t - Tz_t\| &= \|TQ_K(1-t)z_t - TQ_Kz_t\| \\
&\leq \|Q_K(1-t)z_t - Q_Kz_t\| \leq \|(1-t)z_t - z_t\| \\
&\leq t\|z_t\| \longrightarrow 0, \quad (\text{since } t \longrightarrow 0).
\end{aligned}
\tag{3.5}
$$

Next, we show that $\{z_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $t_n \to 0$ as $n \to \infty$. Put $z_n := z_{t_n}$. From (3.5), we obtain that

$$
\|z_n - Tz_n\| \longrightarrow 0.
\tag{3.6}
$$

Remark 3.1. Let $x^* \in F(T)$ and $r_1 > 0$ be sufficiently large such that $z_t \in \overline{B_{r_1}(x^*)} \cap K$ for each $t \in (0,1)$, where $\overline{B_{r_1}(x^*)} := \{z \in E : \|z - x^*\| \leq r_1\}$. For the next theorem, we define $A := \max\{1, 2r_1\}$ and assume that the function β from Lemma 2.3 satisfies the following condition: $\beta(s) \leq s/A$.

Theorem 3.2. *Let E be a real Banach space which is uniformly smooth and uniformly convex and let K be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_K is the sunny nonexpansive retraction of E onto K. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For each $t \in (0,1)$, let $\{z_t\}$ be generated by (3.3), then as $t \to 0$, $\{z_t\}$ converges strongly to a fixed point of T if E admits weak sequential continuous duality mapping j.*

Proof. From (3.3), we get for $u \in F(T)$,

$$
\begin{aligned}
\|z_t - u\|^2 &= \|TQ_K[(1-t)z_t] - TQ_Ku\|^2 \\
&\leq \|(1-t)z_t - u\|^2 = \|z_t - u - tz_t\|^2 \\
&\leq \|z_t - u\|^2 - 2t\langle z_t, j(z_t - u)\rangle + \max\{\|z_t - u\|, 1\}t\|z_t\|\beta(t\|z_t\|) \\
&\leq \|z_t - u\|^2 - 2t\langle z_t, j(z_t - u)\rangle + \max\{2r_1, 1\}t\|z_t\|\beta(t\|z_t\|) \\
&\leq \|z_t - u\|^2 - 2t\langle z_t, j(z_t - u)\rangle + t^2\|z_t\|^2 \\
&= \|z_t - u\|^2 - 2t\langle z_t - u, j(z_t - u)\rangle - 2t\langle u, j(z_t - u)\rangle + t^2\|z_t\|^2 \\
&\leq \|z_t - u\|^2 - 2t\|z_t - u\|^2 - 2t\langle u, j(z_t - u)\rangle + t^2\|z_t\|^2.
\end{aligned}
\tag{3.7}
$$

This implies that

$$\|z_t - u\|^2 \leq \langle u, j(u - z_t) \rangle + \frac{t}{2}\|z_t\|^2. \tag{3.8}$$

In particular,

$$\|z_n - u\|^2 \leq \langle u, j(u - z_n) \rangle + \frac{t_n}{2}\|z_n\|^2. \tag{3.9}$$

Since $\{z_n\}$ is bounded, without loss of generality, we can assume that $\{z_n\}$ converges weakly to z^*. Using the demiclosedness property of $(I - T)$ at zero and the fact that $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, we obtain that $z^* \in F(T)$. Therefore, we can substitute z^* for u in (3.9) to obtain

$$\|z_n - z^*\|^2 \leq \langle z^*, j(z^* - z_n) \rangle + \frac{t_n}{2}\|z_n\|^2. \tag{3.10}$$

Using the fact that j is weakly sequentially continuous, we have from the last inequality that $\{z_n\}$ converges strongly to z^*. We now show that $\{z_t\}$ actually converges to z^*. Suppose that $\{z_{t_m}\}$ converges strongly to x^*. Put $z_m := z_{t_m}$, then since $\|z_m - Tz_m\| \to 0$ as $m \to \infty$ and $I - T$ is demiclosed at zero, we have that $x^* \in F(T)$.

Claim ($z^* = x^*$). Suppose in contradiction that $x^* \neq z^*$. Using (3.3), we obtain using similar argument as above that

$$\|z_m - z^*\|^2 \leq \langle z^*, j(z^* - z_m) \rangle + \frac{t_m}{2}\|z_m\|^2. \tag{3.11}$$

Thus,

$$\|x^* - z^*\|^2 \leq \langle z^*, j(z^* - x^*) \rangle. \tag{3.12}$$

Interchanging x^* and z^*, we obtain

$$\|z^* - x^*\|^2 \leq \langle x^*, j(x^* - z^*) \rangle. \tag{3.13}$$

Adding (3.12) and (3.13) yields

$$2\|x^* - z^*\|^2 \leq \|x^* - z^*\|^2 \tag{3.14}$$

and implies that $x^* = z^*$. This completes the proof. □

Corollary 3.3. *Let $E := l_p, 1 < p < \infty$ and let K be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_K is the sunny nonexpansive retraction of E onto K. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For each $t \in (0,1)$, let $\{z_t\}$ be generated by (3.3) then as $t \to 0$, $\{z_t\}$ converges strongly to a fixed point of T.*

4. Iterative Methods and Convergence Theorems

Theorem 4.1. *Let E be a real Banach space which is uniformly smooth and uniformly convex, and let K be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_K is the sunny nonexpansive retraction of E onto K. For each $r = 1, 2, \ldots, N$, let $A_r : K \to E$ be an m-accretive mapping such that $\bigcap_{r=1}^{N} N(A_r) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For an arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$
\begin{aligned}
y_n &= Q_K[(1 - \alpha_n)x_n], \\
x_{n+1} &= (1 - \beta_n)x_n + \beta_n S_N y_n, \quad n \geq 1,
\end{aligned}
\tag{4.1}
$$

where $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_N J_{A_N}$, with $J_{A_r} := (I + A_r)^{-1}, r = 1, 2, \ldots, N$ for $0 < a_i < 1$, $i = 0, 1, 2, \ldots, N$, $\sum_{i=0}^{N} a_i = 1$. Suppose that the following conditions are satisfied:

(a) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common zero of $\{A_r\}_{r=1}^{N}$ if E admits weakly sequentially continuous duality mapping j.

Proof. By Lemma 2.6, S_N is nonexpansive and $F(S_N) = \bigcap_{r=1}^{N} N(A_r)$. Now, we first show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $x^* \in \bigcap_{r=1}^{N} N(A_r) = F(S_N)$, we have from (4.1) that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \left\| (1 - \beta_n)(x_n - x^*) + \beta_n(S_N y_n - x^*) \right\| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n \|S_N y_n - x^*\| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n[(1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|x^*\|] \\
&= (1 - \alpha_n \beta_n)\|x_n - x^*\| + \alpha_n \beta_n \|x^*\| \\
&\leq \max\{\|x_n - x^*\|, \|x^*\|\} \\
&\ \ \vdots \\
&\leq \max\{\|x_1 - x^*\|, \|x^*\|\}.
\end{aligned}
\tag{4.2}
$$

Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded and $\{S_N x_n\}$ is also bounded. Set $u_n = S_N y_n, n \geq 1$. It follows that

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \|S_N y_{n+1} - S_N y_n\| \\
&\leq \|y_{n+1} - y_n\| \leq \|(1 - \alpha_{n+1})x_{n+1} - (1 - \alpha_n)x_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\|.
\end{aligned}
\tag{4.3}
$$

Hence, $\limsup_{n\to\infty}(\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|) \leq 0$. This together with Lemma 2.2 implies that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Thus,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\beta_n\|x_n - u_n\| = 0,$$

$$\begin{aligned}
\|x_n - S_N x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_N x_n\| \\
&\leq \|x_n - x_{n+1}\| + (1 - \beta_n)\|x_n - S_N x_n\| + \beta_n\|S_N y_n - S_N x_n\| \\
&\leq \|x_n - x_{n+1}\| + (1 - \beta_n)\|x_n - S_N x_n\| + \beta_n\|y_n - x_n\| \\
&= \|x_n - x_{n+1}\| + (1 - \beta_n)\|x_n - S_N x_n\| + \beta_n\|Q_K[(1-\alpha_n)x_n] - Q_K x_n\| \\
&\leq \|x_n - x_{n+1}\| + (1 - \beta_n)\|x_n - S_N x_n\| + \beta_n\|(1-\alpha_n)x_n - x_n\| \\
&= \|x_n - x_{n+1}\| + (1 - \beta_n)\|x_n - S_N x_n\| + \alpha_n\beta_n\|x_n\| \\
&\leq \|x_n - x_{n+1}\| + (1 - \beta_n)\|x_n - S_N x_n\| + \alpha_n\|x_n\|, \quad \text{since } \beta_n \in (0,1),
\end{aligned}$$

(4.4)

that is,

$$\|x_n - S_N x_n\| \leq \frac{1}{\beta_n}\{\|x_n - x_{n+1}\| + \alpha_n\|x_n\|\} \longrightarrow 0. \tag{4.5}$$

Let $\{z_t\}$ be defined by (3.3) for $T = S_N$, then from Theorem 3.2, $z_t \to x^* \in F(S_N) = \bigcap_{r=1}^{N} N(A_r)$ as $t \to 0$ (This is guaranteed because E admits weakly sequentially continuous duality mapping). Next, we show that

$$\limsup_{n\to\infty}\langle x^*, j(x^* - x_n)\rangle \leq 0. \tag{4.6}$$

Now, since $\{x_n\}_{n=1}^{\infty}$ and $\{z_t\}$ are bounded, there exist $r_1 > 0, r_2 > 0$ such that $z_t \in \overline{B_{r_1}(x^*)} \cap K$ for each $t \in (0,1)$ and $x_n \in \overline{B_{r_2}(x^*)} \cap K$ for any $n \geq 1$. Let $\rho = r_1 + r_2$, $A = \max\{2\rho, 1\}$, and $\beta(s) \leq s/A$. Hence, by Lemma 2.3, we have

$$\begin{aligned}
\|z_t - x_n\|^2 &= \|z_t - S_N x_n + S_N x_n - x_n\|^2 \\
&\leq \|z_t - S_N x_n\|^2 + 2\langle S_N x_n - x_n, j(z_t - S_N x_n)\rangle \\
&\quad + \max\{\|z_t - S_N x_n\|, 1\}\|S_N x_n - x_n\|\beta(\|S_N x_n - x_n\|) \\
&\leq \|z_t - S_N x_n\|^2 + 2\langle S_N x_n - x_n, j(z_t - S_N x_n)\rangle \\
&\quad + \max\{\|z_t - x^*\| + \|x_n - x^*\|, 1\}\|S_N x_n - x_n\|\beta(\|S_N x_n - x_n\|) \\
&\leq \|z_t - S_N x_n\|^2 + 2\langle S_N x_n - x_n, j(z_t - S_N x_n)\rangle \\
&\quad + \max\{2\rho, 1\}\|S_N x_n - x_n\|\beta(\|S_N x_n - x_n\|)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|z_t - S_N x_n\|^2 + 2\langle S_N x_n - x_n, j(z_t - S_N x_n)\rangle + \|S_N x_n - x_n\|^2 \\
&\leq \|z_t - S_N x_n\|^2 + 2\|S_N x_n - x_n\|\|z_t - S_N x_n\| + \|S_N x_n - x_n\|^2 \\
&\leq \|z_t - S_N x_n\|^2 + M\|S_N x_n - x_n\| \\
&\leq \|S_N Q_K(1-t)z_t - S_N x_n\|^2 + M\|S_N x_n - x_n\| \\
&\leq \|Q_K(1-t)z_t - x_n\|^2 + M\|S_N x_n - x_n\| \\
&= \|(1-t)z_t - x_n\|^2 + M\|S_N x_n - x_n\| \\
&= \|z_t - x_n - t z_t\|^2 + M\|S_N x_n - x_n\| \\
&\leq \|z_t - x_n\|^2 - 2t\langle z_t, j(z_t - x_n)\rangle \\
&\quad + \max\{\|z_t - x_n\|, 1\} t\|z_t\|\beta(t\|z_t\|) + M\|Tx_n - x_n\| \\
&\leq \|z_t - x_n\|^2 - 2t\langle z_t, j(z_t - x_n)\rangle \\
&\quad + \max\{\|z_t - x^*\| + \|x_n - x^*\|, 1\} t\|z_t\|\beta(t\|z_t\|) + M\|Tx_n - x_n\| \\
&\leq \|z_t - x_n\|^2 - 2t\langle z_t, j(z_t - x_n)\rangle \\
&\quad + \max\{2\rho, 1\} t\|z_t\|\beta(t\|z_t\|) + M\|Tx_n - x_n\| \\
&\leq \|z_t - x_n\|^2 - 2t\langle z_t, j(z_t - x_n)\rangle + t^2\|z_t\|^2 + M\|Tx_n - x_n\| \\
&\leq \|z_t - x_n\|^2 - 2t\langle z_t, j(z_t - x_n)\rangle + t^2 M_1 + M\|Tx_n - x_n\|
\end{aligned}
\tag{4.7}
$$

for some $M > 0$ and $M_1 > 0$. Thus, $\langle z_t, j(z_t - x_n)\rangle \leq M_1 t/2 + (M/2t)\|S_N x_n - x_n\|$. Therefore,

$$
\lim_{t \to 0} \limsup_{n \to \infty} \langle z_t, j(z_t - x_n)\rangle \leq 0. \tag{4.8}
$$

Moreover,

$$
\begin{aligned}
\langle -z_t, j(x_n - z_t)\rangle &= \langle -x^*, j(x_n - x^*)\rangle + \langle -x^*, j(x_n - z_t)\rangle \\
&\quad - \langle -x^*, j(x_n - x^*)\rangle + \langle x^* - z_t, j(x_n - z_t)\rangle \\
&= \langle -x^*, j(x_n - x^*)\rangle + \langle -x^*, j(x_n - z_t) - j(x_n - x^*)\rangle \\
&\quad + \langle x^* - z_t, j(x_n - z_t)\rangle.
\end{aligned}
\tag{4.9}
$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded, we have that $\langle x^* - z_t, j(x_n - z_t)\rangle \to 0$ as $t \to 0$ and since j is norm-to-weak* uniformly continuous on bounded sets, we have $\langle -x^*, j(x_n - z_t) - j(x_n - x^*)\rangle \to 0$ as $t \to 0$. Using (4.8) and (4.9), we obtain

$$
\limsup_{n \to \infty} \langle -x^*, j(x_n - x^*)\rangle \leq 0. \tag{4.10}
$$

From (4.1), we have

$$
\begin{aligned}
\|y_n - x^*\| &= \|S_N Q_K (1 - \alpha_n) x_n - x^*\|^2 \leq \|(1 - \alpha_n) x_n - x^*\|^2 = \|x_n - x^* - \alpha_n x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\alpha_n \langle x_n, j(x_n - x^*) \rangle + \max\{1, 2\rho\} \alpha_n \|x_n\| \beta(\alpha_n \|x_n\|) \\
&\leq \|x_n - x^*\|^2 - 2\alpha_n \langle x_n, j(x_n - x^*) \rangle + \alpha_n^2 \|x_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\alpha_n \langle x_n - x^* + x^*, j(x^* - x_n) \rangle + \alpha_n^2 \|x_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\alpha_n \langle x^*, j(x^* - x_n) \rangle - 2\alpha_n \langle x^* - x_n, j(x^* - x_n) \rangle + \alpha_n^2 \|x_n\|^2 \\
&= \|x_n - x^*\|^2 + 2\alpha_n \langle x^*, j(x^* - x_n) \rangle + \alpha_n^2 \|x_n\|^2 - 2\alpha_n \|x_n - x^*\|^2 \\
&= (1 - \alpha_n) \|x_n - x^*\|^2 - \alpha_n \|x_n - x^*\|^2 + 2\alpha_n \langle x^*, j(x^* - x_n) \rangle + \alpha_n^2 \|x_n\|^2 \\
&\leq (1 - \alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle x^*, j(x^* - x_n) \rangle + \alpha_n^2 \|x_n\|^2 .
\end{aligned}
\tag{4.11}
$$

Also, from (4.1), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|y_n - x^*\|^2 \\
&\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \left[(1 - \alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle x^*, j(x^* - x_n) \rangle + \alpha_n^2 \|x_n\|^2 \right] \\
&\leq (1 - \alpha_n \beta_n) \|x_n - x^*\|^2 + 2\alpha_n \beta_n \langle x^*, j(x^* - x_n) \rangle + \alpha_n^2 \beta_n M_2 \\
&= (1 - \alpha_n \beta_n) \|x_n - x^*\|^2 + \alpha_n \beta_n \left[2\langle x^*, j(x^* - x_n) \rangle + \frac{\alpha_n}{\beta_n} M_2 \right],
\end{aligned}
\tag{4.12}
$$

where $M_2 := \sup_{n \geq 1} \|x_n\|^2$. Using Lemma 2.4, we get that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(S_N) = \bigcap_{r=1}^{N} N(A_r)$. This completes the proof. $\qquad\square$

If in Theorem 4.1, we consider $K = E$, the condition that each A_r, $r = 1, 2, \ldots, N$ is m-accretive may be replaced with continuity of each A_r. Thus, we have this theorem.

Theorem 4.2. *Let E be a real Banach space which is uniformly smooth and also uniformly convex. For each $r = 1, 2, \ldots, N$, let $A_r : E \to E$ be a continuous accretive mapping such that $\bigcap_{r=1}^{N} N(A_r) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0, 1)$. For an arbitrary $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$
\begin{aligned}
y_n &= (1 - \alpha_n) x_n, \\
x_{n+1} &= (1 - \beta_n) x_n + \beta_n S_N y_n, \quad n \geq 1,
\end{aligned}
\tag{4.13}
$$

where $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_N J_{A_N}$, with $J_{A_r} := (I + A_r)^{-1}$, $r = 1, 2, \ldots, N$ for $0 < a_i < 1, i = 0, 1, 2, \ldots, N$, $\sum_{i=0}^{N} a_i = 1$. Suppose that the following conditions are satisfied:

(a) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common zero of $\{A_r\}_{r=1}^{N}$ if E admits weakly sequentially continuous duality mapping j.

Proof. Take $Q_K = I$ in Theorem 4.1. By Lemma 2.5, we have that A_r is m-accretive for each $r = 1, 2, \ldots, N$. Then, the result follows from Theorem 4.1. $\square$

The following theorems give strong convergence to a common fixed point of a finite family of pseudocontractive mappings.

Theorem 4.3. *Let E be a real Banach space which is uniformly smooth and uniformly convex, and let K be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_K is the sunny nonexpansive retraction of E onto K. For each $r = 1, 2, \ldots, N$, let $T_r : K \to E$ be a pseudocontractive mapping such that $(I - T_r)$ is m-accretive on K with $\bigcap_{r=1}^{N} F(T_r) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0, 1)$ and $J_{T_r} := (2I - T_r)^{-1}$ for each $r = 1, 2, \ldots, N$. For an arbitrary $x_1 \in K$ let sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$y_n = Q_K[(1 - \alpha_n)x_n],$$
$$x_{n+1} = (1 - \beta_n)x_n + \beta_n S_N y_n, \quad n \geq 1, \tag{4.14}$$

where $S_N := a_0 I + a_1 J_{T_1} + a_2 J_{T_2} + \cdots + a_N J_{T_N}$ for $0 < a_i < 1, i = 0, 1, 2, \ldots, N, \sum_{i=0}^{N} a_i = 1$. Suppose that the following conditions are satisfied:

(a) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_r\}_{r=1}^{N}$ if E admits weakly sequentially continuous duality mapping j.

Proof. Let $A_r := (I - T_r)$ for each $r = 1, 2, \ldots, N$. Then, clearly, $F(T_r) = N(A_r)$ and hence $\bigcap_{r=1}^{N} N(A_r) = \bigcap_{r=1}^{N} F(T_r) \neq \emptyset$. Furthermore, each A_r for $r = 1, 2, \ldots, N$ is m-accretive. The result follows from Theorem 4.1. $\square$

Theorem 4.4. *Let E be a real Banach space which is uniformly smooth and uniformly convex. For each $r = 1, 2, \ldots, N$, let $T_r : K \to E$ be a continuous pseudocontractive mapping on E such that $\bigcap_{r=1}^{N} F(T_r) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0, 1)$ and $J_{T_r} := (2I - T_r)^{-1}$ for each $r = 1, 2, \ldots, N$. For arbitrary $x_1 \in K$ let sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$y_n = (1 - \alpha_n)x_n,$$
$$x_{n+1} = (1 - \beta_n)x_n + \beta_n S_N y_n, \quad n \geq 1, \tag{4.15}$$

where $S_N := a_0 I + a_1 J_{T_1} + a_2 J_{T_2} + \cdots + a_N J_{T_N}$ for $0 < a_i < 1, i = 0, 1, 2, \ldots, N, \sum_{i=0}^{N} a_i = 1$. Suppose

that the following conditions are satisfied:

    (a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

    (b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a common fixed point of* $\{T_r\}_{r=1}^{N}$ *if E admits weakly sequentially continuous duality mapping j.*

Proof. The proof follows from Theorem 4.2.   □

Acknowledgments

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