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Research Article

Weighted Anisotropic Integral Representations of Holomorphic Functions in the Unit Ball of C^n

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We obtain weighted integral representations for spaces of functions holomorphic in the unit ball B_n and belonging to area-integrable weighted L^p-classes with "anisotropic" weight function of the type $\prod_{i=1}^{n}(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2)^{\alpha_i}$, $w=(w_1,w_2,\ldots,w_n)\in B_n$. The corresponding kernels of these representations are estimated, written in an integral form, and even written out in an explicit form (for $n=2$).

1. Introduction

Denote by B_n the unit ball in the complex n-dimensional space $C^n : B_n = \{w \in C^n : |w| < 1\}$. For $1 \le p < +\infty$ and $\alpha > -1$, denote by $H^p_\alpha(B_n)$ the space of all functions f holomorphic in B_n and satisfying the condition

$$\int_{B_n} |f(w)|^p \left(1-|w|^2\right)^{\alpha} dm(w) < +\infty, \tag{1.1}$$

where dm is the Lebesgue measure in $C^n \equiv R^{2n}$. Further, for a complex number β with $\operatorname{Re}\beta > -1$, put

$$c_n(\beta) = \frac{\Gamma(n+1+\beta)}{\pi^n \cdot \Gamma(1+\beta)}. \tag{1.2}$$

We have the following theorem.

Theorem 1.1. *Assume that $1 \le p < +\infty, \alpha > -1$, and that the complex number β satisfies the condition*

$$\operatorname{Re}\beta \ge \alpha, \qquad p = 1,$$
$$\operatorname{Re}\beta > \frac{\alpha+1}{p} - 1, \qquad 1 < p < \infty. \tag{1.3}$$

Then each function $f \in H^p_\alpha(B_n)$ admits the following integral representations:

$$f(z) = c_n(\beta) \cdot \int_{B_n} \frac{f(w)\left(1-|w|^2\right)^\beta}{(1-\langle z,w\rangle)^{n+1+\beta}} dm(w), \quad z \in B_n,$$
$$\overline{f(0)} = c_n(\beta) \cdot \int_{B_n} \frac{\overline{f(w)}\left(1-|w|^2\right)^\beta}{(1-\langle z,w\rangle)^{n+1+\beta}} dm(w), \quad z \in B_n, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ is the Hermitean inner product in C^n.

For $n = 1$, that is, for the case of the unit disc $D \subset C$, this theorem was established in [1, 2], where the formulas (1.4) are important in the theory of factorization of meromorphic functions in the unit disc.

For $n > 1$, the theorem was proved in [3] (when $\alpha = 0$) and in [4, 5] (when $\alpha > -1$).

In monographs [6, 7], one can find numerous applications of the formulas (1.4) in the complex analysis.

In the present paper, we generalize Theorem 1.1 in the following way.

Assume that $1 \le p < +\infty$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in R^n$ satisfies the conditions

$$\begin{gathered} \alpha_n > -1, \\ \alpha_n + \alpha_{n-1} > -2, \\ \alpha_n + \alpha_{n-1} + \alpha_{n-2} > -3, \\ \vdots \\ \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_2 > -(n-1), \\ \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_2 + \alpha_1 > -n. \end{gathered} \tag{1.5}$$

Then we introduce the spaces $H^p_\alpha(B_n)$ of functions f holomorphic in B_n and satisfying the condition

$$\int_{B_n} |f(w)|^p \cdot \prod_{i=1}^n \left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) < +\infty. \tag{1.6}$$

Section 3 contains detailed investigation of these spaces with "anisotropic" weight function.

For these "anisotropic" spaces, similarities of the integral representations (1.4) are obtained, but this time a special kernels $S_{\beta_1\beta_2\cdots\beta_n}(z;w) \equiv S_\beta(z;w)$ (where β_i, $1 \leq i \leq n$, are associated with $\alpha_i, 1 \leq i \leq n$, and p in a special way) appear instead of $c_n(\beta) \cdot (1-\langle z,w\rangle)^{-(n+1+\beta)}$ (Theorem 4.7). Theorem 4.5 gives the description (in a multiple series form) and the main properties of these kernels. Theorem 4.8 makes it possible to represent the kernels S_β as integrals taken over $[0;1]^{n-1} \subset R^n$. Finally, in the special case $n=2$ we write out these kernels in an explicit form (see Theorem 4.12).

2. Preliminaries

In this section, we present several well-known facts which will be used in what follows.

Fact 1. For $\alpha > -1$ and $k = 0, 1, 2, \ldots$, put

$$J^{(1)}(\alpha;k) = \int_D |w|^{2k}\left(1-|w|^2\right)^\alpha dm(w), \tag{2.1}$$

then

$$J^{(1)}(\alpha;k) = \pi\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+2)}. \tag{2.2}$$

Moreover, if $\alpha > -1$ and $k, l = 0, 1, 2, \ldots$ $(k \neq l)$, then

$$\int_D w^k\overline{w}^l\left(1-|w|^2\right)^\alpha dm(w) = 0. \tag{2.3}$$

Remark 2.1. Assume that $\rho > 0$ and φ is a continuous positive (i.e., $\varphi > 0$) function in $(0;\rho)$. If $k, l = 0, 1, 2, \ldots$ $(k \neq l)$, then

$$\int_{|w|<\rho} w^k\overline{w}^l\varphi(|w|)dm(w) = 0, \tag{2.4}$$

when the corresponding integral exists.

Fact 2. For $\lambda \neq 0, -1, -2, \ldots$,

$$\frac{1}{(1-x)^\lambda} = \sum_{k=0}^{\infty}\frac{\Gamma(k+\lambda)}{\Gamma(\lambda)\Gamma(k+1)}x^k, \quad |x| < 1. \tag{2.5}$$

Fact 3. If $\operatorname{Re} p > 0, \operatorname{Re} q > 0$, then

$$\int_0^1 t^{p-1}(1-t)^{q-1}dt \equiv B(p;q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.6}$$

Fact 4. If $g \in C[0;T]$ $(T > 0)$, then

$$\lim_{\beta \downarrow 0} \frac{1}{\Gamma(\beta)} \int_0^T g(x)(T-x)^{\beta-1} dx = g(T). \tag{2.7}$$

As a consequence of Stirling's Formula, we have the following fact.

Fact 5. For arbitrary $\mu \in C$ and for $R \geq 0$, $R \to +\infty$

$$\left|\Gamma(\mu+R)\right| \asymp e^{-R} \cdot R^{\operatorname{Re}\mu + R - 1/2}. \tag{2.8}$$

In addition, if $\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0$, and $R \geq 0$, $R \to +\infty$, then

$$\begin{gathered} \left|\Gamma(\mu+R)\right| \asymp \left|\Gamma(\operatorname{Re}\mu + R)\right|, \\ \left|\frac{\Gamma(\mu+R)}{\Gamma(\nu+R)}\right| \asymp \frac{1}{R^{\operatorname{Re}\nu - \operatorname{Re}\mu}}. \end{gathered} \tag{2.9}$$

Fact 6. Assume that $p > 0$, $\rho > 0$, and $f \in H(|w| < \rho)$; then $\int_0^{2\pi} |f(re^{i\vartheta})|^p d\vartheta$ is a nondecreasing function of r, that is,

$$\int_0^{2\pi} \left|f\left(r_1 e^{i\vartheta}\right)\right|^p d\vartheta \leq \int_0^{2\pi} \left|f\left(r_2 e^{i\vartheta}\right)\right|^p d\vartheta, \quad 0 \leq r_1 < r_2 < \rho. \tag{2.10}$$

Corollary 2.2. *Assume that* $p > 0$, $\rho > 0$, $f \in H(|w| < \rho)$, *and* φ *is a continuous positive (i.e.,* $\varphi > 0$*) function in* $(0;\rho)$. *Then*

$$\int_{\rho_1 < |w| < \rho_2} \left|f(r \cdot w)\right|^p \cdot \varphi(|w|) dm(w) \leq \int_{\rho_1 < |w| < \rho_2} \left|f(w)\right|^p \cdot \varphi(|w|) dm(w) \tag{2.11}$$

if $0 \leq \rho_1 < \rho_2 \leq \rho$ *and* $0 \leq r \leq 1$. *In particular,*

$$\int_{|w| < \rho} \left|f(r \cdot w)\right|^p \cdot \varphi(|w|) dm(w) \leq \int_{|w| < \rho} \left|f(w)\right|^p \cdot \varphi(|w|) dm(w). \tag{2.12}$$

3. Main Function Spaces

Suppose that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$. We put

$$m_\alpha = \sum_{\alpha_i < 0} -\alpha_i, \qquad l_\alpha = \sum_{\alpha_i > 0} \alpha_i. \tag{3.1}$$

Further, we shall write $\alpha \prec (\star)$ only if the following conditions are satisfied:

$$\alpha_n + \alpha_{n-1} + \cdots + \alpha_i > -(n+1-i) \quad (1 \leq i \leq n). \tag{$\star$}$$

Similarly, if $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in C^n$, then we shall write $\beta \prec (\star)$ if only $\operatorname{Re}\beta \equiv (\operatorname{Re}\beta_1, \operatorname{Re}\beta_2, \ldots, \operatorname{Re}\beta_n) \prec (\star)$.

The following multidimensional analogue of Fact 1 is valid.

Lemma 3.1. *For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \prec (\star)$ and for arbitrary multi-index $k = (k_1, k_2, \ldots, k_n) \in Z_+^n$, put*

$$J^{(n)}(\alpha; k) \equiv J^{(n)}(\alpha_1, \alpha_2, \ldots, \alpha_n; k_1, k_2, \ldots, k_n) = \int_{B_n} \prod_{i=1}^{n} |w_i|^{2k_i} \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w); \tag{3.2}$$

then

$$\begin{aligned} J^{(n)}(\alpha; k) = \pi^n \cdot \frac{\Gamma(k_n + \alpha_n + \alpha_{n-1} + 2)}{\Gamma(k_n + \alpha_n + 2)} & \\ \times \frac{\Gamma(k_n + k_{n-1} + \alpha_n + \alpha_{n-1} + \alpha_{n-2} + 3)}{\Gamma(k_n + k_{n-1} + \alpha_n + \alpha_{n-1} + 3)} & \\ \times \cdots \times \frac{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_3 + \alpha_2 + n - 1)}{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_3 + n - 1)} & \\ \times \frac{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + k_2 + \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_3 + \alpha_2 + \alpha_1 + n)}{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + k_2 + \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_3 + \alpha_2 + n)} & \\ \times \frac{\Gamma(k_n+1) \cdot \Gamma(k_{n-1}+1) \cdot \Gamma(k_{n-2}+1) \cdot \cdots \cdot \Gamma(k_3+1) \cdot \Gamma(k_2+1) \cdot \Gamma(k_1+1) \cdot \Gamma(\alpha_n+1)}{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_3 + k_2 + k_1 + \alpha_n + \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_3 + \alpha_2 + \alpha_1 + n + 1)}. & \end{aligned} \tag{3.3}$$

Moreover, if $\alpha \prec (\star)$ and $k = (k_1, k_2, \ldots, k_n)$, $l = (l_1, l_2, \ldots, l_n)$ are arbitrary multi-indices such that $k \neq l$, then

$$\int_{B_n} \prod_{i=1}^{n} w_i^{k_i} \prod_{i=1}^{n} \overline{w_i}^{l_i} \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) = 0. \tag{3.4}$$

Proof. We intend to establish (3.3) by induction in n. For $n = 1$ we simply arrived at (2.2). Assume the validity of (3.3) for some n and proceed to the case of $n + 1$. Note that arbitrary

$w \in B_{n+1}$ can be written as $w = (w', w_{n+1})$, where $w' \in B_n$, $|w_{n+1}| < \sqrt{1-|w'|^2}$. Consequently, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}) \prec (\star)$ and for multi-index $k = (k_1, k_2, \ldots, k_n, k_{n+1}) \in Z_+^{n+1}$, we have

$$\begin{aligned} J^{(n+1)}(\alpha; k) &\equiv J^{(n+1)}(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}; k_1, k_2, \ldots, k_n, k_{n+1}) \\ &= \int_{B_{n+1}} \prod_{i=1}^{n+1} |w_i|^{2k_i} \prod_{i=1}^{n+1} \left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \\ &= \int_{B_n} \prod_{i=1}^{n} |w_i|^{2k_i} \prod_{i=1}^{n} \left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} \\ &\quad \times \int_{|w_{n+1}|<\sqrt{1-|w'|^2}} |w_{n+1}|^{2k_{n+1}} \left(1-\left|w'\right|^2-|w_{n+1}|^2\right)^{\alpha_{n+1}} dm(w_{n+1}) dm(w'). \end{aligned} \tag{3.5}$$

A simple change of variable: $w_{n+1} = \sqrt{1-|w'|^2} \cdot \zeta$ ($\zeta \in D$) in the inner integral gives the following recurrent relation:

$$\begin{aligned} J^{(n+1)}(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}; k_1, k_2, \ldots, k_n, k_{n+1}) &= \pi \frac{\Gamma(k_{n+1}+1)\Gamma(\alpha_{n+1}+1)}{\Gamma(k_{n+1}+\alpha_{n+1}+2)} \\ \times J^{(n)}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n+\alpha_{n+1}+k_{n+1}+1; k_1, k_2, \ldots, k_n). \end{aligned} \tag{3.6}$$

Note that $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n + \alpha_{n+1} + k_{n+1} + 1) \prec (\star)$ (for all $k_{n+1} \in Z$) due to the condition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}) \prec (\star)$. Consequently, in (3.6) we can apply (3.3) to $J^{(n)}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n + \alpha_{n+1} + k_{n+1} + 1; k_1, k_2, \ldots, k_n)$ due to our inductive assumption. As a result, we obtain (3.3) but this time for $n+1$. Thus, the inductive argument is completed and the formula (3.3) is established.

Now suppose that $k = (k_1, k_2, \ldots, k_n)$, $l = (l_1, l_2, \ldots, l_n) \in Z_+^n$, and $k \neq l \Rightarrow$ there exists i_0 $(1 \le i_0 \le n)$ such that $k_{i_0} \neq l_{i_0}$. Then we can split arbitrary $w = (w_1, \ldots, w_n) \in B_n$ in $w_{\hat{i_0}} = (w_1, \ldots \hat{i_0} \ldots, w_n) \in C^{n-1}$ and $w_{i_0} \in C$ so that $|w_{\hat{i_0}}|^2 + |w_{i_0}|^2 < 1$ and $dm(w) = dm(w_{i_0}) dm(w_{\hat{i_0}})$. Hence

$$\begin{aligned} &\int_{B_n} \prod_{i=1}^{n} w_i^{k_i} \prod_{i=1}^{n} \overline{w_i}^{l_i} \prod_{i=1}^{n} \left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \\ &\quad = \int_{|w_{\hat{i_0}}|^2<1} w_1^{k_1} \cdot \overline{w_1}^{l_1} \cdot \ldots \hat{i_0} \ldots \cdot w_n^{k_n} \overline{w_n}^{l_n} \cdot \prod_{i=1}^{i_0-1} \left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} \\ &\quad\quad \times \int_{|w_{i_0}|^2<1-|w_{\hat{i_0}}|^2} w_{i_0}^{k_{i_0}} \cdot \overline{w_{i_0}}^{l_{i_0}} \cdot \prod_{i=i_0}^{n} \left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w_{i_0}) dm\left(w_{\hat{i_0}}\right). \end{aligned} \tag{3.7}$$

In view of (2.4), the inner integral in (3.7) is equal to 0, so (3.4) is also proved. □

Corollary 3.2. *If $\alpha \prec (\star)$, then*

$$\int_{B_n} \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) = \frac{\pi^n}{\prod_{i=1}^{n} (\alpha_n + \alpha_{n-1} + \cdots + \alpha_i + n + 1 - i)} < +\infty. \tag{3.8}$$

Remark 3.3. In the integrals $J^{(n)}(\alpha; k)$ (see (3.2)) instead of $\alpha \prec (\star)$, arbitrary $\beta \prec (\star)$ $(\beta \in C^n)$ can be considered and the formulas (3.3), (3.4), and (3.8) remain true after the replacement of α by β.

Definition 3.4. Assume that $0 < p < +\infty$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \prec (\star)$. Denote by $L^p_\alpha(B_n)$ the space of all complex-valued functions f in B_n with

$$M^p_\alpha(f) \equiv \int_{B_n} |f(w)|^p \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) < +\infty. \tag{3.9}$$

We obviously have

$$M^p_\alpha(f + g) \leq 2^p \left\{ M^p_\alpha(f) + M^p_\alpha(g) \right\}, \qquad M^p_\alpha(c \cdot f) = c^p \cdot M^p_\alpha(f). \tag{3.10}$$

Correspondingly, denote by $H^p_\alpha(B_n) \subset L^p_\alpha(B_n)$ the subspace of functions holomorphic in B_n. Note that $1 \leq p < q < +\infty \Rightarrow L^q_\alpha(B_n) \subset L^p_\alpha(B_n), H^q_\alpha(B_n) \subset H^p_\alpha(B_n)$.

Lemma 3.5. *Assume that $0 < p < +\infty$, $\alpha \prec (\star)$, and $f \in H(B_n)$. Then*

$$|f(z)|^p \leq \frac{2^{2n+l_\alpha}}{c_n \cdot (1 - |z|)^{2n+l_\alpha}} \cdot M^p_\alpha(f), \quad \forall z \in B_n, \tag{3.11}$$

where $c_n = m(B_n)$ is the volume of the unit ball of C^n.

Proof. Fix an arbitrary $z \in B_n$; put $r = (1 - |z|)/2$ and $B(z; r) = \{w \in C^n : |w - z| < r\} \subset B_n$. Since $|f|^p$ is subharmonic in B_n, we have

$$|f(z)|^p \leq \frac{1}{c_n \cdot r^{2n}} \int_{B(z;r)} |f(w)|^p dm(w). \tag{3.12}$$

Note that

$$\begin{aligned} w \in B(z; r) &\Longrightarrow |w| < r + |z| = \frac{1 - |z|}{2} + |z| = \frac{1 + |z|}{2} \Longrightarrow 1 - |w| > \frac{1 - |z|}{2} \\ &\Longrightarrow 1 - |w|^2 \geq 1 - |w| > \frac{1 - |z|}{2} \\ &\Longrightarrow 1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2 > \frac{1 - |z|}{2} \quad (1 \leq i \leq n). \end{aligned} \tag{3.13}$$

Hence for $w \in B(z;r)$

$$\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} \geq \begin{cases} 1, & \alpha_i \leq 0, \\ \dfrac{(1-|z|)^{\alpha_i}}{2^{\alpha_i}}, & \alpha_i > 0. \end{cases} \tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$\begin{aligned} |f(z)|^p &\leq \frac{1}{c_n \cdot ((1-|z|)/2)^{2n}} \cdot \frac{\prod_{\alpha_i>0}((1-|z|)^{\alpha_i}/2^{\alpha_i})}{\prod_{\alpha_i>0}((1-|z|)^{\alpha_i}/2^{\alpha_i})} \int_{B(z;r)} |f(w)|^p dm(w) \\ &\leq \frac{2^{2n+l_\alpha}}{c_n \cdot (1-|z|)^{2n+l_\alpha}} \int_{B(z;r)} |f(w)|^p \cdot \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \\ &\leq \frac{2^{2n+l_\alpha}}{c_n \cdot (1-|z|)^{2n+l_\alpha}} \cdot M_\alpha^p(f). \end{aligned} \tag{3.15}$$

□

Corollary 3.6. *$H_\alpha^p(B_n)$ is a closed subspace in $L_\alpha^p(B_n)$.*

Definition 3.7. If $f \in H(B_n)$ and $0 \leq r_i \leq 1$ $(1 \leq i \leq n)$, then put

$$f_{r_1 r_2 \cdots r_n}(w) \equiv f(r_1 \cdot w_1, r_2 \cdot w_2, \ldots, r_n \cdot w_n), \quad w = (w_1, w_2, \ldots, w_n) \in B_n. \tag{3.16}$$

Similarly, if $f \in H(B_n)$ and $0 \leq r \leq 1$, then put

$$f_r(w) \equiv f(r \cdot w_1, r \cdot w_2, \ldots, r \cdot w_n), \quad w = (w_1, w_2, \ldots, w_n) \in B_n. \tag{3.17}$$

In particular, if $r_i \equiv r \in [0;1]$ $(1 \leq i \leq n) \Rightarrow f_{r_1 r_2 \cdots r_n}(w) \equiv f_r(w)$, note also that $f_{11\cdots 1}(w) \equiv f_1(w) \equiv f(w)$.

Lemma 3.8. *Assume that $0 < p < +\infty$, $\alpha \prec (\star)$, and $f \in H(B_n)$. Then for all r_i, $0 \leq r_i \leq 1$ $(1 \leq i \leq n)$,*

$$M_\alpha^p(f_{r_1 r_2 \cdots r_n}) \leq M_\alpha^p(f). \tag{3.18}$$

In particular, $f \in H_\alpha^p(B_n) \Rightarrow f_{r_1 r_2 \cdots r_n} \in H_\alpha^p(B_n)$.

Proof. Evidently, it suffices to fix i_0 $(1 \leq i_0 \leq n)$ and consider the case $0 \leq r_{i_0} \leq 1$, $r_i = 1$ $(1 \leq i \leq n, i \neq i_0)$. In other words, it is sufficient to show that

$$M_\alpha^p\left(f_{11\cdots 1 r_{i_0} 1\cdots 11}\right) \leq M_\alpha^p(f). \tag{3.19}$$

To this end, we proceed as follows:

$$
\begin{aligned}
M_\alpha^p\left(f_{11\cdots 1 r_{i_0} 1\cdots 11}\right) &= \int_{B_n} \left|f(w_1,\ldots,w_{i_0-1}, r_{i_0}\cdot w_{i_0}, w_{i_0+1},\ldots,w_n)\right|^p \\
&\quad \times \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \\
&= \int_{|w_{\hat{i_0}}|^2<1} \prod_{i=1}^{i_0-1}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} \\
&\quad \times \int_{|w_{i_0}|^2<1-|w_{\hat{i_0}}|^2} \left|f(w_1,\ldots,w_{i_0-1}, r_{i_0}\cdot w_{i_0}, w_{i_0+1},\ldots,w_n)\right|^p \\
&\quad \times \prod_{i=i_0}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w_{i_0})\,dm\left(w_{\hat{i_0}}\right).
\end{aligned}
\tag{3.20}
$$

An application of Corollary 2.2 (see (2.12)) to the inner integral gives the desired inequality. □

Corollary 3.9. *Assume that* $0 < p < +\infty$, $\alpha \prec (\star)$, *and* $f \in H(B_n)$. *Then for all* r, $0 \le r \le 1$,

$$
M_\alpha^p(f_r) \le M_\alpha^p(f). \tag{3.21}
$$

In particular, $f \in H_\alpha^p(B_n) \Rightarrow f_r \in H_\alpha^p(B_n)$.

Lemma 3.10. *Assume that* $0 < p < +\infty$, $\alpha \prec (\star)$, *and* $f \in H(B_n)$. *Then for all* r_i, $0 \le r_i \le 1$ $(1 \le i \le n)$ *and for arbitrary* ε $(0 < \varepsilon \le 1)$,

$$
\begin{aligned}
&\int_{1-\varepsilon<|w|<1} \left|f_{r_1 r_2\cdots r_n}(w)\right|^p \cdot \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \\
&\quad \le \int_{1-\varepsilon<|w|<1} |f(w)|^p \cdot \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w).
\end{aligned}
\tag{3.22}
$$

Proof. Since the case $\varepsilon = 1$ coincides with (3.18), we can suppose that $0 < \varepsilon < 1$. Further, similarly to the proof of Lemma 3.8, it suffices to fix i_0 $(1 \le i_0 \le n)$ and consider the case $0 \le r_{i_0} \le 1, r_i = 1$ $(1 \le i \le n, i \ne i_0)$:

$$
\begin{aligned}
&\int_{1-\varepsilon<|w|<1} \left|f(w_1,\ldots,w_{i_0-1}, r_{i_0}\cdot w_{i_0}, w_{i_0+1},\ldots,w_n)\right|^p \\
&\quad \times \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \\
&\equiv I_1 + I_2,
\end{aligned}
\tag{3.23}
$$

where

$$I_1 = \int_{|w_{\hat{i_0}}|^2<(1-\varepsilon)^2} \prod_{i=1}^{i_0-1}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i}$$
$$\times \int_{(1-\varepsilon)^2-|w_{\hat{i_0}}|^2<|w_{i_0}|^2<1-|w_{\hat{i_0}}|^2} \left|f(w_1,\ldots,w_{i_0-1},r_{i_0}\cdot w_{i_0},w_{i_0+1},\ldots,w_n)\right|^p \tag{3.24}$$
$$\times \prod_{i=i_0}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w_{i_0})\,dm\left(w_{\hat{i_0}}\right),$$

$$I_2 = \int_{(1-\varepsilon)^2<|w_{\hat{i_0}}|^2<1} \prod_{i=1}^{i_0-1}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i}$$
$$\times \int_{|w_{i_0}|^2<1-|w_{\hat{i_0}}|^2} \left|f(w_1,\ldots,w_{i_0-1},r_{i_0}w_{i_0},w_{i_0+1},\ldots,w_n)\right|^p \tag{3.25}$$
$$\times \prod_{i=i_0}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w_{i_0})\,dm\left(w_{\hat{i_0}}\right).$$

It remains to apply Corollary 2.2. More exactly, let us apply (2.11) to the inner integral of (3.24) and (2.12) to the inner integral of (3.25). By this, the proof is complete. □

Corollary 3.11. *Assume that* $0 < p < +\infty$, $\alpha \prec (\star)$, *and* $f \in H(B_n)$. *Then for all* r, $0 \le r \le 1$ *and for arbitrary* ε $(0 < \varepsilon \le 1)$,

$$\int_{1-\varepsilon<|w|<1} \left|f_r(w)\right|^p \cdot \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w)$$
$$\le \int_{1-\varepsilon<|w|<1} \left|f(w)\right|^p \cdot \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w). \tag{3.26}$$

Lemma 3.12. *Assume that* $0 < p < +\infty$, $\alpha \prec (\star)$, *and* $f \in H^p_\alpha(B_n)$. *Then*

$$M^p_\alpha\left(f - f_{r_1 r_2 \cdots r_n}\right) \longrightarrow 0 \quad \text{as } r_i \uparrow 1 \ (1 \le i \le n). \tag{3.27}$$

Proof. For arbitrary fixed $\varepsilon \in (0;1)$ and for all r_i, $0 \le r_i \le 1$ $(1 \le i \le n)$, we have

$$M^p_\alpha\left(f - f_{r_1 r_2 \cdots r_n}\right) = \int_{B_n} \left|f(w_1,w_2,\ldots,w_n) - f(r_1\cdot w_1, r_2\cdot w_2,\ldots,r_n\cdot w_n)\right|^p$$
$$\times \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) \tag{3.28}$$
$$= \int_{|w|<1-\varepsilon} + \int_{1-\varepsilon<|w|<1} \equiv J_1 + J_2.$$

In view of the inequality $(a+b)^p \le 2^p(a^p+b^p)$ $(a,b \ge 0,\ p>0)$ and due to (3.22),

$$J_2 \le 2^{p+1} \cdot \int_{1-\varepsilon<|w|<1} |f(w_1,w_2,\dots,w_n)|^p \times \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w). \tag{3.29}$$

Note that the condition $f \in H^p_\alpha(B_n)$ implies $J_2 \to 0$ as $\varepsilon \downarrow 0$. Besides, it is evident that for a fixed $\varepsilon \in (0;1) : J_1 \to 0$ as $r_i \uparrow 1$ $(1 \le i \le n)$. All these complete the proof. □

Corollary 3.13. *Assume that* $0<p<+\infty$, $\alpha \prec (\star)$, *and* $f \in H^p_\alpha(B_n)$. *Then*

$$M^p_\alpha(f-f_r) \longrightarrow 0 \quad \text{as } r \uparrow 1. \tag{3.30}$$

4. Main Integral Representations

In fact, Lemma 3.1 asserts that the system

$$\left\{w_1^{k_1} \cdot w_2^{k_2} \cdot \cdots \cdot w_n^{k_n}\right\}_{k_1,k_2,\dots,k_n=0}^{\infty} \tag{4.1}$$

is orthogonal in the Hilbert space

$$L^2\left\{B_n; \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w)\right\}. \tag{4.2}$$

And what about completeness of this system?

Proposition 4.1. *Assume that* $1 \le p < +\infty$, $\alpha \prec (\star)$, *and* $f \in H^p_\alpha(B_n)$. *If for arbitrary multi-index* $k=(k_1,k_2,\dots,k_n) \in Z^n_+$

$$\int_{B_n} f(w) \cdot \overline{w_1}^{k_1} \cdot \overline{w_2}^{k_2} \cdot \cdots \cdot \overline{w_n}^{k_n} \cdot \prod_{i=1}^{n}\left(1-|w_1|^2-|w_2|^2-\cdots-|w_i|^2\right)^{\alpha_i} dm(w) = 0, \tag{4.3}$$

then $f \equiv 0$.

Proof. Since $f \in H(B_n) \Rightarrow$

$$\begin{aligned} f(w) &= \sum_{l_1,l_2,\dots,l_n \ge 0} a_{l_1 l_2 \cdots l_n} \cdot w_1^{l_1} w_2^{l_2} \cdots w_n^{l_n} \\ &\equiv \sum_{|l| \ge 0} a_l \cdot w^l, \quad w \in B_n, \end{aligned} \tag{4.4}$$

where $|l| = l_1 + l_2 + \cdots + l_n$ for arbitrary multi-index $l = (l_1, l_2, \ldots, l_n) \in Z_+^n$. Moreover, for arbitrary compact $K \subset B_n$ there exist a positive series $\sum_{|l|\geq 0} c_l < +\infty$ such that $|a_l \cdot w^l| \leq c_l$ $(|l| \geq 0)$ uniformly in $w \in K$. Consequently, for arbitrary fixed $r \in (0;1)$

$$f(r \cdot w) = \sum_{|l|\geq 0} a_l \cdot r^{|l|} w^l, \quad w \in \overline{B_n}, \tag{4.5}$$

and, in addition, there exist a positive series $\sum_{|l|\geq 0} d_l < +\infty$ such that $|a_l \cdot r^{|l|} \cdot w^l| \leq d_l$ $(|l| \geq 0)$ uniformly in $w \in \overline{B_n}$.

Now let us fix arbitrary $k_1 \geq 0,\ k_2 \geq 0, \ldots, k_n \geq 0$. Since $f_r \longrightarrow f$ (as $r \uparrow 1$) in the space $H_\alpha^p(B_n)$ (Corollary 3.13), we have (as $r \uparrow 1$)

$$\begin{aligned} &\int_{B_n} f(r \cdot w) \cdot \prod_{i=1}^n \overline{w_i}^{k_i} \cdot \prod_{i=1}^n \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) \\ &\longrightarrow \int_{B_n} f(w) \cdot \prod_{i=1}^n \overline{w_i}^{k_i} \cdot \prod_{i=1}^n \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) = 0. \end{aligned} \tag{4.6}$$

On the other hand,

$$\begin{aligned} &\int_{B_n} f(r \cdot w) \cdot \prod_{i=1}^n \overline{w_i}^{k_i} \cdot \prod_{i=1}^n \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) \\ &= \sum_{|l|\geq 0} a_l \cdot r^{|l|} \int_{B_n} \prod_{i=1}^n w_i^{l_i} \cdot \prod_{i=1}^n \overline{w_i}^{k_i} \cdot \prod_{i=1}^n \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) \\ &= a_k \cdot r^{|k|} \cdot J^{(n)}(\alpha_1, \ldots, \alpha_n; k_1, \ldots, k_n). \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we obtain that $a_k \cdot r^{|k|} \cdot J^{(n)}(\alpha_1, \ldots, \alpha_n; k_1, \ldots, k_n) \longrightarrow 0$ (as $r \uparrow 1$) $\Rightarrow a_k = 0$ and this takes place for any multi-index $k \Rightarrow f \equiv 0$. □

Corollary 4.2. *If $\alpha \prec (\star)$, then the system*

$$\left\{ \frac{w_1^{k_1} \cdot w_2^{k_2} \cdot \cdots \cdot w_n^{k_n}}{\sqrt{J^{(n)}(\alpha_1, \ldots, \alpha_n; k_1, \ldots, k_n)}} \right\}_{k_1, k_2, \ldots, k_n = 0}^{\infty} \tag{4.8}$$

is an orthonormal basis in the space $H_\alpha^2(B_n)$.

Corollary 4.3. *If $\alpha \prec (\star)$, then the reproducing kernel (i.e., the kern-function) of the space $H_\alpha^2(B_n)$ is given by the formula*

$$S_\alpha(z; w) = \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} \frac{(z_1 \overline{w_1})^{k_1} \cdot (z_2 \overline{w_2})^{k_2} \cdot \cdots \cdot (z_n \overline{w_n})^{k_n}}{J^{(n)}(\alpha_1, \ldots, \alpha_n; k_1, \ldots, k_n)}, \quad z, w \in B_n. \tag{4.9}$$

Definition 4.4. Assume that $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in C^n$ and $\beta \prec (\star) \Leftrightarrow \operatorname{Re}\beta = (\operatorname{Re}\beta_1, \operatorname{Re}\beta_2, \ldots, \operatorname{Re}\beta_n) \prec (\star)$. For arbitrary $z \in B_n,\ w \in \overline{B_n}$, let us introduce the following kernel (motivated by (4.9)):

$$S_\beta(z;w) = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} \frac{(z_1\overline{w_1})^{k_1} \cdot (z_2\overline{w_2})^{k_2} \cdot \cdots \cdot (z_n\overline{w_n})^{k_n}}{J^{(n)}(\beta_1,\ldots,\beta_n;k_1,\ldots,k_n)} \tag{4.9$'$}$$

or, in a slightly different form,

$$S_\beta(z;w) = \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} \frac{(z_1\overline{w_1})^{k_1} \cdot (z_2\overline{w_2})^{k_2} \cdot \cdots \cdot (z_n\overline{w_n})^{k_n}}{J^{(n)}(\beta_1,\ldots,\beta_n;k_1,\ldots,k_n)} \equiv \sum_{k=0}^{\infty} P_k(z;w), \tag{4.9$''$}$$

where

$$\begin{aligned}
P_k(z;w) &\\
= \frac{1}{\pi^n} &\cdot \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} (z_1\overline{w_1})^{k_1} \cdot (z_2\overline{w_2})^{k_2} \cdot \cdots \cdot (z_n\overline{w_n})^{k_n} \\
&\times \frac{\Gamma(k_n+\beta_n+2)}{\Gamma(k_n+\beta_n+\beta_{n-1}+2)} \times \frac{\Gamma(k_n+k_{n-1}+\beta_n+\beta_{n-1}+3)}{\Gamma(k_n+k_{n-1}+\beta_n+\beta_{n-1}+\beta_{n-2}+3)} \\
&\times \cdots \times \frac{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_3+n-1)}{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_3+\beta_2+n-1)} \\
&\times \frac{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+k_2+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_3+\beta_2+n)}{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+k_2+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_3+\beta_2+\beta_1+n)} \\
&\times \frac{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+k_2+k_1+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_3+\beta_2+\beta_1+n+1)}{\Gamma(k_n+1)\cdot\Gamma(k_{n-1}+1)\cdot\Gamma(k_{n-2}+1)\cdot\cdots\cdot\Gamma(k_3+1)\cdot\Gamma(k_2+1)\cdot\Gamma(k_1+1)\cdot\Gamma(\beta_n+1)}.
\end{aligned} \tag{4.10}$$

Note that if we take $\operatorname{Re}\beta_n > -1,\ \beta_i = 0\ (1 \le i \le n-1)$ in (4.9$''$)-(4.10),then

$$S_\beta(z;w) = \frac{\Gamma(\beta_n+n+1)}{\pi^n \cdot \Gamma(\beta_n+1)} \cdot \frac{1}{(1 - z_1\overline{w_1} - z_2\overline{w_2} - z_3\overline{w_3} - \cdots - z_n\overline{w_n})^{\beta_n+n+1}}, \tag{4.11}$$

that is, we arrive at the kernel of the integral representations (1.4).

We are going to estimate $P_k(z;w),\ k = 0,1,2,\ldots \Rightarrow$ to estimate the kernel $S_\beta(z;w)$. Let us put (compare with (3.1))

$$m_\beta^\star = \sum_{\operatorname{Re}\beta_i<0,\,1\le i\le n-1} (-\operatorname{Re}\beta_i), \qquad l_\beta^\star = \sum_{\operatorname{Re}\beta_i>0,\,1\le i\le n-1} (\operatorname{Re}\beta_i). \tag{4.12}$$

Then in view of (2.9), we have ($z \in B_n, w \in \overline{B_n}, k = 0,1,2,\ldots$)

$$
\begin{aligned}
|P_k(z;w)| &\le \operatorname{const}(n;\beta)\cdot \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} \frac{1}{(k_n+1)^{\operatorname{Re}\beta_{n-1}}} \times \frac{1}{(k_n+k_{n-1}+1)^{\operatorname{Re}\beta_{n-2}}} \\
&\quad \times\cdots\times \frac{1}{(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+1)^{\operatorname{Re}\beta_2}} \times \frac{1}{(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+k_2+1)^{\operatorname{Re}\beta_1}} \\
&\quad \times \frac{\left|\Gamma\left(k_n+k_{n-1}+\cdots+k_2+k_1+\beta_n+\beta_{n-1}+\cdots+\beta_2+\beta_1+n+1\right)\right|}{\left|\Gamma\left(\beta_n+1\right)\right|\cdot\Gamma(k_n+k_{n-1}+\cdots+k_2+k_1+1)} \\
&\quad \times \left[\frac{\Gamma(k_n+k_{n-1}+\cdots+k_2+k_1+1)\cdot|z_1\overline{w_1}|^{k_1}\cdot|z_2\overline{w_2}|^{k_2}\cdot\cdots\cdot|z_n\overline{w_n}|^{k_n}}{\Gamma(k_n+1)\cdot\Gamma(k_{n-1}+1)\cdot\cdots\cdot\Gamma(k_2+1)\cdot\Gamma(k_1+1)}\right] \\
&\le \operatorname{const}(n;\beta)\cdot \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} \frac{(k_n+k_{n-1}+k_{n-2}+\cdots+k_3+k_2+k_1+1)^{m_\beta^\star}}{(k_n+1)^{l_\beta^\star}} \\
&\quad \times \frac{\left|\Gamma\left(k_n+k_{n-1}+\cdots+k_2+k_1+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+1\right)\right|}{\left|\Gamma\left(\operatorname{Re}\beta_n+1\right)\right|\cdot\Gamma(k_n+k_{n-1}+\cdots+k_2+k_1+1)} \\
&\quad \times \left[\frac{\Gamma(k_n+k_{n-1}+\cdots+k_2+k_1+1)\cdot|z_1\overline{w_1}|^{k_1}\cdot|z_2\overline{w_2}|^{k_2}\cdot\cdots\cdot|z_n\overline{w_n}|^{k_n}}{\Gamma(k_n+1)\cdot\Gamma(k_{n-1}+1)\cdot\cdots\cdot\Gamma(k_2+1)\cdot\Gamma(k_1+1)}\right] \\
&\le \operatorname{const}(n;\beta)\cdot \frac{(k+1)^{m_\beta^\star}\cdot\Gamma\left(k+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+1\right)}{\Gamma\left(\operatorname{Re}\beta_n+1\right)\cdot\Gamma(k+1)} \\
&\quad \times \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} \frac{\Gamma(k+1)\cdot|z_1\overline{w_1}|^{k_1}\cdot|z_2\overline{w_2}|^{k_2}\cdot\cdots\cdot|z_n\overline{w_n}|^{k_n}}{\Gamma(k_n+1)\cdot\Gamma(k_{n-1}+1)\cdot\cdots\cdot\Gamma(k_2+1)\cdot\Gamma(k_1+1)} \\
&\le \operatorname{const}(n;\beta)\cdot \frac{(k+1)^{m_\beta^\star}\cdot\Gamma\left(k+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+1\right)}{\Gamma\left(\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+1\right)\cdot\Gamma(k+1)} \\
&\quad \times \left(|z_1\overline{w_1}|+|z_2\overline{w_2}|+\cdots+|z_n\overline{w_n}|\right)^k.
\end{aligned}
\tag{4.13}
$$

Further, due to (2.8),

$$
\begin{aligned}
&\Gamma\left(k+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+1\right) \\
&\quad \asymp e^{-(k+1)}\cdot(k+1)^{\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+k+1-1/2} \quad (k=0,1,2,\ldots).
\end{aligned}
\tag{4.14}
$$

Consequently,

$$\begin{aligned}(k+1)^{m_\beta^\star}\cdot\Gamma(k+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+1)\\ \asymp e^{-(k+1)}\cdot(k+1)^{\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+n+m_\beta^\star+k+1-1/2}\\ \asymp\Gamma\Big(k+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+m_\beta^\star+n+1\Big).\end{aligned}\tag{4.15}$$

Combining (4.13), (4.15) and taking into account the inequality

$$|z_1\overline{w_1}|+|z_2\overline{w_2}|+\cdots+|z_n\overline{w_n}|\le|z|\cdot|w|\le|z|<1\quad\Big(z\in B_n,w\in\overline{B_n}\Big),\tag{4.16}$$

we obtain

$$|P_k(z;w)|\le\operatorname{const}(n;\beta)\cdot\frac{\Gamma\Big(k+\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+m_\beta^\star+n+1\Big)\cdot|z|^k}{\Gamma\Big(\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+m_\beta^\star+n+1\Big)\cdot\Gamma(k+1)}.\tag{4.17}$$

In view of the formula (2.5), the summation of (4.17) over $k=0,1,2,\dots$ gives

$$\Big|S_\beta(z;w)\Big|\le\frac{\operatorname{const}(n;\beta)}{(1-|z|)^{\operatorname{Re}\beta_n+\operatorname{Re}\beta_{n-1}+\cdots+\operatorname{Re}\beta_2+\operatorname{Re}\beta_1+m_\beta^\star+n+1}}=\frac{\operatorname{const}(n;\beta)}{(1-|z|)^{\operatorname{Re}\beta_n+l_\beta^\star+n+1}}.\tag{4.18}$$

Thus, we have the following theorem.

Theorem 4.5. *If $\beta\prec(\star)$, then the kernel $S_\beta(z;w)$ satisfies the following properties ($z\in B_n,w\in\overline{B_n}$):*

(a) *the series (4.9′) converges absolutely for arbitrary fixed $z\in B_n,w\in\overline{B_n}$;*

(b) *in the expansion $S_\beta(z;w)=\sum_{k=0}^{\infty}P_k(z;w)$ (see (4.9″)), the series is majorated by a convergent positive numerical series $\sum_{k=0}^{\infty}b_k$ uniformly in $z\in K\subset\subset B_n,w\in\overline{B_n}$;*

(c) *$S_\beta(z;w)$ is holomorphic in $z\in B_n$;*

(d) *$S_\beta(z;w)$ is antiholomorphic in $w\in B_n$ and continuous in $w\in\overline{B_n}$;*

(e) *the estimate*

$$\Big|S_\beta(z;w)\Big|\le\frac{const(n;\beta)}{(1-|z|)^{n+1+\operatorname{Re}\beta_n+l_\beta^\star}}\tag{4.19}$$

is valid uniformly in $z\in B_n,w\in\overline{B_n}$.

Definition 4.6. Assume that $1\le p<+\infty$ and $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\prec(\star)$. For $\beta=(\beta_1,\beta_2,\dots,\beta_n)\in C^n$, we shall write $\beta\succ(p;\alpha)$ if only (compare with (1.3))

(i) when $p = 1$, then

$$\operatorname{Re}\beta_i \geq \alpha_i \quad (1 \leq i \leq n); \tag{4.20}$$

(ii) when $1 < p < +\infty$, then

$$\operatorname{Re}\beta_n + \operatorname{Re}\beta_{n-1} + \cdots + \operatorname{Re}\beta_i > \frac{\alpha_n + \alpha_{n-1} + \cdots + \alpha_i + n + 1 - i}{p} - (n + 1 - i). \tag{4.21}$$

Obviously, $\beta \succ (p;\alpha) \Rightarrow \beta \prec (\star)$.

Theorem 4.7. *Assume that* $1 \leq p < +\infty$, $\alpha \prec (\star)$, *and* $\beta \succ (p;\alpha)$. *Then arbitrary* $f \in H^p_\alpha(B_n)$ *admits the following integral representations:*

$$f(z) = \int_{B_n} f(w) S_\beta(z;w) \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w), \quad z \in B_n, \tag{4.22}$$

$$\overline{f(0)} = \int_{B_n} \overline{f(w)} S_\beta(z;w) \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w), \quad z \in B_n. \tag{4.23}$$

Proof. Let us fix an arbitrary $f \in H^p_\alpha(B_n)$ and an arbitrary $z \in B_n$. Further,

$$f(w) = \sum_{|l| \geq 0} a_l \cdot w^l = \sum_{m=0}^{\infty} F_m(w), \quad w \in B_n, \tag{4.24}$$

where

$$F_m(w) = \sum_{l_1 + l_2 + \cdots + l_{n-1} + l_n = m} a_l \cdot w^l \quad (m = 0, 1, 2, \ldots). \tag{4.25}$$

Hence, for arbitrary fixed $r \in (0;1)$,

$$f_r(w) = \sum_{|l| \geq 0} a_l \cdot r^{|l|} \cdot w^l = \sum_{m=0}^{\infty} r^m \cdot F_m(w), \quad w \in \overline{B_n}. \tag{4.26}$$

Obviously,

$$\overline{f_r(w)} = \sum_{|l| \geq 0} \overline{a_l} \cdot r^{|l|} \cdot \overline{w}^l = \sum_{m=0}^{\infty} r^m \cdot \overline{F_m(w)}, \quad w \in \overline{B_n}. \tag{4.27}$$

Note that the first series in (4.26) and (4.27) and the second series in (4.26) and (4.27) are majorated (uniformly in $w \in \overline{B_n}$) by positive convergent numerical series $\sum_{|l| \geq 0} c_l < +\infty$ and $\sum_{m=0}^{\infty} \widetilde{c_m} < +\infty$, respectively.

First we intend to prove both formulas (4.22), (4.23) for the function f_r, that is,

$$f_r(z) = \int_{B_n} f_r(w) S_\beta(z;w) \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w), \tag{4.22$'$}$$

$$\overline{f(0)} = \int_{B_n} \overline{f_r(w)} S_\beta(z;w) \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w). \tag{4.23$'$}$$

In view of Theorem 4.5(b), (4.26)-(4.27) and due to Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\int_{B_n} f_r(w) S_\beta(z;w) \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w) \\ &\quad = \sum_{k,m=0}^{\infty} \int_{B_n} r^m F_m(w) \cdot P_k(z;w) \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w), \end{aligned} \tag{4.28}$$

$$\begin{aligned} &\int_{B_n} \overline{f_r(w)} S_\beta(z;w) \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w) \\ &\quad = \sum_{k,m=0}^{\infty} \int_{B_n} r^m \overline{F_m(w)} \cdot P_k(z;w) \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w), \end{aligned} \tag{4.29}$$

and the series $\sum_{k,m=0}^{\infty} \ldots$ in (4.28) and (4.29) converge absolutely.

Taking into account (3.4) and Remark 3.3, we conclude that all terms in the right-hand side of (4.29) (except of the case $k = m = 0$) are equal to zero. Hence the left-hand side of (4.29) is equal to

$$\begin{aligned} &\int_{B_n} \overline{a_{00\cdots0}} \cdot \frac{1}{\pi^n} \cdot \frac{\Gamma(\beta_n + 2)}{\Gamma(\beta_n + \beta_{n-1} + 2)} \times \frac{\Gamma(\beta_n + \beta_{n-1} + 3)}{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + 3)} \\ &\qquad \times \cdots \times \frac{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + n - 1)}{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + n - 1)} \\ &\qquad \times \frac{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + n)}{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + \beta_1 + n)} \\ &\qquad \times \frac{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + \beta_1 + n + 1)}{\Gamma(\beta_n + 1)} \\ &\qquad \times \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w) \\ &\quad = \overline{f(0)} \cdot \frac{\int_{B_n} \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w)}{J^{(n)}(\beta_1, \beta_2, \ldots, \beta_n; 0, 0, \ldots, 0)} = \overline{f(0)}. \end{aligned} \tag{4.30}$$

Thus, (4.23$'$) has been proved.

Now let us proceed to (4.22′). If $k \neq m$, then in view of (3.4) and Remark 3.3

$$\int_{B_n} r^m F_m(w) \cdot P_k(z;w) \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w) = 0. \tag{4.31}$$

If $k = m$, then in view of (3.4), (3.2), (3.3) and Remark 3.3

$$\begin{aligned}
&\int_{B_n} r^k F_k(w) \cdot P_k(z;w) \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\beta_i} dm(w) \\
&= \frac{r^k}{\pi^n} \cdot \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} a_{k_1 k_2 \cdots k_{n-1} k_n} \frac{z_1^{k_1} \cdot z_2^{k_2} \cdot \cdots \cdot z_{n-1}^{k_{n-1}} \cdot z_n^{k_n}}{\Gamma(k_n+1) \cdot \Gamma(k_{n-1}+1) \cdot \cdots \cdot \Gamma(k_2+1) \cdot \Gamma(k_1+1)} \\
&\times \frac{\Gamma(k_n + \beta_n + 2)}{\Gamma(k_n + \beta_n + \beta_{n-1} + 2)} \times \frac{\Gamma(k_n + k_{n-1} + \beta_n + \beta_{n-1} + 3)}{\Gamma(k_n + k_{n-1} + \beta_n + \beta_{n-1} + \beta_{n-2} + 3)} \\
&\times \cdots \times \frac{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + \beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + n - 1)}{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + \beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + n - 1)} \\
&\times \frac{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + k_2 + \beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + n)}{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + k_2 + \beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + \beta_1 + n)} \\
&\times \frac{\Gamma(k_n + k_{n-1} + k_{n-2} + \cdots + k_3 + k_2 + k_1 + \beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_3 + \beta_2 + \beta_1 + n + 1)}{\Gamma(\beta_n + 1)} \\
&\times J^{(n)}(\beta_1, \beta_2, \ldots, \beta_n; k_1, k_2, \ldots, k_n) \\
&= r^k \cdot \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} a_{k_1 k_2 \cdots k_{n-1} k_n} z_1^{k_1} \cdot z_2^{k_2} \cdot \cdots \cdot z_{n-1}^{k_{n-1}} \cdot z_n^{k_n} \cdot 1 = r^k \cdot F_k(z).
\end{aligned} \tag{4.32}$$

Consequently, (4.28), (4.31), and (4.32) together yield

$$\int_{B_n} f_r(w) S_\beta(z;w) \prod_{k=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_k|^2\right)^{\beta_k} dm(w) = \sum_{k=0}^{\infty} r^k \cdot F_k(z) = f_r(z). \tag{4.33}$$

Thus, (4.22′) also has been proved.

Now we intend to make passage $r \uparrow 1$ in (4.22′) and (4.23′). Evidently, the left-hand sides tend to $f(z)$ and $\overline{f(0)}$ correspondingly. It remains to show that the right-hand sides of

(4.22$'$) and (4.23$'$) tend to the right-hand sides of (4.22) and (4.23), respectively. In view of the estimate (4.19), it suffices to show that

$$I(r) \equiv \int_{B_n} \left|f(w) - f_r(w)\right| \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\operatorname{Re}\beta_i} dm(w) \xrightarrow{r\uparrow 1} 0. \tag{4.34}$$

If $p = 1$, then due to the condition (4.20) and Corollary 3.13

$$\begin{aligned} I(r) &= \int_{B_n} \left|f(w) - f_r(w)\right| \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\operatorname{Re}\beta_i - \alpha_i + \alpha_i} dm(w) \\ &\leq \int_{B_n} \left|f(w) - f_r(w)\right| \cdot \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{\alpha_i} dm(w) \\ &= M_\alpha^1(f - f_r) \xrightarrow{r\uparrow 1} 0. \end{aligned} \tag{4.35}$$

If $1 < p < \infty$, then an application of Hölder integral inequality to $I(r)$ gives

$$I(r) \leq \left[M_\alpha^p(f - f_r)\right]^{1/p} \cdot \left(\int_{B_n} \prod_{i=1}^{n} \left(1 - |w_1|^2 - |w_2|^2 - \cdots - |w_i|^2\right)^{q(\operatorname{Re}\beta_i - \alpha_i/p)} dm(w)\right)^{1/q}. \tag{4.36}$$

Here the integral over B_n is finite due to Corollary 3.2 and the condition $\beta \succ (p;\alpha)$. Consequently, $I(r) \to 0,\ r \uparrow 1$, in view of Corollary 3.13. Thus, the theorem is proved. □

Theorem 4.8. *For* $\operatorname{Re}\beta_n > -1, \operatorname{Re}\beta_i > 0\ (1 \leq i \leq n-1)$*, and for* $z \in B_n, w \in \overline{B_n}$*,*

$$\begin{aligned} S_\beta(z;w) &= \frac{\Gamma(\beta_n + \beta_{n-1} + \cdots + \beta_2 + \beta_1 + n + 1)}{\pi^n \cdot \Gamma(\beta_1)\Gamma(\beta_2) \cdot \cdots \cdot \Gamma(\beta_{n-1})\Gamma(\beta_n + 1)} \\ &\quad \times \int_{[0;1]^{n-1}} \frac{\prod_{i=n}^{2} s_i^{\beta_n + \beta_{n-1} + \cdots + \beta_i + n + 1 - i} \cdot \prod_{i=n}^{2} (1 - s_i)^{\beta_{i-1} - 1} ds_n ds_{n-1} \cdots ds_3 ds_2}{(1 - z_1\overline{w_1} - s_2 z_2 \overline{w_2} - s_2 s_3 z_3 \overline{w_3} - \cdots - s_2 s_3 \cdots s_n z_n \overline{w_n})^{\beta_n + \beta_{n-1} + \cdots + \beta_2 + \beta_1 + n + 1}}. \end{aligned} \tag{4.37}$$

Proof. For $k = 0, 1, 2, 3, \ldots$ according to (4.10) and (2.6), we have

$$
\begin{aligned}
P_k(z;w) = {} & \frac{1}{\pi^n} \cdot \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} \frac{(z_1\overline{w_1})^{k_1} \cdot (z_2\overline{w_2})^{k_2} \cdot \cdots \cdot (z_{n-1}\overline{w_{n-1}})^{k_{n-1}} \cdot (z_n\overline{w_n})^{k_n}}{\Gamma(k_1+1) \cdot \Gamma(k_2+1) \cdot \cdots \cdot \Gamma(k_{n-1}+1) \cdot \Gamma(k_n+1)} \\
& \times \frac{1}{\Gamma(\beta_{n-1})} \int_0^1 s_n^{k_n+\beta_n+1} (1-s_n)^{\beta_{n-1}-1} ds_n \\
& \times \frac{1}{\Gamma(\beta_{n-2})} \int_0^1 s_{n-1}^{k_n+k_{n-1}+\beta_n+\beta_{n-1}+2} (1-s_{n-1})^{\beta_{n-2}-1} ds_{n-1} \\
& \times \cdots \times \frac{1}{\Gamma(\beta_1)} \int_0^1 s_2^{k_n+k_{n-1}+k_{n-2}+\cdots+k_2+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_2+n-1} (1-s_2)^{\beta_1-1} ds_2 \\
& \times \frac{\Gamma(k_n+k_{n-1}+k_{n-2}+\cdots+k_2+k_1+\beta_n+\beta_{n-1}+\beta_{n-2}+\cdots+\beta_2+\beta_1+n+1)}{\Gamma(\beta_n+1)} \\
= {} & \frac{1}{\pi^n \cdot \Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_{n-1})\Gamma(\beta_n+1)} \\
& \times \int_{[0;1]^{n-1}} s_n^{\beta_n+1} \cdot s_{n-1}^{\beta_n+\beta_{n-1}+2} \cdot \cdots \cdot s_2^{\beta_n+\beta_{n-1}+\cdots+\beta_2+n-1} \\
& \times (1-s_n)^{\beta_{n-1}-1} \cdot (1-s_{n-1})^{\beta_{n-2}-1} \cdot \cdots \cdot (1-s_2)^{\beta_1-1} \\
& \times \sum_{k_1+k_2+\cdots+k_{n-1}+k_n=k} \frac{(z_1\overline{w_1})^{k_1} \cdot (s_2z_2\overline{w_2})^{k_2} \cdot (s_2s_3z_3\overline{w_3})^{k_3} \cdot \cdots \cdot (s_2s_3\cdots s_nz_n\overline{w_n})^{k_n}}{\Gamma(k_n+1) \cdot \Gamma(k_{n-1}+1) \cdot \cdots \cdot \Gamma(k_2+1) \cdot \Gamma(k_1+1)} \\
& \times \frac{\Gamma(k+1)}{\Gamma(k+1)} \cdot \Gamma(k+\beta_n+\beta_{n-1}+\cdots+\beta_2+\beta_1+n+1) ds_n ds_{n-1} \cdots ds_3 ds_2 \\
= {} & \frac{\Gamma(k+\beta_n+\beta_{n-1}+\cdots+\beta_2+\beta_1+n+1)}{\pi^n \cdot \Gamma(k+1)\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_{n-1})\Gamma(\beta_n+1)} \\
& \times \int_{[0;1]^{n-1}} s_n^{\beta_n+1} \cdot s_{n-1}^{\beta_n+\beta_{n-1}+2} \cdot \cdots \cdot s_2^{\beta_n+\beta_{n-1}+\cdots+\beta_2+n-1} \\
& \times (1-s_n)^{\beta_{n-1}-1} \cdot (1-s_{n-1})^{\beta_{n-2}-1} \cdot \cdots \cdot (1-s_2)^{\beta_1-1} \\
& \times (z_1\overline{w_1} + s_2z_2\overline{w_2} + s_2s_3z_3\overline{w_3} + \cdots + s_2s_3\cdots s_nz_n\overline{w_n})^k ds_n ds_{n-1} \cdots ds_3 ds_2 \\
= {} & P_k(z;w).
\end{aligned}
\tag{4.38}
$$

The summation of these relations over $k = 0, 1, 2, 3, \ldots$ yields (see (2.5))

$$
\begin{aligned}
S_\beta(z;w) = \sum_{k=0}^{\infty} P_k(z;w) &= \frac{\Gamma(\beta_n + \beta_{n-1} + \beta_{n-2} + \cdots + \beta_2 + \beta_1 + n + 1)}{\pi^n \cdot \Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_{n-1})\Gamma(\beta_n + 1)} \\
&\times \int_{[0;1]^{n-1}} s_n^{\beta_n+1} \cdot s_{n-1}^{\beta_n+\beta_{n-1}+2} \cdot \cdots \cdot s_2^{\beta_n+\beta_{n-1}+\cdots+\beta_2+n-1} \\
&\times (1 - s_n)^{\beta_{n-1}-1} \cdot (1 - s_{n-1})^{\beta_{n-2}-1} \cdot \cdots \cdot (1 - s_2)^{\beta_1 - 1} \\
&\times \frac{ds_n ds_{n-1} \cdots ds_3 ds_2}{(1 - z_1\overline{w_1} - s_2 z_2 \overline{w_2} - s_2 s_3 z_3 \overline{w_3} - \cdots - s_2 s_3 \cdots s_n z_n \overline{w_n})^{\beta_n+\beta_{n-1}+\cdots+\beta_2+\beta_1+n+1}}.
\end{aligned}
\tag{4.39}
$$

Thus, (4.37) is proved. □

Remark 4.9. Under the conditions of the theorem, the formula (4.37) easily implies the assertions (c), (d), and (e) of Theorem 4.5.

Remark 4.10. If we take $\operatorname{Re}\beta_n > -1$, $\beta_i \downarrow 0$ $(1 \le i \le n-1)$ in (4.37), then the formal application of (2.7) gives the following formula:

$$
S_\beta(z;w) = \frac{\Gamma(\beta_n + n + 1)}{\pi^n \cdot \Gamma(\beta_n + 1)} \cdot \frac{1}{(1 - z_1\overline{w_1} - z_2\overline{w_2} - z_3\overline{w_3} - \cdots - z_n\overline{w_n})^{\beta_n+n+1}},
\tag{4.40}
$$

that is, we arrive at the kernel of the integral representations (1.4).

Remark 4.11. In fact, the kernel $S_\beta(z;w)$, defined for $\beta \prec (\star)$ by $(4.9')$-$(4.9'')$, can be considered as an analytic continuation in β of the integral $S_\beta(z;w)$, defined for $\operatorname{Re}\beta_n > -1, \operatorname{Re}\beta_i > 0$ $(1 \le i \le n-1)$ by (4.37).

Now we consider an interesting special case $n = 2$, when (in comparison with $(4.9')$-$(4.9'')$ or (4.37)) the kernel $S_\beta(z;w)$ can be written out in an explicit form.

Theorem 4.12. *Assume that* $\beta = (\beta_1, \beta_2) \prec (\star)$*, that is,* $\operatorname{Re}\beta_2 > -1, \operatorname{Re}\beta_2 + \operatorname{Re}\beta_1 > -2$*. Then for* $z = (z_1, z_2) \in B_2, w = (w_1, w_2) \in \overline{B_2}$,

$$
\begin{aligned}
S_\beta(z;w) = \frac{1}{\pi^2} &\cdot \frac{\beta_1(\beta_2 + 1)}{(1 - z_1\overline{w_1})^{\beta_1+1}(1 - z_1\overline{w_1} - z_2\overline{w_2})^{\beta_2+2}} \\
&+ \frac{1}{\pi^2} \cdot \frac{(\beta_2 + 1)(\beta_2 + 2)}{(1 - z_1\overline{w_1})^{\beta_1}(1 - z_1\overline{w_1} - z_2\overline{w_2})^{\beta_2+3}}.
\end{aligned}
\tag{4.41}
$$

Proof. According to (4.9′)-(4.9″)-(4.10) and in view of the formula (2.5),

$$
\begin{aligned}
S_\beta(z;w) &= \frac{1}{\pi^2}\cdot\sum_{k_1,k_2=0}^{\infty}\frac{(z_1\overline{w_1})^{k_1}\cdot(z_2\overline{w_2})^{k_2}}{\Gamma(k_1+1)\Gamma(k_2+1)}\frac{\Gamma(\beta_2+k_2+2)}{\Gamma(\beta_2+\beta_1+k_2+2)}\cdot\frac{\Gamma(\beta_2+\beta_1+k_2+k_1+3)}{\Gamma(\beta_2+1)}\\
&= \frac{1}{\pi^2}\cdot\sum_{k_2=0}^{\infty}\frac{(z_2\overline{w_2})^{k_2}}{\Gamma(k_2+1)}\cdot\frac{\Gamma(\beta_2+k_2+2)}{\Gamma(\beta_2+\beta_1+k_2+2)\cdot\Gamma(\beta_2+1)}\\
&\quad\times\sum_{k_1=0}^{\infty}\frac{(z_1\overline{w_1})^{k_1}}{\Gamma(k_1+1)}\cdot\Gamma(\beta_2+\beta_1+k_2+k_1+3)\\
&= \frac{1}{\pi^2}\cdot\sum_{k_2=0}^{\infty}\frac{(z_2\overline{w_2})^{k_2}}{\Gamma(k_2+1)}\cdot\frac{\Gamma(\beta_2+k_2+2)\cdot\Gamma(\beta_2+\beta_1+k_2+3)}{\Gamma(\beta_2+\beta_1+k_2+2)\cdot\Gamma(\beta_2+1)}\frac{1}{(1-z_1\overline{w_1})^{\beta_2+\beta_1+k_2+3}}\\
&= \frac{1}{\pi^2}\cdot\frac{1}{(1-z_1\overline{w_1})^{\beta_2+\beta_1+3}}\cdot\sum_{k_2=0}^{\infty}\left(\frac{z_2\overline{w_2}}{1-z_1\overline{w_1}}\right)^{k_2}\frac{\Gamma(\beta_2+k_2+2)}{\Gamma(k_2+1)\Gamma(\beta_2+1)}\cdot(\beta_2+\beta_1+k_2+2)\\
&= \frac{1}{\pi^2}\cdot\frac{1}{(1-z_1\overline{w_1})^{\beta_2+\beta_1+3}}\cdot\sum_{k_2=0}^{\infty}\left(\frac{z_2\overline{w_2}}{1-z_1\overline{w_1}}\right)^{k_2}\\
&\quad\times\left\{\beta_1\frac{\Gamma(\beta_2+k_2+2)}{\Gamma(k_2+1)\Gamma(\beta_2+1)}+\frac{\Gamma(\beta_2+k_2+3)}{\Gamma(k_2+1)\Gamma(\beta_2+1)}\right\}\\
&= \frac{1}{\pi^2}\cdot\frac{1}{(1-z_1\overline{w_1})^{\beta_2+\beta_1+3}}\left\{\frac{\beta_1(\beta_2+1)}{(1-z_2\overline{w_2}/(1-z_1\overline{w_1}))^{\beta_2+2}}+\frac{(\beta_2+1)(\beta_2+2)}{(1-z_2\overline{w_2}/(1-z_1\overline{w_1}))^{\beta_2+3}}\right\}\\
&= \frac{1}{\pi^2}\cdot\frac{1}{(1-z_1\overline{w_1})^{\beta_2+\beta_1+3}}\left\{\frac{\beta_1(\beta_2+1)(1-z_1\overline{w_1})^{\beta_2+2}}{(1-z_1\overline{w_1}-z_2\overline{w_2})^{\beta_2+2}}+\frac{(\beta_2+1)(\beta_2+2)(1-z_1\overline{w_1})^{\beta_2+3}}{(1-z_1\overline{w_1}-z_2\overline{w_2})^{\beta_2+3}}\right\}\\
&= \frac{1}{\pi^2}\cdot\left\{\frac{\beta_1(\beta_2+1)}{(1-z_1\overline{w_1})^{\beta_1+1}(1-z_1\overline{w_1}-z_2\overline{w_2})^{\beta_2+2}}+\frac{(\beta_2+1)(\beta_2+2)}{(1-z_1\overline{w_1})^{\beta_1}(1-z_1\overline{w_1}-z_2\overline{w_2})^{\beta_2+3}}\right\}.
\end{aligned}
\tag{4.42}
$$

Thus, (4.41) is established. □

Remark 4.13. During the proof we regroup the double series: $\sum_{k_1,k_2=0}^{\infty}\{\dots\} = \sum_{k_2=0}^{\infty}\{\sum_{k_1=0}^{\infty}\{\dots\}\}$, which is legitimate in view of Theorem 4.5(a). In fact, we apply Fubini's theorem for double series.

Remark 4.14. The analysis of the proof shows that we have established (4.41) only for those $z=(z_1,z_2)\in B_2$ and $w=(w_1,w_2)\in\overline{B_2}$, which satisfy the condition $|z_2\overline{w_2}/(1-z_1\overline{w_1})|<1$. In fact, this is quite sufficiently since both sides of (4.41) are holomorphic in $z\in B_n$, antiholomorphic in $w\in B_n$, and continuous in $w\in\overline{B_n}$ (see Theorem 4.5(c), (d)).

Remark 4.15. If one takes $\mathrm{Re}\,\beta_2 > -1$, $\beta_1 = 0$ in Theorem 4.12, then

$$S_{\beta_1;\beta_2}(z_1, z_2; w_1, w_2) = \frac{1}{\pi^2} \cdot \frac{(\beta_2 + 1)(\beta_2 + 2)}{(1 - z_1\overline{w_1} - z_2\overline{w_2})^{\beta_2+3}}, \tag{4.43}$$

which coincides with (4.11) (or (4.40)) for the case $n = 2$.

Remark 4.16. Note that for the same case $n = 2$ and under slightly restrictive conditions $\mathrm{Re}\,\beta_2 > -1, \mathrm{Re}\,\beta_1 > 0$, the formula (4.37) gives ($z \in B_2, w \in \overline{B_2}$)

$$S_\beta(z; w) = \frac{\Gamma(\beta_2 + \beta_1 + 3)}{\pi^2 \cdot \Gamma(\beta_2 + 1)\Gamma(\beta_1)} \cdot \int_0^1 \frac{s^{\beta_2+1} \cdot (1 - s)^{\beta_1-1}}{(1 - z_1\overline{w_1} - s \cdot z_2\overline{w_2})^{\beta_2+\beta_1+3}} ds. \tag{4.44}$$

Remark 4.17. It can be shown (we omit the proof) that under the conditions $\mathrm{Re}\,\beta_2 > 0, \mathrm{Re}\,\beta_2 + \mathrm{Re}\,\beta_1 > -2$ the following interesting formula is valid ($z \in B_2, w \in \overline{B_2}$):

$$S_{(\beta_1,\beta_2)}(z_1, z_2; w_1, w_2) = \left\{ \frac{z_2\overline{w_2} \cdot \partial/\partial(z_2\overline{w_2}) + \beta_2 + 1}{\beta_2} \right\} \cdot S_{(\beta_1+1,\beta_2-1)}(z_1, z_2; w_1, w_2). \tag{4.45}$$

References

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