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Research Article

The Essential Norm of the Generalized Hankel Operators on the Bergman Space of the Unit Ball in C^n

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In 1993, Peloso introduced a kind of operators on the Bergman space $A^2(B)$ of the unit ball that generalizes the classical Hankel operator. In this paper, we estimate the essential norm of the generalized Hankel operators on the Bergman space $A^p(B)$ $(p > 1)$ of the unit ball and give an equivalent form of the essential norm.

1. Introduction

Let B be the open unit ball in C^n, m the Lebesgue measure on C^n normalized so that $m(B) = 1$, $H(B)$ denotes the class of all holomorphic functions on B. The Bergman space $A^2(B)$ is the Banach space of all holomorphic functions f on B such that $\int_B |f(z)|^2 dm(z) < \infty$. It is easy to show that $A^2(B)$ is a closed subspace of $L^2(B, dm)$.

There is an orthogonal projection of $L^2(B, dm)$ onto $A^2(B)$, denoted by P and

$$Pf(z) = \int_B K(z,w) f(w) dm(w), \tag{1.1}$$

where $K(z,w) = 1/(1 - \langle z,w\rangle)^{n+1}$ is the Bergman kernel on B.

For a function $f \in H(B)$, define the Hankel operator $H_f : A^2(B) \to A^2(B)^\perp$ with symbol f by

$$H_f g = (I - P)\left(\overline{f} g\right) = \int_B \overline{(f(z) - f(w))} K(z,w) g(w) dm(w), \tag{1.2}$$

where I is the identity operator.

Since the Hankel operator H_f is connected with the Toeplitz operator, the commutator, the Bloch space, and the Besov space, it has been extensively studied. Important papers in this context are [1, 2] for the case $n = 1$ and [3–5] for the case $n > 1$. It is known that H_f is bounded on $A^2(B)$ if and only if $f \in \beta(B)$ and H_f is compact $A^2(B)$ if and only if $f \in \beta_0(B)$, where

$$\beta(B) = \left\{ f \in H(B) : \sup_{z\in B}\left(1 - |z|^2\right)\left|Rf(z)\right| < \infty \right\},$$
$$\beta_0(B) = \left\{ f \in H(B) : \left(1 - |z|^2\right)\left|Rf(z)\right| \longrightarrow 0, \text{as } |z| \longrightarrow 1 \right\}. \tag{1.3}$$

Rf is the radial derivative of f defined by

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}. \tag{1.4}$$

$\beta(B)$ is called the Bloch space, and $\beta_0(B)$ is called the little Bloch space.

For $n = 1$, $f \in H(D)$ (D is the open unit disc), H_f is in the Schatten class S_p $(1 < p < \infty)$ if and only if $f \in B_p(D)$; $H_f \in S_p$ $(0 < p \le 1)$ if and only if f is a constant, where

$$B_p(D) = \left\{ f \in H(D) : f'(z)\left(1 - |z|^2\right) \in L^p(d\lambda) \right\}, \quad p > 1, \tag{1.5}$$

and $d\lambda(z) = (1 - |z|^2)^{-2} dm(z)$ is the invariant volume measure on D, $B_p(D)$ is called the Besov space on D. This theorem expresses that there is a cutoff of H_f at $p = 1$.

For $n > 1$, $f \in H(B)$, $H_f \in S_p$ $(2n < p < \infty)$ if and only if $f \in B_p(B)$, $H_f \in S_p$ $(0 < p \le 2n)$ if and only if f is a constant, where

$$B_p(B) = \left\{ f \in H(B) : \left(1 - |z|^2\right) Rf(z) \in L^p(d\lambda) \right\}, \quad p > n, \tag{1.6}$$

and $d\lambda(z) = (1 - |z|^2)^{-(n+1)} dm(z)$ is the invariant volume measure on B. $B_p(B)$ is called the Besov space on B. Then, the cutoff phenomenon of H_f appears at $p = 2n$. If $c(n)$ denotes the value of "cutoff," then

$$c(n) = \begin{cases} 1, & n = 1, \\ 2n, & n > 1. \end{cases} \tag{1.7}$$

Obviously, $c(n)$ depends on the dimension n of the unit ball.

In 1993, Peloso [3] replaced $f(z) - f(w)$ with

$$\Delta_j f(w, z) = f(w) - \sum_{|\alpha|<j} \frac{D^\alpha f(z)}{\alpha!}(w - z)^\alpha \tag{1.8}$$

to define a kind of generalized Hankel operator:

$$H_{f,j}g(z) = \int_B \overline{-\Delta_j f(w,z)} K(z,w) g(w) dm(w),$$
$$H'_{f,j}g(z) = \int_B \overline{\Delta_j f(z,w)} K(z,w) g(w) dm(w). \quad (1.9)$$

Here, $(D^\alpha f)(z) = (\partial^{|\alpha|} f(z))/(\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n})$. Clearly, if $j = 1$, $H_{f,1}$ and $H'_{f,1}$ are just the classical Hankel operator H_f. He proved that $H_{f,j}$ has the same boundedness and compactness properties as H_f, but the Schatten class property of $H_{f,j}$ is different from that of H_f. If $n \geq 2$, $f \in H(B)$, $H_{f,j} \in S_p((2n/j) < p < \infty)$ if and only if $f \in B_p(B)$; if $0 < p \leq (2n/j)$, $H_{f,j} \in S_p$ if and only if f is a polynomial of degree at most $j - 1$. So the value of "cutoff" of $H_{f,j}$ is $2n/j$; this means that the cutoff constant $c(n)$ depends not only on the dimension but also on the degree of the polynomial

$$\sum_{|\alpha|<j} \frac{D^\alpha f(z)}{\alpha!} (w - z)^\alpha \ , \quad (1.10)$$

and we are able to lower the cutoff constant by increasing j.

The cutoff phenomenon expressed that the generalized Hankel operator $H_{f,j}$ defined by Peloso and the classical Hankel operator H_f are different.

In the present paper, we will consider the generalized Hankel operators $H_{f,j}$ defined by Peloso on the Bergman space $A^p(B)$ which is the Banach space of all holomorphic functions f on B such that $\int_B |f(z)|^p dm(z) < \infty$, for $p > 1$.

For $f(z) \in H(B)$, j is a positive integer, and we define the generalized Hankel operators $H_{f,j}$ and $H'_{f,j}$ of order j with symbol f by

$$H_{f,j}g(z) = \int_B \overline{-\Delta_j f(w,z)} K(z,w) g(w) dm(w),$$
$$H'_{f,j}g(z) = \int_B \overline{\Delta_j f(z,w)} K(z,w) g(w) dm(w), \quad (1.11)$$

where $g \in A^p(B)$,

$$\Delta_j f(w,z) = f(w) - \sum_{|\alpha|<j} \frac{D^\alpha f(z)}{\alpha!} (w - z)^\alpha,$$
$$\Delta_j f(z,w) = f(w) - \sum_{|\alpha|<j} \frac{D^\alpha f(w)}{\alpha!} (z - w)^\alpha. \quad (1.12)$$

Luo and Ji-Huai [6] studied the boundedness, compactness, and the Schatten class property of the generalized Hankel operator $H_{f,j}$ on the Bergman space $A^p(B)$ $(p > 1)$, which extended the known results.

We will study the essential norm of this kind of generalized Hankel operators $H_{f,j}$ and $H'_{f,j}$. We recall that the essential norm of a bounded linear operator T is the distance from T to the compact operators; that is,

$$\|T\|_{\mathrm{ess}} = \inf\{\|T - K\| : K \text{ is a compact operator}\}. \tag{1.13}$$

The essential norm of a bounded linear operator T is connected with the compactness of the operator T and the spectrum of the operator T.

We know that $\|T\|_{\mathrm{ess}} = 0$ if and only if T is compact, so that estimates on $\|T\|_{\mathrm{ess}}$ lead to conditions for T to be compact. Thus, we will obtain a different proof of the compactness of the generalized Hankel operators $H_{f,j}$ and $H'_{f,j}$.

Throughout the paper, C denotes a positive constant, whose value may change from one occurrence to the next one.

2. Preliminaries

For any fixed point $a \in B - \{0\}$, $z \in B$, define the Möbius transformation φ_a by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \tag{2.1}$$

where $s_a = \sqrt{1 - |a|^2}$ and P_a is the orthogonal projection from C^n onto the one-dimensional subspace $[a]$ generated by a, Q_a is the orthogonal projection from C^n onto $C^n ! [a]$. It is clear that

$$\begin{aligned} P_a(z) &= \frac{\langle z, a\rangle}{|a|^2} a, \quad z \in C^n, \\ Q_a(z) &= z - \frac{\langle z, a\rangle}{|a|^2} a, \quad z \in B. \end{aligned} \tag{2.2}$$

Lemma 2.1. *For every $a \in B$, φ_a has the following properties:*

(1) $\varphi_a(0) = a$ *and* $\varphi_a(a) = 0$,

(2) $\varphi_a \circ \varphi_a(z) = z,\ z \in B$,

(3) $1/(1 - \langle \varphi_a(z), a\rangle) = (1 - \langle z, a\rangle)/(1 - |a|^2),\ z \in B$.

Proof. The proofs can be found in [7]. □

Lemma 2.2. *For $s > -1$, t real, define*

$$I_t(z) = \int_B \frac{\left(1 - |w|^2\right)^s}{|1 - \langle z, w\rangle|^{n+1+s+t}} dm(w), \quad z \in B. \tag{2.3}$$

Then,

(1) $t < 0$, $I_t(z)$ *is bounded in* B,

(2) $t = 0$, $I_t(z) \sim \log(1/(1-|z|^2))$ *as* $|z| \to 1^-$,

(3) $t > 0$, $I_t(z) \sim (1-|z|^2)^{-t}$ *as* $|z| \to 1^-$.

Here, the notation $a(z) \sim b(z)$ means that the ratio $a(z)/b(z)$ has a positive finite limit as $|z| \to 1^-$.

Proof. This is in [7, Theorem 1.12]. □

Lemma 2.3. *Let* $k_\xi(z) = K(z,\xi)/\|K_\xi\|_{L^p(dm)}$, *where* $K_\xi(z) = K(z,\xi) = 1/(1-\langle z,\xi\rangle)^{n+1}$, *then* $k_\xi(z)$ *has the following properties:*

(1) $\|k_\xi\|_{L^p(dm)} = 1$,

(2) $k_\xi(z) \to 0$ *at every point* $z \in B$ *as* $|\xi| \to 1^-$.

Proof. It is obvious. □

Lemma 2.4. *Let* $K_\xi(z) = K(z,\xi)$. *Then, for any positive integer* j,

(1) $H_{f,j}K_\xi = \overline{-\Delta_j f(\xi,\cdot)}K_\xi$,

(2) $H'_{f,j}K_\xi = \overline{\Delta_j f(\cdot,\xi)}K_\xi$.

Proof. The proof is obtained by the definition of $H_{f,j}$ and $H'_{f,j}$ and the reproducing property of $K(z,\xi)$, through the direct computation to get them. □

Lemma 2.5. *Let* j *be any positive integer,* $f \in H(B)$, *and* $0 < q < \infty$, *then there is a constant* C *independent of* f, *such that*

(1) $(1-|z|^2)^j|R^j f(z)| \le C\{\int_B |\Delta_j f(\varphi_z(w),z)|^q dm(w)\}^{1/q}$,

(2) $(1-|z|^2)^j|R^j f(z)| \le C\{\int_B |\Delta_j f(z,\varphi_z(w))|^q dm(w)\}^{1/q}$,

where $R^j f$ is the jth order radial derivative of f,

$$R^j f(z) = \sum_{k=1}^{\infty} k^j f_k(z), \tag{2.4}$$

and $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion.

Proof. This is in [3, Proposition 3.2]. □

Lemma 2.6. *Let* j *be any positive integer,* $f \in H(B)$, *and* $0 < \rho < 1$, $p > 1$, *then*

(1) $\int_B |\Delta_j f(w,z)|^p((1-|w|^2)^{-\rho}/|1-\langle z,w\rangle|^{n+1})dm(w) \le C(1-|z|^2)^{-\rho}(\sup_{z\in B}(1-|z|^2)^j|R^j f(z)|)^p$,

(2) $\int_B |\Delta_j f(z,w)|^p((1-|w|^2)^{-\rho}/|1-\langle z,w\rangle|^{n+1})dm(w) \le C(1-|z|^2)^{-\rho}(\sup_{z\in B}(1-|z|^2)^j|R^j f(z)|)^p$.

Proof. (1) Write $F(w,z)$ for $\Delta_j f(w,z)$. Using the change of variables $w=\varphi_z(\xi)$, we obtain

$$\begin{aligned}
&\int_B |F(w,z)|^p \frac{\left(1-|w|^2\right)^{-\rho}}{|1-\langle z,w\rangle|^{n+1}} dm(w)\\
&\quad= \int_B \left|F(\varphi_z(\xi),z)\right|^p \frac{\left(1-\left|\varphi_z(\xi)\right|^2\right)^{-\rho}}{\left|1-\langle z,\varphi_z(\xi)\rangle\right|^{n+1}} \frac{\left(1-|z|^2\right)^{n+1}}{|1-\langle \xi,z\rangle|^{2(n+1)}} dm(\xi) \qquad (*)\\
&\quad= \left(1-|z|^2\right)^{-\rho} \int_B \left|F(\varphi_z(\xi),z)\right|^p \frac{\left(1-|\xi|^2\right)^{-\rho}}{|1-\langle \xi,z\rangle|^{n+1-2\rho}} dm(\xi).
\end{aligned}$$

Let

$$1 < q' < min\left(\frac{1}{\rho}, \frac{n+1}{n+1-\rho}\right) \tag{2.5}$$

and set $q = q'/(q'-1)$. Then, applying Hölder's inequality to $(*)$, we obtain

$$\begin{aligned}
&\int_B |F(w,z)|^p \frac{\left(1-|w|^2\right)^{-\rho}}{|1-\langle z,w\rangle|^{n+1}} dm(w)\\
&\quad\le \left(1-|z|^2\right)^{-\rho} \left(\int_B \left|F(\varphi_z(\xi),z)\right|^{pq} dm(\xi)\right)^{1/q} \left(\int_B \frac{\left(1-|\xi|^2\right)^{-\rho q'}}{|1-\langle \xi,z\rangle|^{(n+1-2\rho)q'}} dm(\xi)\right)^{1/q'}.
\end{aligned} \tag{2.6}$$

Because of our choice of q', it follows that $-\rho q' > -1$ and $(n+1-2\rho)q' < n+1-\rho q'$. Now, Lemma 2.2 implies that

$$\int_B \frac{\left(1-|\xi|^2\right)^{-\rho q'}}{|1-\langle \xi,z\rangle|^{(n+1-2\rho)q'}} dm(\xi) \tag{2.7}$$

is bounded by a constant. Therefore, applying [3, Theorem 3.4], we get

$$\int_B \left|\Delta_j f(w,z)\right|^p \frac{\left(1-|w|^2\right)^{-\rho}}{|1-\langle z,w\rangle|^{n+1}} dm(w) \le C\left(1-|z|^2\right)^{-\rho} \left(\sup_{z\in B}\left(1-|z|^2\right)^j \left|R^j f(z)\right|\right)^p. \tag{2.8}$$

(2) The proof of (2) is similar to that of (1). □

3. The Main Result and Its Proof

Theorem 3.1. *Let $f \in H(B)$, j any positive integer, $p > 1$, and the generalized Hankel operators $H_{f,j}$, $H'_{f,j}$ defined on $A^p(B)$ by*

$$H_{f,j}g(z) = \int_B \overline{-\Delta_j f(w,z)} K(z,w) g(w) dm(w),$$
$$H'_{f,j}g(z) = \int_B \overline{\Delta_j f(z,w)} K(z,w) g(w) dm(w). \tag{3.1}$$

Suppose that $H_{f,j}$ and $H'_{f,j}$ are bounded on $A^p(B)$, then the following quantities are equivalent:

(1) $\|H_{f,j}\|_{ess}$ *and* $\|H'_{f,j}\|_{ess}$,

(2) $\overline{\lim}_{|z|\to 1^-}(1-|z|^2)^j|R^j f(z)|$,

(3) $\overline{\lim}_{|z|\to 1^-}(1-|z|^2)|Rf(z)|$.

Particularly, $H_{f,j}$ and $H'_{f,j}$ are compact on $A^p(B)$ if and only if $\overline{\lim}_{|z|\to 1^-}(1-|z|^2)|Rf(z)| = 0$.

Proof. First, we will prove that $\|H_{f,j}\|_{\text{ess}} \ge C\,\overline{\lim}_{|z|\to 1^-}(1-|z|^2)^j|R^j f(z)|$. By the definition of $k_\xi(z)$ of Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
\|H_{f,j}k_\xi\|^p_{L^p(dm)} &= \int_B |H_{f,j}k_\xi(z)|^p dm(z) \\
&= \int_B \left| H_{f,j}\frac{K(z,\xi)}{\|K_\xi\|_{L^p(dm)}}\right|^p dm(z) \\
&= \frac{1}{\|K_\xi\|^p_{L^p(dm)}}\int_B |H_{f,j}K(z,\xi)|^p dm(z) \\
&= \frac{1}{\|K_\xi\|^p_{L^p(dm)}}\int_B |\Delta_j f(\xi,z)|^p |K(z,\xi)|^p dm(z) \\
&= \frac{1}{\|K_\xi\|^p_{L^p(dm)}}\cdot I,
\end{aligned} \tag{3.2}$$

here $I = \int_B |\Delta_j f(\xi,z)|^p|K(z,\xi)|^p dm(z)$.

Use the change of variables $z = \varphi_\xi(\tau)$ in the integral I, and recall that

$$dm(z) = \left(\frac{1-|\xi|^2}{|1-\langle \tau,\xi\rangle|^2}\right)^{n+1} dm(\tau). \tag{3.3}$$

Thus

$$
\begin{aligned}
I &= \int_B \frac{\left|\Delta_j f(\xi,\varphi_\xi(\tau))\right|^p}{\left|1-\langle\varphi_\xi(\tau),\xi\rangle\right|^{p(n+1)}}\cdot\left(\frac{1-|\xi|^2}{|1-\langle\tau,\xi\rangle|^2}\right)^{n+1} dm(\tau)\\
&= \frac{1}{\left(1-|\xi|^2\right)^{(n+1)(p-1)}}\int_B \frac{\left|\Delta_j f(\xi,\varphi_\xi(\tau))\right|^p}{|1-\langle\tau,\xi\rangle|^{(2-p)(n+1)}} dm(\tau)\\
&\geq \frac{1}{\left(1-|\xi|^2\right)^{(n+1)(p-1)}}\left(\int_B \left|\Delta_j f(\xi,\varphi_\xi(\tau))\right| dm(\tau)\right)^p\\
&\quad\times\left(\int_B \frac{1}{|1-\langle\tau,\xi\rangle|^{(2-p)(n+1)/(1-p)}} dm(\tau)\right)^{1-p}\\
&\geq \frac{C}{\left(1-|\xi|^2\right)^{(n+1)(p-1)}}\left(\int_B \left|\Delta_j f(\xi,\varphi_\xi(\tau))\right| dm(\tau)\right)^p\\
&\geq \frac{C}{\left(1-|\xi|^2\right)^{(n+1)(p-1)}}\left[\left(1-|\xi|^2\right)^j\left|R^j f(\xi)\right|\right]^p.
\end{aligned}
\tag{3.4}
$$

Here, we have used (3) of Lemma 2.1, Hölder's inequality for the indexes p and $p/(p-1)$, (1) of Lemma 2.2, and (2) of Lemma 2.5.

Therefore,

$$
\begin{aligned}
\|H_{f,j}k_\xi\|_{L^p(dm)}^p &\geq \frac{1}{\|K_\xi\|_{L^p(dm)}^p}\frac{C}{\left(1-|\xi|^2\right)^{(n+1)(p-1)}}\left[\left(1-|\xi|^2\right)^j\left|R^j f(\xi)\right|\right]^p\\
&\geq C\left(1-|\xi|^2\right)^{(n+1)(p-1)}\frac{1}{\left(1-|\xi|^2\right)^{(n+1)(p-1)}}\left[\left(1-|\xi|^2\right)^j\left|R^j f(\xi)\right|\right]^p\\
&= C\left[\left(1-|\xi|^2\right)^j\left|R^j f(\xi)\right|\right]^p.
\end{aligned}
\tag{3.5}
$$

So $\|H_{f,j}k_\xi\|_{L^p(dm)} \geq C(1-|\xi|^2)^j|R^j f(\xi)|$.

For any compact operator T, by (2) of Lemma 2.3, we have $\|Tk_\xi\|_{L^p(dm)} \to 0$ as $|\xi| \to 1^-$. Then,

$$\begin{aligned}\|H_{f,j} - T\| &\geq \overline{\lim_{|\xi|\to 1^-}} \|(H_{f,j} - T)k_\xi\|_{L^p(dm)} \\ &\geq \overline{\lim_{|\xi|\to 1^-}} \left(\|H_{f,j}k_\xi\|_{L^p(dm)} - \|Tk_\xi\|_{L^p(dm)}\right) \\ &= \overline{\lim_{|\xi|\to 1^-}} \|H_{f,j}k_\xi\|_{L^p(dm)} \\ &\geq C\overline{\lim_{|\xi|\to 1^-}} \left(1-|\xi|^2\right)^j \left|R^j f(\xi)\right|.\end{aligned} \tag{3.6}$$

Thus, $\|H_{f,j}\|_{\text{ess}} \geq C\,\overline{\lim}_{|z|\to 1^-}(1-|z|^2)^j|R^j f(z)|$.

Now, we will prove that $\|H_{f,j}\|_{\text{ess}} \leq C\,\overline{\lim}_{|z|\to 1^-}(1-|z|^2)^j|R^j f(z)|$.

Write $F(z,w)$ for $-\Delta_j f(z,w)$. For $0<\rho<1$ and $g \in A^p(B)$, let $B(0,\rho)$ and $B(0;\rho,1)$ denote the ball $|z| \leq \rho$ and the ring $\rho < |z| < 1$, respectively, then we have

$$\begin{aligned}H_{f,j}g(z) &= \chi_{B(0,\rho)}(z)\int_B \overline{F(w,z)}K(z,w)g(w)dm(w) \\ &\quad + \chi_{B(0;\rho,1)}(z)\int_B \overline{F(w,z)}K(z,w)g(w)dm(w) \\ &= T_1 g(z) + T_2 g(z).\end{aligned} \tag{3.7}$$

Here,

$$\begin{aligned}T_1 g(z) &= \chi_{B(0,\rho)}(z)\int_B \overline{F(w,z)}K(z,w)g(w)dm(w), \\ T_2 g(z) &= \chi_{B(0;\rho,1)}(z)\int_B \overline{F(w,z)}K(z,w)g(w)dm(w).\end{aligned} \tag{3.8}$$

We first show that T_1 is compact. Let $\{g_l\}$ be a sequence weakly converging to 0 and $p' = p/(p-1)$, by Hölder's inequality, then we have

$$\begin{aligned}\left|T_1 g_l(z)\right|^p &= \left|\chi_{B(0,\rho)}(z)\int_B \overline{F(w,z)}K(z,w)g_l(w)dm(w)\right|^p \\ &\leq \chi_{B(0,\rho)}(z)\left(\int_B |F(w,z)|\frac{\left|g_l(w)\right|}{|1-\langle z,w\rangle|^{n+1}}dm(w)\right)^p \\ &\leq \chi_{B(0,\rho)}(z)\left(\int_B \frac{|F(w,z)|^{p'}\left(1-|w|^2\right)^{-1/p}}{|1-\langle z,w\rangle|^{n+1}}dm(w)\right)^{p/p'} \\ &\quad \times \int_B \frac{\left|g_l(w)\right|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w).\end{aligned} \tag{3.9}$$

By Lemma 2.6, we get

$$\begin{aligned}\left|T_1 g_l(z)\right|^p \leq C\chi_{B(0,\rho)}(z)\left(\sup_{z\in B}\left(1-|z|^2\right)^j\left|R^j f(z)\right|\right)^p \\ \times\left(1-|z|^2\right)^{-1/p'}\int_B \frac{\left|g_l(w)\right|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w).\end{aligned} \tag{3.10}$$

Thus,

$$\begin{aligned}\left\|T_1 g_l\right\|^p_{L^p(dm)} &= \int_B \left|T_1 g_l(z)\right|^p dm(z) \\ &\leq C\int_{|z|<\rho}\left(\sup_{z\in B}\left(1-|z|^2\right)^j\left|R^j f(z)\right|\right)^p\left(1-|z|^2\right)^{-1/p'} \\ &\quad\times\int_B \frac{\left|g_l(w)\right|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w)dm(z) \\ &\leq C\left(1-|\rho|^2\right)^{-1/p'}\left(\sup_{z\in B}\left(1-|z|^2\right)^j\left|R^j f(z)\right|\right)^p \\ &\quad\times\int\int_B \frac{\left|g_l(w)\right|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w)dm(z) \\ &= C\left(1-|\rho|^2\right)^{-1/p'}\left(\sup_{z\in B}\left(1-|z|^2\right)^j\left|R^j f(z)\right|\right)^p \\ &\quad\cdot\int_B \left|g_l(w)\right|^p\left(1-|w|^2\right)^{1/p'}\int_B \frac{1}{|1-\langle z,w\rangle|^{n+1}}dm(z)dm(w) \\ &\leq C\left(1-|\rho|^2\right)^{-1/p'}\left(\sup_{z\in B}\left(1-|z|^2\right)^j\left|R^j f(z)\right|\right)^p \\ &\quad\times\int_B \left|g_l(w)\right|^p\left(1-|w|^2\right)^{1/p'}\log\left(1-|w|^2\right)dm(w) \\ &\longrightarrow 0, \quad \text{as } l\longrightarrow\infty.\end{aligned} \tag{3.11}$$

So, T_1 is compact.

For $g\in A^p$ and $p' = p/(p-1)$, by Hölder's inequality,

$$\begin{aligned}\left|T_2 g(z)\right|^p &= \left|\chi_{B(0;\rho,1)}(z)\int_B \overline{F(w,z)}K(z,w)g(w)dm(w)\right|^p \\ &\leq \left(\int_B \chi_{B(0;\rho,1)}(z)|F(w,z)|\frac{\left|g(w)\right|}{|1-\langle z,w\rangle|^{n+1}}dm(w)\right)^p\end{aligned}$$

$$\le \left(\int_B \chi_{B(0;\rho,1)}(z)\frac{|F(w,z)|^{p'}\left(1-|w|^2\right)^{-1/p}}{|1-\langle z,w\rangle|^{n+1}}dm(w)\right)^{p/p'}$$
$$\times\int_B \frac{\left|g(w)\right|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w). \tag{3.12}$$

So

$$\|T_2g\|^p_{L^p(dm)}=\int_B\left|T_2g(z)\right|^p dm(z)\le\int_B\left(\int_B \chi_{B(0;\rho,1)}(z)\frac{|F(w,z)|^{p'}\left(1-|w|^2\right)^{-1/p}}{|1-\langle z,w\rangle|^{n+1}}dm(w)\right)^{p/p'}$$
$$\cdot\int_B \frac{\left|g(w)\right|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w)dm(z). \tag{3.13}$$

Change the variables $w=\varphi_z(\xi)$, let

$$1<q'<min\left(p,\frac{n+1}{n+1-1/p}\right), \tag{3.14}$$

and set $q=q'/(q'-1)$, by Lemmas 2.1 and 2.2, then we obtain

$$\int_B \chi_{B(0;\rho,1)}(z)\frac{|F(w,z)|^{p'}(1-|w|^2)^{-1/p}}{|1-\langle z,w\rangle|^{n+1}}\ dm(w)$$
$$=\int_B \chi_{B(0;\rho,1)}(z)\frac{\left|F(\varphi_z(\xi),z)\right|^{p'}\left(1-\left|\varphi_z(\xi)\right|^2\right)^{-1/p}}{\left|1-\langle z,\varphi_z(\xi)\rangle\right|^{n+1}}\frac{\left(1-|z|^2\right)^{n+1}}{|1-\langle \xi,z\rangle|^{2(n+1)}}dm(\xi)$$
$$=\int_B \chi_{B(0;\rho,1)}(z)\left(1-|z|^2\right)^{-1/p}\left|F(\varphi_z(\xi),z)\right|^{p'}\frac{\left(1-|\xi|^2\right)^{-1/p}}{|1-\langle \xi,z\rangle|^{n+1-2/p}}dm(\xi)$$
$$\le\left(1-|z|^2\right)^{-1/p}\left(\int_B \chi_{B(0;\rho,1)}(z)\left|F(\varphi_z(\xi),z)\right|^{p'q}dm(\xi)\right)^{1/q} \tag{3.15}$$
$$\times\left(\int_B\frac{\left(1-|\xi|^2\right)^{-q'/p}}{|1-\langle \xi,z\rangle|^{(n+1-2/p)q'}}dm(\xi)\right)^{1/q'}$$
$$\le C\left(1-|z|^2\right)^{-1/p}\left(\int_B \chi_{B(0;\rho,1)}(z)\left|F(\varphi_z(\xi),z)\right|^{p'q}dm(\xi)\right)^{1/q}.$$

By the same argument of [3, Theorem 3.4], we know that

$$\left(\int_B \chi_{B(0;\rho,1)}(z)\left|F\left(\varphi_z(\xi),z\right)\right|^{p'q}dm(\xi)\right)^{1/q} \le C\left(\sup_{\rho<|z|<1}\left(1-|z|^2\right)^j\left|R^jf(z)\right|\right)^{p'}. \tag{3.16}$$

Applying Fubini's theorem and Lemma 2.2, we have

$$\begin{aligned}\|T_2g\|^p_{L^p(dm)} &\le C\left(\sup_{\rho<|z|<1}\left(1-|z|^2\right)^j\left|R^jf(z)\right|\right)^p \\ &\quad\times\int_B\left(1-|z|^2\right)^{-1/p'}\int_B\frac{|g(w)|^p\left(1-|w|^2\right)^{1/p'}}{|1-\langle z,w\rangle|^{n+1}}dm(w)dm(z) \\ &\le C\left(\sup_{\rho<|z|<1}\left(1-|z|^2\right)^j\left|R^jf(z)\right|\right)^p\|g\|^p_{L^p(dm)}.\end{aligned} \tag{3.17}$$

So

$$\|T_2\| \le C\sup_{\rho<|z|<1}\left(1-|z|^2\right)^j\left|R^jf(z)\right|. \tag{3.18}$$

Thus, by the definition of the essential norm, we have

$$\left\|H_{f,j}\right\|_{\text{ess}} \le \|T_1+T_2\|_{\text{ess}} \le \|T_2\| \le C\sup_{\rho<|z|<1}\left(1-|z|^2\right)^j\left|R^jf(z)\right|. \tag{3.19}$$

As $\rho\to 1$, we have

$$\left\|H_{f,j}\right\|_{\text{ess}} \le C\,\overline{\lim_{|z|\to1^-}}\left(1-|z|^2\right)^j\left|R^jf(z)\right|. \tag{3.20}$$

Similarly, we get the equality of $\|H'_{f,j}\|_{\text{ess}}$ and $\overline{\lim}_{|z|\to1^-}(1-|z|^2)^j|R^jf(z)|$.

By [7, Theorems 3.4 and 3.5], we obtain the equality of $\overline{\lim}_{|z|\to1^-}(1-|z|^2)^j|R^jf(z)|$ and $\overline{\lim}_{|z|\to1^-}(1-|z|^2)|Rf(z)|$.

We complete the proof of Theorem 3.1. □

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